The Radius of Analyticity for Real Analytic Functions

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NonLinear Functional Analysis

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E: a real or complex Banach space.

Power series at the origin:

$$f(x) = \sum_{n=0}^{\infty} P_n(x)$$

where P_n are bounded n-homogeneous polynomials:

$$P_n(x) = A_n(x, x, \ldots, x) = A_n(x)^n$$

where A_n are bounded, symmetric n-linear forms on E.

Norms:

$$\|P_n\| = \sup\{|P_n(x)| : \|x\| \leqslant 1\}$$

$$\|A_n\| = \sup\{|A_n(x_1, \ldots, x_n)| : \|x_j\| \leqslant 1\}$$

The Radius of Uniform Convergence

$R = R(f, 0)$: the largest $R \geqslant 0$ such that the power series

$$f(x) = \sum_{n=0}^{\infty} P_n(x)$$

converges uniformly in the ball $B_\rho(0)$ for every $\rho < R$.

The Cauchy-Hadamard formula:

$$R = \left(\limsup \|P_n\|^{1/n}\right)^{-1}$$

The Taylor series of f at a point $a \in B_R(0)$:

$$T_a f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{d}^n f(a)(x - a)$$

The complex case:
We have the Cauchy Inequalities:

$$\left\| \frac{1}{n!} \hat{d}^n f(a) \right\| \leqslant \frac{\|f\|_{B_\rho(a)}}{\rho^n}$$

and it follows that the radius of uniform convergence of the Taylor series $T_a f(x)$ is at least $R - \|a\|$.

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The real case:
We no longer have the Cauchy Integral.

Analytic and Fully Analytic Functions

Let E be a real Banach space and U an open subset. A function $f\colon U \to \mathbb{R}$ is

1. **Analytic at** $a \in U$ if there is a power series at a with positive radius of uniform convergence that converges to f within the ball of uniform convergence.
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2. **Analytic in** U if f is analytic at every point in U.

3. **Fully analytic in** U if it is analytic in U and for every $a \in U$, the Taylor series at a converges uniformly in *every* closed ball centered at a that is contained in U.

The Radius of Analyticity

Let

$$f(x) = \sum_{n=0}^{\infty} P_n(x)$$

be a power series with radius of uniform convergence R.

The **radius of analyticity** R_A is the largest $r > 0$ such that f is fully analytic in the ball $B_r(0)$.

Clearly, we have $R_A \leqslant R$.

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Question:

$$R_A = R\ ?$$

First Approach: Complexification

Given a power series

$$f(x) = \sum_{n=0}^{\infty} P_n(x)$$

with radius of uniform convergence R.

Complexify the space: $E_{\mathbb{C}} = E + iE$ with a suitable norm.

Compexify the polynomials to get a power series in $z \in E_{\mathbb{C}}$:

$$f(z) = \sum_{n=0}^{\infty} \big(P_n\big)_{\mathbb{C}}(z)\,.$$

Use the Cauchy estimates and restrict back to E.

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This gives the estimate

$$R_A \geqslant \frac{R}{2}$$

Second Approach: Polarization

Expand around the point a:

$$
\begin{aligned}
f(x) = \sum_{n=0}^{\infty} P_n(x) &= \sum_{n=0}^{\infty} A_n(a + (x-a))^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} A_n(a^{n-k}(x-a)^k) \\
&= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \binom{n}{k} A_n(a)^{n-k}(x-a)^k \\
&= \sum_{k=0}^{n} \frac{1}{k!} \hat{d}^k f(a)(x-a)
\end{aligned}
$$

Absolute convergence will follow if

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}|A_n(a^{n-k}(x-a)^k)|$$

$$\leqslant \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\|A_n\|\|a\|^{n-k}\|x-a\|^k)$$

$$\leqslant \sum_{n=0}^{\infty}\|A_n\|(\|a\|+\|x-a\|)^n \leqslant \sum_{n=0}^{\infty} e^n\|P_n\|(\|a\|+\|x-a\|)^n < \infty$$

and this happens when

$$\|a\| + \|x-a\| < \frac{R}{e}$$

The Curse of Polarization

The factor e has appeared because of the Polarization Inequality:

$$\|A_n\| \leqslant \frac{n^n}{n!}\|P_n\|\,.$$

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Banach (and others) showed that, in the case of a Hilbert space,

$$\|A_n\| = \|P_n\|.$$

So, for power series on Hilbert spaces,

$$R_A = R.$$

In the general case, we can do a little better than e — the full polarization inequality is not required. Instead of $\|A_n\|$, we only have to estimate

$$\|A_n\|_{(2)} = \sup\{|A_n(u^k v^{n-k})| : 0 \leqslant k \leqslant n, \|u\|, \|v\| \leqslant 1\}$$

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L. Harris 1972, P. Hájek & M. Johanis 2014:

$$R_A \geqslant \frac{R}{\sqrt{2}}$$

(P. Hájek & M. Johanis, *Smooth Analysis in Banach Spaces*, 2014)

Also proved by **Papadiamantis & Sarantopoulos 2016**, using an improvement of Nguyen's techniques.

The Constant of Analyticity for a Banach space

Let E be a real Banach space.

We define the **constant of analyticity of** E, denoted $\mathcal{A}(E)$, to be the supremum of the set of positive real numbers ρ for which every power series at the origin in E, with unit radius of uniform convergence, has radius of analyticity at least ρ.

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In general,

$$\frac{1}{\sqrt{2}} \leqslant \mathcal{A}(\mathrm{E}) \leqslant 1$$

and when E is a Hilbert space, $\mathcal{A}(\mathrm{E}) = 1$

ℓ_1 is the worst (or best) case.

