Polynomial Daugavet Property for Representable Spaces

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Valencia, October 19th, 2017

DAUGAVET EQUATION

Daugavet Equation Appearance

- **I. K. Daugavet, 1963:** Every compact linear operator T on $C[0,1]$ satisfies the equation $\|\mathrm{Id} + T\| = 1 + \|T\|$.

Definition

*Let X be a Banach space and let $T : X \to X$ be a bounded linear operator. We say that T satisfies the **Daugavet equation** if*

$$\|\mathrm{Id} + T\| = 1 + \|T\|. \tag{DE}$$

*We say that X has the **Daugavet property** (DP) if every rank-one operator on X satisfies the (DE).*

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Classical examples

- **J. R. Holub, 1987:**
  Weakly compact linear operators on $L_1(\mu)$, where μ is an atomless σ-finite measure, satisfy (DE).

- **D. Werner, 1996:**
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Daugavet Equation for Polynomials

Generalization of (DE)

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*Let X be a Banach space and let Φ be a bounded mapping from the closed unit ball B_X into X. We say that Φ satisfies the **Daugavet equation** if*

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*Let X be a Banach space. We say that X has the **polynomial Daugavet property** (PDP) if every weakly compact polynomial on X satisfies (DE).*

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- **Y. Choi, D. García, M. Maestre and M. Martín, 2007:**
  If Ω is a completely regular Hausdorff space without isolated points, then $C_b(\Omega, X)$ has the polynomial Daugavet property.
- **Y. Choi, D. García, M. Maestre and M. Martín, 2008:**
  If μ is an atomless σ-finite measure, then $L_\infty(\mu, X)$ has the polynomial Daugavet property.
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Daugavet Property on Representable Spaces

Representable spaces

Definition

*Let K be a compact Hausdorff space. A Banach space X is said **K-representable** if there exists a family $(X_k)_{k\in K}$ of Banach spaces such that X is (linearly isometric to) a closed $C(K)$-submodule of the $C(K)$-module $\prod_{k\in K}^{\infty} X_k$ in such a way that, for every $x \in S_X$ and every $\varepsilon > 0$, the set $\{k \in K : \|x(k)\| > 1-\varepsilon\}$ is infinite.*

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J. B. Guerrero and A. Rodrígues-Palacios, 2008

Proposition

Every representable Banach space has the Daugavet property.

G. Botelho and S., 2016

Theorem

Every representable Banach space has the polynomial Daugavet property.

Proposition

Let X be a Banach space. Suppose that for every $x, z \in S_X$, $\omega \in \mathbb{T}$ and $\varepsilon > 0$, there exists a sequence (z_n) in X such that $\sum_n z_n$ is weakly unconditionally Cauchy and

$$\limsup \|z + z_n\| \leq 1 \quad \text{and} \quad \|x + \omega(z + z_n)\| > 2 - \varepsilon$$

for every $n \in \mathbb{N}$. Then X has the polynomial Daugavet property.

Let X be a K-representable Banach space and let $(X_k)_{k\in K}$ be as in definition of representable space. Fix $x, z \in S_X$, $\omega \in \mathbb{T}$ and $\varepsilon > 0$. Then the set

$$V = \left\{k \in K : \|x(k)\| > 1 - \frac{\varepsilon}{2}\right\}$$

is infinite and there exist a sequence $(k_n)_{n\in\mathbb{N}}$ in V and a sequence $(V_n)_{n\in\mathbb{N}}$ of pairwise disjoint nonempty open subsets of K, such that k_n belongs to V_n for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, apply Urysohn's lemma to find a continuous function $f_n : K \longrightarrow [0,1]$ such that $f_n(k_n) = 1$ and $f_n(k) = 0$ for every $k \in K \setminus V_n$. Now, define

$$z_n = f_n(\omega^{-1}x - z) \in X.$$

By disjointness of the supports, the series $\sum_n z_n$ is weakly unconditionally Cauchy.

Proof of the Theorem

Let X be a K-representable Banach space and let $(X_k)_{k\in K}$ be as in definition of representable space. Fix $x, z \in S_X$, $\omega \in \mathbb{T}$ and $\varepsilon > 0$. Then the set

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Moreover, for every $n \in \mathbb{N}$ and $k \in K$,

$$\|[z + z_n](k)\| = \left\| (1 - f_n(k))\, z(k) + f_n(k)\omega^{-1} x(k) \right\|,$$

so $\|z + z_n\| \leq 1$ by convexity. Also, for every $n \in \mathbb{N}$,

$$\begin{aligned} \|x + \omega(z + z_n)\| &\geq \|x(k_n) + \omega\,(z(k_n) + z_n(k_n))\| \\ &= \|2x(k_n)\| > 2\left(1 - \frac{\varepsilon}{2}\right) = 2 - \varepsilon. \end{aligned}$$

The result follows from the previous proposition.

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Results

- Consider Z a norming subspace of Y^* for Y, and τ a vector space topology on Y with $\sigma(Y,Z) \leq \tau \leq n$, where n denotes the norm topology on Y and $\sigma(Y,Z)$ denotes the weak topology on Y relative to its duality with Z. If K is a perfect compact Hausdorff topological space, then $C(K,(Y,\tau))$ is K-representable by ([5], Theorem 3.1).

Corollary

Let K be a perfect compact Hausdorff topological space, let Y be a non-zero Banach space, let Z be a norming subspace of Y^* for Y, and let τ be a vector space topology on Y with $\sigma(Y,Z) \leq \tau \leq n$. Then $C(K,(Y,\tau))$ satisfies the polynomial Daugavet property.

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Results

- Given a Banach space X, a representable Banach space Y and a closed subspace M of $\mathcal{L}(X, Y)$ such that $\mathcal{L}(Y) \circ M \subset M$, ([5], Lemma 2.5 and Corollary 2.6) proved that M and $X \widehat{\otimes}_\epsilon Y$ are representable spaces.

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Let X be a Banach space, let Y be a representable Banach space, and let M be a closed subspace of $\mathcal{L}(X, Y)$ such that $\mathcal{L}(Y) \circ M \subset M$. Then M has the polynomial Daugavet property.

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*Let X be a Banach space. A closed subspace J of X will be called an **M-summand** if there is a closed subspace $J^\perp$ of X such that X is the algebraic direct sum of J and $J^\perp$, and*

$$\|x + x^\perp\| = \max\{\|x\|, \|x^\perp\|\} \quad \text{for } x \in J, x^\perp \in J^\perp.$$

- By ([5], Theorem 4.3) we know that every dual Banach space Y without minimal M-summands is a representable space.

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Every dual Banach space without minimal M-summands has the polynomial Daugavet property.

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*Let X be a Banach space. A closed subspace J of X will be called an **L-summand** if there is a closed subspace $J^\perp$ of X such that X is the algebraic direct sum of J and $J^\perp$, and*

$$\|x + x^\perp\| = \|x\| + \|x^\perp\| \quad \text{for } x \in J, x^\perp \in J^\perp.$$

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Main References

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Thanks!