On the extension of Whitney ultrajets

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General notation

E will always denote a compact set in $\mathbb{R}^n$.

j^∞_E denotes the mapping which sends a smooth function to the infinite jet consisting of its partial derivatives of all orders restricted to E.

For a weight function $\omega : [0, +\infty) \to [0, +\infty)$ (satisfying some basic growth properties) we denote by $\mathcal{B}^{\{\omega\}}(\mathbb{R}^n)$ the associated space of ultradifferentiable functions f on $\mathbb{R}^n$; the growth rate of the sequence $(\|f^{(\alpha)}\|_{L^\infty(\mathbb{R}^n)})_{\alpha\in\mathbb{N}^n}$ is regulated in terms of ω. (The letter $\mathcal{B}$ emphasizes that the bounds are global in $\mathbb{R}^n$.)

Similarly $\mathcal{B}^{\{\omega\}}(E)$ is the space of jets on the compact subset $E \subseteq \mathbb{R}^n$ with a growth rate regulated by ω, so-called ultrajets.

Characterization result for preserving the class - BBMT '91

Theorem

Let ω be a weight function, TFAE:

(*i*) *For every compact $E \subseteq \mathbb{R}^n$ the jet mapping $j_E^\infty : \mathcal{B}^{\{\omega\}}(\mathbb{R}^n) \to \mathcal{B}^{\{\omega\}}(E)$ is surjective.*

(*ii*) *ω is strong, i.e., $\int_1^\infty \frac{\omega(tu)}{u^2} du \leq C\omega(t) + C$ for all $t > 0$ and some $C > 0$.*

Remark: This result holds true for the so-called Beurling case as well.

Main problem/question

Let ω be a non-quasianalytic (i.e. $\int_1^\infty \frac{\omega(t)}{t^2}dt < +\infty$) weight function, let σ be another weight function.
Under which conditions is the jet mapping j^∞_E defined on $\mathcal{B}^{\{\omega\}}(\mathbb{R}^n)$ surjective onto $\mathcal{B}^{\{\sigma\}}(E)$ for all compact sets $E \subseteq \mathbb{R}^n$?

Known results concerning our main problem/question

(*i*) Bonet, Meise, and Taylor '92 - for the one-point set $E = \{0\}$
(*ii*) Langenbruch '94 - for compact convex E

The mapping $j^\infty_E : \mathcal{B}^{\{\omega\}}(\mathbb{R}^n) \to \mathcal{B}^{\{\sigma\}}(E)$ is surjective if and only if

$$\exists\, C > 0\ \forall\, t \geq 0 : \quad \int_1^\infty \frac{\omega(tu)}{u^2}du \leq C\sigma(t) + C. \qquad (1)$$

For the proofs many tools from Functional Analysis have been applied.

Question: What can be said about general compact sets $E \subseteq \mathbb{R}^n$?

Our result I

We give a complete answer assuming three additional conditions:

(*i*) ω is concave (each strong weight function is equivalent to a concave one by Meise, and Taylor '88).

(*ii*) $\sigma(t) = o(t)$ as $t \to \infty$ (any strong weight function has this property).

(*iii*) (!!!) The weight matrix $\mathcal{S} = \{S^x : x > 0\}$ associated with σ satisfies the "good property":

$$\forall\, x > 0\ \exists\, y > 0\ \exists\, C \geq 1\ \forall\, 1 \leq j \leq k : \quad \frac{S_j^x}{jS_{j-1}^x} \leq C \frac{S_k^y}{kS_{k-1}^y}. \tag{2}$$

Using the weight matrix notation we have (Rainer, S. '14):

$$\mathcal{B}^{\{\sigma\}}(\mathbb{R}^n) = \varinjlim_{x>0} \mathcal{B}^{\{S^x\}}(\mathbb{R}^n). \tag{3}$$

Our result II

Theorem

Let ω be a non-quasianalytic concave weight function. Let σ be a weight function satisfying $\sigma(t) = o(t)$ as $t \to \infty$ and $\mathcal{S} = \{S^x : x > 0\}$ has the good property.
Then the following conditions are equivalent:

- *(i) For every compact $E \subseteq \mathbb{R}^n$ the jet mapping $j^\infty_E : \mathcal{B}^{\{\omega\}}(\mathbb{R}^n) \to \mathcal{B}^{\{\sigma\}}(E)$ is surjective.*
- *(ii) There is $C > 0$ such that $\int_1^\infty \frac{\omega(tu)}{u^2} du \le C\sigma(t) + C$ for all $t > 0$.*

Note: Concavity and the good property are not invariant under equivalence, it is enough that the assumptions are satisfied up to equivalence of weight functions.

