Non-symmetric polarization

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Introduction: Polynomials and Polarization

Homogeneous polynomials

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An m–form which is somewhat naturally associated to P is given by

$$L_P(x^{(1)}, \ldots, x^{(m)}) := \sum_{1 \le j_1 \le \ldots \le j_m \le n} c_{(j_1, \ldots, j_m)} x_{j_1}^{(1)} \cdots x_{j_m}^{(m)} ,$$

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and the symmetrization of L_P,

$$\mathcal{S}L_P(x^{(1)}, \ldots, x^{(m)}) := \frac{1}{m!} \sum_{\sigma \in \Sigma_m} L_P(x^{(\sigma(1))}, \ldots, x^{(\sigma(m))}) ,$$

where Σ_m denotes the set of all permutations of $\{1, \ldots, m\}$, is symmetric and likewise defines P.

Norm inequalities

With a norm $\|\cdot\|$ on $\mathbb{C}^n$, the space of all m–homogeneous polynomials P and the space of all m–linear forms L become Banach spaces, when equipped with the supremum norms

$$\|P\|_\infty := \sup_{\|x\|\leq 1} |P(x)|$$

and

$$\|L\|_\infty := \sup_{\|x^{(k)}\|\leq 1} \left| L\big(x^{(1)}, \dots, x^{(m)}\big) \right| .$$

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As $P(x) = L_P(x, \ldots, x) = \mathcal{S}L_P(x, \ldots, x)$ for all $x \in \mathbb{C}^n$, it is easy to see that $\|P\|_\infty \leq \|L_P\|_\infty$ and $\|P\|_\infty \leq \|\mathcal{S}L_P\|_\infty$.

The classical polarization formula

Theorem (Polarization formula)

For each m–homogeneous polynomial $P : \mathbb{C}^n \to \mathbb{C}$ and every choice of $x^{(1)}, \ldots, x^{(m)} \in \mathbb{C}^n$,

$$\mathcal{S}L_P\big(x^{(1)}, \ldots, x^{(m)}\big) = \frac{1}{2^m m!} \sum_{\varepsilon_k = \pm 1} \varepsilon_1 \cdots \varepsilon_m P\Big(\sum_{k=1}^{m} \varepsilon_k x^{(k)}\Big) .$$

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As an easy consequence,

$$\|P\|_\infty \le \|\mathcal{S}L_P\|_\infty \le \tfrac{m^m}{m!}\|P\|_\infty \le e^m\|P\|_\infty\,.$$

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The question

Can we compare the norms of a m–homogeneous polynomial P with norms of *non-symmetric* m–forms, which define P?

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Consider for example

$$L : (\mathbb{C}^2)^2 \to \mathbb{C}\,, \quad (x, y) \mapsto x_1 y_2 - x_2 y_1$$

and $P(x) := L(x, x)$. Then $L \neq 0$, but $P = 0$, i.e. $\|L\|_\infty > 0$ and $\|P\|_\infty = 0$.

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Good news: The mappings L_P have a special structure...

Our results

Main theorem

We consider 1–unconditional norms on $\mathbb{C}^n$, that is $|x_k| \leq |y_k|$ for all k implies $\|x\| \leq \|y\|$.

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Theorem

There exists a universal constant $c_1 \geq 1$ such that for every m–homogeneous polynomial $P : \mathbb{C}^n \to \mathbb{C}$ and every 1–unconditional norm $\|\cdot\|$ on $\mathbb{C}^n$

$$\|L_P\|_\infty \leq c_1^{m^2} (\log n)^{m-1} \cdot \|P\|_\infty \, .$$

Moreover, if $\|\cdot\| = \|\cdot\|_p$ for $1 \leq p < 2$, then there even is a constant $c_2 = c_2(p) \geq 1$ for which

$$\|L_P\|_\infty \leq c_2^{m^2} \cdot \|P\|_\infty \, .$$

Partial symmetrization

Keeping the polarization formula in mind, we have to show that $\|L_P\|_\infty \leq c \cdot \|\mathcal{S}L_P\|_\infty$ with a suitable constant c. We will use an iterative approach.

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For $1 \leq k \leq n$ define the partial symmetrization

$$\begin{aligned}\mathcal{S}_k L_P(x^{(1)}, \ldots, x^{(m)}) \\ := \frac{1}{k!} \sum_{\sigma \in \Sigma_k} L_P(x^{(\sigma(1))}, \ldots, x^{(\sigma(k))}, x^{(k+1)}, \ldots, x^{(\sigma(m))}) .\end{aligned}$$

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Note that $\mathcal{S}_1 L_P = L_P$ and $\mathcal{S}_m L_P = \mathcal{S}L_P$.

Partial symmetrization (cont.)