Theorem (C. Boyd, RR & N. Snigireva)

Let E, F *be real Banach spaces such that* E *is a quotient space of* F. *Then* $\mathcal{A}(\mathrm{E}) \geqslant \mathcal{A}(\mathrm{F})$.

Every Banach space is a quotient of $\ell_1(\mathrm{I})$ for a suitable indexing set I.

The value of the constant of analyticity of $\ell_1(\mathrm{I})$ is essentially determined by a countable set of points in the space. Each point in $\ell_1(\mathrm{I})$ is supported by a countable subset of I. Hence

Theorem (C. Boyd, RR & N. Snigireva)

Let E *be any real Banach space. Then*

$$\mathcal{A}(\mathrm{E}) \geqslant \mathcal{A}(\ell_1)\,.$$

Harmonic functions on $\mathbb{R}^n$

Let f be a harmonic function on the open euclidean ball $\|x\| < R$ in $\mathbb{R}^n$. Then f is real analytic in this ball and has a power series expansion

$$f(x) = \sum_{n=0}^{\infty} P_n(x)$$

(where P_n are harmonic n-homogeneous polynomials) with radius of convergence at least equal to R.

What happens if we expand using monomials?

$$f(x) = \sum_{n=0}^{\infty} \sum_{\substack{\alpha\in\mathbb{N}^{(\mathbb{N})} \\ |\alpha|=n}} c_\alpha x^\alpha = \sum_{\alpha\in\mathbb{N}^{(\mathbb{N})}} c_\alpha x^\alpha \text{ ?}$$

W.K. Hayman 1970:

1. *Let* f *be a harmonic function on the open euclidean ball at the origin in* $\mathbb{R}^n$ *of radius* R*. Then the monomial expansion of* f *is absolutely uniformly convergent for* $\|x\| \leqslant \rho$*, where*

$$\rho < \frac{R}{\sqrt{2}},$$

but this expansion may diverge at some points on the sphere $\|x\| = R/\sqrt{2}$*.*

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$$\rho < \frac{R}{\sqrt{2}},$$

but this expansion may diverge at some points on the sphere $\|x\| = R/\sqrt{2}$.

2. *Suppose that* f *is harmonic in the open euclidean disc at the origin in* $\mathbb{R}^2$ *of radius* R, *but not in any larger open disc centered at the origin.*
Then the monomial expansion of f *is absolutely uniformly convergent on every compact subset of the open square* $\|x\|_1 < R$. *It diverges at all points outside this square that do not lie on the coordinate axes.*

The Banach Lattice Viewpoint

Let E, F be Banach lattices
An n-homogeneous polynomial $P_n = \hat{A}_n$ is **positive** if the unique symmetric n-linear form that generates P_n is positive in each variable:

$$A_n(x_1, \ldots, x_n) \geqslant 0 \quad \text{if} \quad x_1, \ldots, x_n \geqslant 0\,.$$

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An n-homogeneous polynomial P_n is said to be **regular** if it is the difference of two positive n-homogeneous polynomials. If F is Dedekind complete, this condition is equivalent to the existence of the absolute value (modulus): this is the smallest positive n-homogeneous polynomial satisfying $\pm P_n \leqslant |P_n|$. In particular,

$$|P_n(x)| \leqslant |P_n|(|x|) \quad \text{for every } x \in E.$$

B. Grecu & RR 2005:

Let E be a Banach space with a 1-unconditional Schauder basis. A bounded n-homogeneous polynomial on E is regular if and only if its monomial expansion is unconditionally convergent at every point in E. And if

$$P_n(x) = \sum_{|\alpha|=n} c_\alpha x^\alpha$$

is regular, then $|P_n|$ is given by

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Hayman's result:

If $\sum_{n=0}^{\infty} P_n(x)$ has radius of convergence R,

then $\sum_{n=0}^{\infty} |P_n|(x)$ has radius of convergence at least $R/\sqrt{2}$.

Power series on Banach lattices

Every Banach lattice E can be **complexified**: on the algebraic complexification $E_{\mathbb{C}} = E + iE$, a modulus is defined by

$$|z| = |x + iy| = \sup\{x \cos\theta + y \sin\theta : 0 \leqslant \theta \leqslant 2\pi\}.$$

It can be shown that this supremum always exists. We also have

$$|z| = \sqrt{x^2 + y^2}$$

where this expression is defined using the Krivine functional calculus.

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A **complex Banach lattice** is a complex Banach space of the form $E_{\mathbb{C}}$, where E is a Banach lattice and the norm is given by

$$\|z\| = \|\,|z|\,\|.$$

Proposition (C. Boyd, RR & N. Snigireva)

Let E, F *be Banach lattices, with* F *Dedekind complete and let* $P_n : E \to F$ *be a regular* n*-homogeneous polynomial.*

(a) *The complexification of* P_n *satisfies*

$$|(P_n)_{\mathbb{C}}(z)| \leqslant |P_n|(|z|)$$

for every $z \in E_{\mathbb{C}}$.

(b) *In particular, if* P_n *is positive, then*

$$|(P_n)_{\mathbb{C}}(z)| \leqslant P_n(|z|)$$

and so $\|(P_n)_{\mathbb{C}}\| = \|P_n\|$.

Theorem (C. Boyd, RR & N. Snigireva)

Let E *be a real Banach lattice and let*

$$f(x) = \sum_{n=0}^{\infty} P_n(x)$$

be a power series with positive terms. Then the radius of analyticity and the radius of uniform convergence are equal.