General remarks on the proof of the main theorem I

We combine methods/techniques from BBMT '91 and Chaumat, and Chollet '94, more precisely:

(i) generalize special cut-off functions as constructed in BBMT '91 to a mixed setting,

(ii) combine the resulting partition of unity $\{\varphi_i\}_i$ subordinate to a collection of *Whitney cubes* $(Q_i)_i$ with centers $(x_i)_i$ with the technique of Ch./Ch. '94 - *mixed weight sequence setting* (based on an extension method of Dynkin '80),

(iii) the extension of an ultrajet $F \in \mathcal{B}^{\{\sigma\}}$ is defined as a linear combination

$$\sum_i \varphi_i T_{\hat{x}_i}^{p(x_i)} F$$

of Taylor polynomials, where the degree $p(x_i)$ depends on the distance of x_i to E and $\hat{x}_i \in E$ realizes this distance.

General remarks on the proof of the main theorem II

The dependence of p is given through a counting function corresponding to the sequences in $\mathcal{S} = \{S^x :> 0\}$, the good property is only needed here.

We will have to work with two counting functions, generalizing the technique of Ch./Ch. '94.

In a recent paper (Rainer, S. '16) we have generalized Ch./Ch. '94 to *admissible* (large) unions of Denjoy-Carleman classes. But modifying the construction of special cut-off functions in Ch./Ch. '94 yields an undesired restrictive condition on the weight functions/matrices.

Weight functions

By a weight function we mean a continuous increasing function $\omega : [0,\infty) \to [0,\infty)$ with $\omega(0) = 0$ and $\lim_{t\to\infty} \omega(t) = \infty$ that satisfies

$$\omega(2t) = O(\omega(t)) \quad \text{as } t \to \infty, \tag{4}$$

$$\omega(t) = O(t) \quad \text{as } t \to \infty, \tag{5}$$

$$\log t = o(\omega(t)) \quad \text{as } t \to \infty, \tag{6}$$

$$\varphi_\omega(t) := \omega(e^t) \text{ is convex.} \tag{7}$$

A weight function is called non-quasianalytic if

$$\int_0^\infty \frac{\omega(t)}{1+t^2}\, dt < \infty. \tag{8}$$

Two weight functions ω and σ are said to be equivalent if $\omega(t) = O(\sigma(t))$ and $\sigma(t) = O(\omega(t))$ as $t \to \infty$.

Note: For each ω there is an equivalent $\tilde{\omega}$ such that $\omega(t) = \tilde{\omega}(t)$ for large $t > 0$ and $\tilde{\omega}|_{[0,1]} = 0$.

The *Young conjugate* φ_ω^* of φ is defined by

$$\varphi_\omega^*(t) := \sup_{s \geq 0}\{st - \varphi_\omega(s)\}, \quad t \geq 0.$$

Assuming $\omega|_{[0,1]} = 0$, we have that φ_ω^* is a convex increasing function satisfying $\varphi_\omega^*(0) = 0$, $t/\varphi_\omega^*(t) \to 0$ as $t \to \infty$, and $\varphi_\omega^{**} = \varphi$.

The space $\mathcal{B}^{\{\omega\}}(\mathbb{R}^n)$

Let ω be a weight function and $l > 0$. Consider the Banach space

$$\mathcal{B}_l^{\omega}(\mathbb{R}^n) := \{f \in \mathcal{C}^{\infty}(\mathbb{R}^n) : \|f\|_l^{\omega} < \infty\},$$

where

$$\|f\|_l^{\omega} := \sup_{x \in \mathbb{R}^n, \alpha \in \mathbb{N}^n} \frac{|f^{(\alpha)}(x)|}{\exp(\frac{1}{l}\varphi_{\omega}^*(l|\alpha|))},$$

and the inductive limit

$$\mathcal{B}^{\{\omega\}}(\mathbb{R}^n) := \varinjlim_{l \in \mathbb{N}_{>0}} \mathcal{B}_l^{\omega}(\mathbb{R}^n).$$

For weight functions ω and σ we have $\mathcal{B}^{\{\omega\}} \subseteq \mathcal{B}^{\{\sigma\}}$ if and only if $\sigma(t) = O(\omega(t))$ as $t \to \infty$.

Weight sequences

Let $\mu = (\mu_k)_k$ be a positive increasing sequence with $1 = \mu_0$, the sequences $M = (M_k)_k$ and $m = (m_k)_k$ are defined by

$$\mu_0\mu_1\mu_2\cdots\mu_k = M_k = k!\cdot m_k. \tag{9}$$

We call M a *weight sequence* if $M_k^{1/k} \to \infty$ as $k \to \infty$.

(*i*) M is log-convex,

(*ii*) $(M_k)^{1/k} \le \mu_k$ follows,

(*iii*) but M is NOT assumed necessarily to be strongly log-convex, i.e. m is log-convex.

A weight sequence M is called *non-quasianalytic* if

$$\sum_k \frac{1}{\mu_k} < \infty. \tag{10}$$

The space $\mathcal{B}^{\{M\}}(\mathbb{R}^n)$

Let $M = (M_k)$ be a weight sequence and $\varrho > 0$. We consider the Banach space

$$\mathcal{B}^M_\varrho(\mathbb{R}^n) := \{f \in \mathcal{C}^\infty(\mathbb{R}^n) : \|f\|^M_\varrho < \infty\},$$

where

$$\|f\|^M_\varrho := \sup_{x\in\mathbb{R}^n, \alpha\in\mathbb{N}^n} \frac{|f^{(\alpha)}(x)|}{\varrho^{|\alpha|} M_{|\alpha|}},$$

and the inductive limit

$$\mathcal{B}^{\{M\}}(\mathbb{R}^n) := \varinjlim_{\varrho\in\mathbb{N}_{>0}} \mathcal{B}^M_\varrho(\mathbb{R}^n).$$

Traditionally, $\mathcal{B}^{\{M\}}(\mathbb{R}^n)$ is called a *Denjoy–Carleman class*.

The connection between $\mathcal{B}^{\{\omega\}}(\mathbb{R}^n)$ and $\mathcal{B}^{\{M\}}(\mathbb{R}^n)$

With each ω we associate a weight matrix $\mathcal{W}=\{W^x : x>0\}$ by

$$W^x_k := \exp(\frac{1}{x}\varphi^*(x\cdot k)).$$

We say that ω is good, if its associated weight matrix satisfies the good property.

Theorem

Let ω be a weight function, then, as locally convex spaces,

$$\mathcal{B}^{\{\omega\}}(\mathbb{R}^n) = \varinjlim_{x>0} \mathcal{B}^{\{W^x\}}(\mathbb{R}^n) = \varinjlim_{x>0} \varinjlim_{\varrho>0} \mathcal{B}^{W^x}_{\varrho}(\mathbb{R}^n). \tag{11}$$

We have $\mathcal{B}^{\{\omega\}}(\mathbb{R}^n) = \mathcal{B}^{\{W^x\}}(\mathbb{R}^n)$ for all $x>0$ if and only if

$$\exists\, H \geq 1\ \forall\, t \geq 0 : \quad 2\omega(t) \leq \omega(Ht) + H. \tag{12}$$

Remarks on condition (mg)

M is said to have *moderate growth*, if

$$\exists\, C \geq 1\ \forall\, j,k \in \mathbb{N}: \quad M_{j+k} \leq C^{j+k} M_j M_k.$$

We know:

(*i*) ω has $\exists\, H \geq 1\ \forall\, t \geq 0: \quad 2\omega(t) \leq \omega(Ht) + H$ if and only if some (equivalently each) W^x has (mg).

(*ii*) There exist many equivalent reformulations of (mg), e.g. M has (mg) if and only if

$$\exists\, C \geq 1\ \forall\, k \in \mathbb{N}_{>0}: \quad \mu_k \leq C(M_k)^{1/k}. \quad \text{(!!!)}$$

(*iii*) For any weight ω the associated matrix $\mathcal{W}$ satisfies generalized/mixed (mg)-conditions (by convexity of φ^*_ω), but the generalization of (*ii*) is not clear in general!

Whitney ultrajets I

We denote by $\mathcal{J}^\infty(E)$ the vector space of all jets $F = (F^\alpha)_{\alpha\in\mathbb{N}^n} \in \mathcal{C}^0(E,\mathbb{R})^{\mathbb{N}^n}$ on E.
For $a \in E$ and $p \in \mathbb{N}$ we associate the Taylor polynomial

$$T_a^p : \mathcal{J}^\infty(E) \to \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}), \quad F \mapsto T_a^p F(x) := \sum_{|\alpha|\le p} \frac{(x-a)^\alpha}{\alpha!} F^\alpha(a),$$

and the remainder $R_a^p F = ((R_a^p F)^\alpha)_{|\alpha|\le p}$ with

$$(R_a^p F)^\alpha(x) := F^\alpha(x) - \sum_{|\beta|\le p-|\alpha|} \frac{(x-a)^\beta}{\beta!} F^{\alpha+\beta}(a), \quad a, x \in E.$$

Whitney ultrajets II

Let M be a weight sequence. For fixed $\varrho > 0$ we denote by $\mathcal{B}^M_\varrho(E)$ the set of all jets F such that there exists $C > 0$ with

$$|F^\alpha(a)| \le C\varrho^{|\alpha|} M_{|\alpha|}, \quad \alpha \in \mathbb{N}^n, \ \ a \in E,$$

$$|(R^p_a F)^\alpha(b)| \le C\varrho^{p+1} M_{p+1} \frac{|b-a|^{p+1-|\alpha|}}{(p+1-|\alpha|)!} \quad p \in \mathbb{N}, \ |\alpha| \le p, \ \ a, b \in E.$$

We define

$$\mathcal{B}^{\{M\}}(E) := \varinjlim_{\varrho \in \mathbb{N}_{>0}} \mathcal{B}^M_\varrho(E).$$

$F \in \mathcal{B}^{\{M\}}(E)$ is called a *Whitney ultrajet of class* $\mathcal{B}^{\{M\}}$ on E.

Whitney ultrajets III

Let ω be a weight function, $\mathcal{W} = \{W^x : x > 0\}$ the associated weight matrix. A jet F is said to be a *Whitney ultrajet* of class $\mathcal{B}^{\{\omega\}}$ on E if $F \in \mathcal{B}^{\{W^x\}}(E)$ for some $x > 0$.
We set

$$\mathcal{B}^{\{\omega\}}(E) = \mathcal{B}^{\{\mathcal{W}\}}(E) = \varinjlim_{x>0} \mathcal{B}^{\{W^x\}}(E) = \varinjlim_{x>0} \varinjlim_{\varrho>0} \mathcal{B}^{W^x}_{\varrho}(E).$$

This definition coincides with the one given in BBMT '91.

Associated weight functions and counting functions

Let $m = (m_k)$ be a positive sequence satisfying $m_0 = 1$ and $m_k^{1/k} \to \infty$ (not necessarily log-convex). We associate:

$$h_m(t) := \inf_{k\in\mathbb{N}} m_k t^k,\ t > 0,\ h_m(0) := 0, \tag{13}$$

$$\omega_m(t) := -\log h_m(1/t) = \sup_{k\in\mathbb{N}} \log\left(\frac{t^k}{m_k}\right),\ t > 0,\ \omega_m(0) := 0, \tag{14}$$

$$\overline{\Gamma}_m(t) := \min\{k : h_m(t) = m_k t^k\},\ t > 0, \tag{15}$$

and, provided that $m_{k+1}/m_k \to \infty$,

$$\underline{\Gamma}_m(t) := \min\left\{k \in \mathbb{N} : \frac{m_{k+1}}{m_k} \geq \frac{1}{t}\right\},\ t > 0. \tag{16}$$

Possible loss of strong log-convexity

(*i*) We want to work with sequences w^x. In general we do not know if they are log-convex, so if W^x is strongly log-convex - as assumed in Ch./Ch. '94.

(*ii*) If m is log-convex, then $\overline{\Gamma}_m = \underline{\Gamma}_m$.

(*iii*) Central new idea: We work with both counting functions simultaneously.

Importance of the good property

Lemma

Let M, N be weight sequences satisfying $m_k^{1/k} \to \infty$, $n_k^{1/k} \to \infty$. Assume

$$\exists\, C \geq 1 \;\forall\, 1 \leq j \leq k : \quad \frac{\mu_j}{j} \leq C \frac{\nu_k}{k},$$

then

$$\forall\, t > 0 : \quad \overline{\Gamma}_n(Ct) \leq \underline{\Gamma}_m(t). \tag{17}$$

The conjugate of ω and its connection to $h_{w^\times}$

Let $\omega : [0,\infty) \to [0,\infty)$ with $\omega(t) = o(t)$ as $t \to \infty$. We define

$$\omega^\star(t) := \sup_{s\geq 0}\{\omega(s) - st\}, \; t > 0. \tag{18}$$

Lemma

Let M be a weight sequence such that $m_k^{1/k} \to \infty$ (but not necessarily s.l.c.), then

$$\forall\, t > 0 : \quad \omega_M^\star(t) \leq \omega_m\left(\frac{1}{t}\right) \leq \omega_M^\star\left(\frac{t}{e}\right). \tag{19}$$

Importance: $\omega, \omega_{W^\times} \to \omega^\star, \omega^\star_{W^\times} \leftrightarrow \omega_{w^\times}, h_{w^\times}$

The heirs of a n.q.a. weight function

Let ω be a non-quasianalytic weight function. Then

$$\kappa(t) = \kappa_\omega(t) := \int_1^\infty \frac{\omega(tu)}{u^2} du = t \int_t^\infty \frac{\omega(u)}{u^2} du, \quad t > 0, \qquad (20)$$

defines a concave weight function (possibly quasianalytic !) and satisfying $\kappa(t) = o(t)$ as $t \to \infty$, $\kappa \geq \omega$.
A weight σ is called a heir of ω, if $\sigma(t) = o(t)$ and $\kappa(t) = O(\sigma(t))$ as $t \to \infty$, i.e.

$$\exists\, C > 0\ \forall\, t > 0: \quad \int_1^\infty \frac{\omega(tu)}{u^2} du \leq C\sigma(t) + C. \qquad (21)$$

Some further notation

$d(Q, E)$ denotes the Euclidean distance of a closed set $Q \subseteq \mathbb{R}^n$ to E; i.e. (if E is fixed):

$$d(x) := d(x, E) = \inf\{|x - y| : y \in E\}.$$

Given $x \in \mathbb{R}^n$, we denote by $\hat{x}$ any point in E with $|x - \hat{x}| = d(x, E)$.

Construction of special/good bump functions

Generalizing the construction presented in BBMT '91 we have:

Proposition

Let ω be a non-quasianalytic concave weight function and let σ be a heir of ω. Then for each $n \in \mathbb{N}_{>0}$ there exist $m \in \mathbb{N}_{>0}$, $M > 0$, and $0 < r_0 < 1/2$ such that for all $0 < r < r_0$ there are functions $f_{n,r} \in \mathcal{C}^\infty(\mathbb{R})$ satisfying the following properties:

$$0 \le f_{n,r} \le 1, \quad \operatorname{supp} f_{n,r} \subseteq \left[-\frac{9}{8}r, \frac{9}{8}r\right], \quad f_{n,r}|_{[-r,r]} = 1, \tag{22}$$

$$\sup_{x\in\mathbb{R}, j\in\mathbb{N}} \frac{|f_{n,r}^{(j)}(x)|}{W_j^m} \le M \exp\left(\frac{1}{n}\sigma^\star(nr)\right). \tag{23}$$

The proof will show that $m = cn$ for some $c \in \mathbb{N}_{>0}$ independent of n.

A special/convenient partition of unity

Proposition

Let $E \subseteq \mathbb{R}^n$ be a non-empty compact set and let $\{Q_i\}_{i\in\mathbb{N}}$ be the family of Whitney cubes. Let ω be a non-quasianalytic concave weight function and let σ be a heir of ω. Then for all $p \in \mathbb{N}_{>0}$ there exist $m \in \mathbb{N}_{>0}$, $M > 0$, $0 < r_0 < 1/2$, and a family of smooth functions $\{\varphi_{i,p}\}_{i\in\mathbb{N}}$ satisfying

1. $0 \leq \varphi_{i,p} \leq 1$ *for all* $i \in \mathbb{N}$,
2. $\operatorname{supp} \varphi_{i,p} \subseteq Q_i^*$ *(cube* Q_i *expanded by* $9/8$*) for all* $i \in \mathbb{N}$,
3. $\sum_{i\in\mathbb{N}} \varphi_{i,p}(x) = 1$ *for all* $x \in \mathbb{R}^n \setminus E$,
4. *if* $d(Q_i, E) \leq r_0/B_1$, *then for all* $\beta \in \mathbb{N}^n$ *and* $x \in \mathbb{R}^n \setminus E$,

$$|\varphi_{i,p}^{(\beta)}(x)| \leq MW_{|\beta|}^m \exp\left(\frac{A_1(n)}{p}\sigma^\star\Big(\frac{b_1 p}{A_2(n)} \operatorname{diam} Q_i\Big)\right).$$

Our main theorem

Theorem

Let ω be a non-quasianalytic concave weight function and let σ be a good heir of ω. Let E be a compact subset of $\mathbb{R}^n$. Then the jet mapping $j_E^\infty : \mathcal{B}^{\{\omega\}}(\mathbb{R}^n) \to \mathcal{B}^{\{\sigma\}}(E)$ is surjective.

Let $S = (S_k)$ be a weight sequence satisfying $s_k^{1/k} \to \infty$ and $F \in \mathcal{B}^{\{S\}}(E)$ be a Whitney ultrajet. The extension of F will be of the form

$$\sum_{i \in \mathbb{N}} \varphi_{i,p}(x) \cdot T_{\hat{x}_i}^{2\overline{\Gamma}_{s'}(Ld(x_i))} F(x), \quad x \in \mathbb{R}^n \setminus E, \tag{24}$$

where S' is a suitable weight sequence depending on S (and $L \geq 1$ a constant dep. on S).

Actually the proof shows:

For each $\varrho > 0$ there exist $M(\varrho) > 0$ and a continuous linear extension operator $T_\varrho : \mathcal{B}^S_\varrho(E) \to \mathcal{B}^W_{M(\varrho)}(\mathbb{R}^n)$ depending on given ϱ and S.

Consequence of our main result - reformulating Ch./Ch. '94

Let M and N be weight sequences,

(i) both having (mg),

(ii)

$$\exists\, C \geq 1\ \forall\, k \in \mathbb{N}: \quad \mu_k \leq C\nu_k,$$

(iii) $\mu_k/k \to \infty$, $\nu_k/k \to \infty$,

(iv)

$$\exists\, C \geq 1\ \forall\, 1 \leq j \leq k: \quad \frac{\mu_j}{j} \leq C\frac{\mu_k}{k}, \tag{25}$$

resp. equivalently (by (mg)!)

$$\exists\, C \geq 1\ \forall\, 1 \leq j \leq k: \quad m_j^{1/j} \leq C m_k^{1/k}. \tag{26}$$

Then TFAE:

1. For every compact $E \subseteq \mathbb{R}^n$ the jet mapping $j_E^\infty : \mathcal{B}^{\{N\}}(\mathbb{R}^n) \to \mathcal{B}^{\{M\}}(E)$ is surjective.
2. There is a $C > 0$ such that $\int_1^\infty \frac{\omega_N(tu)}{u^2} du \le C\omega_M(t) + C$ for all $t > 0$.
3. There is a $C > 0$ such that $\sum_{j \ge k} \frac{1}{\nu_j} \le C \frac{k}{\mu_k}$ for all $k \in \mathbb{N}_{>0}$ (the so-called mixed strong non-quasianalyticity condition for weight sequ., resp. mixed (γ_1)-condition!).

Comparison with Ch./Ch. '94

M and N were assumed to be strongly log-convex

But N was not assumed to have (mg)

We had to assume (mg) for N since we have used for the proof:

Lemma

Let M be a weight sequence of moderate growth. Then $\mathcal{B}^{\{M\}}(\mathbb{R}^n) = \mathcal{B}^{\{\omega_M\}}(\mathbb{R}^n)$ and $\mathcal{B}^{\{M\}}(E) = \mathcal{B}^{\{\omega_M\}}(E)$ for each compact $E \subseteq \mathbb{R}^n$.

Open questions

Q: Is every concave weight function (equivalent to) a good one?

Q: Is every strong weight function (equivalent to) a good one?

What do we know so far?

Theorem

Let ω be a weight function and let $\mathcal{W} = \{W^x : x > 0\}$ be the associated weight matrix, TFAE:

(i) *ω is equivalent to its least concave majorant.*

(ii) $\forall\, x > 0\ \exists\, y > 0\ \exists\, D \geq 1\ \forall\, 1 \leq j \leq k : \quad (w_j^x)^{1/j} \leq D(w_k^y)^{1/k}$.

Theorem

Let ω be a weight function, assume the strange growth property

$$\forall\, x>0\ \exists\, y>0\ \exists\, C\geq 1\ \forall\, k\in\mathbb{N}_{\geq 1}:\quad \frac{W^x_k}{W^x_{k-1}}\leq C(W^y_k)^{1/k}. \tag{27}$$

Then ω is a good weight function if and only if it is equivalent to its least concave majorant.

Corollary

Let ω be a n.q.a. weight function, then κ_ω is a good heir if (27) holds for $\mathcal{S}=\{S^x:x>0\}$ (matrix associated to κ_ω).

For this recent work see [1].

For our first generalization of Ch./Ch. '94 (dealing with large weight matrices resp. unions of D.-C. classes) see [3].

For more information of weight functions and the connections to their associated weight matrices see [2].

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