Theorem

There exists a universal constant $c_1 \geq 1$ *such that for every* m*–homogeneous polynomial* $P : \mathbb{C}^n \to \mathbb{C}$*, every* 1*–unconditional norm* $\|\cdot\|$ *on* $\mathbb{C}^n$ *and* $2 \leq k \leq m$

$$\|\mathcal{S}_{k-1} L_P\|_\infty \leq (c_1 \log n)^k \cdot \|\mathcal{S}_k L_P\|_\infty .$$

Moreover, if $\|\cdot\| = \|\cdot\|_p$ *for* $1 \leq p < 2$*, then there even is a constant* $c_2 = c_2(p) \geq 1$ *for which*

$$\|\mathcal{S}_{k-1} L_P\|_\infty \leq c_2^k \cdot \|\mathcal{S}_k L_P\|_\infty .$$

Ideas of the proof

Coefficients of m–forms and Schur multiplication

An m–form $L : (\mathbb{C}^n)^m \to \mathbb{C}$ is uniquely determined by its coefficients

$$c_{\boldsymbol{i}} = c_{\boldsymbol{i}}(L) := L(e_{i_1}, \dots, e_{i_m}), \quad \boldsymbol{i} \in \mathcal{I}(n,m) := \{1, \dots, n\}^m .$$

With $L_{\boldsymbol{i}} : (\mathbb{C}^n)^m \to \mathbb{C}$, $(x^{(1)}, \dots, x^{(m)}) \mapsto x^{(1)}_{i_1} \cdots x^{(m)}_{i_m}$ we have

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We call $A \in \mathbb{C}^{\mathcal{I}(n,m)}$ an (m–dimensional) matrix and by $c_{\boldsymbol{i}}(A)$ we denote its $\boldsymbol{i}^{\text{th}}$ entry. For $A, B \in \mathbb{C}^{\mathcal{I}(n,m)}$ we define the Schur product $A * B$ (entry wise) by

$$c_{\boldsymbol{i}}(A * B) := c_{\boldsymbol{i}}(A) \cdot c_{\boldsymbol{i}}(B) .$$

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For a norm $\| \cdot \|$ on $\mathbb{C}^n$ and $A \in \mathbb{C}^{\mathcal{I}(n,m)}$ we define the Schur norm $\mu^m_{\| \cdot \|}(A)$ as the best constant c, such that

$$\|A * L\|_\infty \le c \cdot \|L\|_\infty$$

for any m–form $L : (\mathbb{C}^n)^m \to \mathbb{C}$.

Plan for the proof

1. Write $\mathcal{S}_{k-1}L_P$ as the Schur product of a (suitable) matrix and $\mathcal{S}_kL_P$, i.e.

$$\mathcal{S}_{k-1}L_P = \mathfrak{A}^k * \mathcal{S}_kL_P \, .$$

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2. Estimate the Schur norm of $\mathfrak{A}^k$
   2.1 Decompose $\mathfrak{A}^k$ into a sum and product of more handily pieces.
   2.2 Generalize results of Kwapień and Pełczyński (1970) and Bennett (1976) (for the case $m = 2$) to any m.
   2.3 Use the compatibility of addition and Schur multiplication with the Schur norm to estimate $\mu^m_{\|\cdot\|}(\mathfrak{A}^k)$.

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Proposition

Let $P : \mathbb{C}^n \to \mathbb{C}$ *be a* m*–homogeneous polynomial,* $\boldsymbol{i} \in \mathcal{I}(n,m)$ *and* $k \in \{2, \dots, n\}$*. Then*

$$c_{\boldsymbol{i}}(\mathcal{S}_{k-1}L_P) = \frac{k}{|\{1 \leq u \leq k \,|\, i_u = i_k\}|} \cdot c_{\boldsymbol{i}}(\mathcal{S}_k L_P)$$

provided $\max\{i_1, \dots, i_{k-1}\} \leq i_k$*; and*

$$c_{\boldsymbol{i}}(\mathcal{S}_{k-1}L_P) = 0 \cdot c_{\boldsymbol{i}}(\mathcal{S}_k L_P)$$

otherwise.

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Thus,

$$\mathcal{S}_{k-1}L_P = \mathfrak{A}^k * \mathcal{S}_k L_P\,,$$

with $c_{\boldsymbol{i}}(\mathfrak{A}^k)$ given by the proposition.

Decomposing $\mathfrak{A}^k$

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Lemma

For $1 \leq k \leq m$ *we have*

$$\mathfrak{A}^k = \left(\mathop{\ast}_{u=1}^{k-1} T^{u,k}\right) \ast \left(\sum_{u=1}^{k} \frac{k}{u} \cdot A^{k,u}\right)$$

with

$$A^{k,u} := \sum_{\substack{Q \subset \{1,\dots,k\} \\ |Q|=u}} \left(\mathop{\ast}_{q\in Q} D^{q,k}\right) \ast \left(\mathop{\ast}_{q\in Q^c} (\mathbf{1} - D^{q,k})\right),$$

where Q^c *denotes the complement of* Q *in* $\{1,\dots,k\}$.

The matrices $\mathbf{1}, D^{u,v}, T^{u,v} \in \mathbb{C}^{\mathcal{I}(n,m)}$ $(u, v \in \{1,\dots,m\}, u \neq v)$ are defined by

$$c_{\boldsymbol{i}}(\mathbf{1}) := 1\,, \quad c_{\boldsymbol{i}}(D^{u,v}) := \begin{cases} 1, & i_u = i_v \\ 0, & i_u \neq i_v \end{cases}, \quad c_{\boldsymbol{i}}(T^{u,v}) := \begin{cases} 1, & i_u \leq i_v \\ 0, & i_u > i_v \end{cases}.$$

Classical Schur multipliers

In the case $m = 2$ Kwapień and Pełczyński (1970) and Bennett (1976) obtained for these matrices:

$$\mu^2_{\|\cdot\|_\infty}(D^{1,2}) \leq 1\,,$$
$$\mu^2_{\|\cdot\|_\infty}(T^{1,2}) \leq \log_2(2n)\,,$$

and, moreover, for $1 \leq p < 2$ there is a constant $c_3 = c_3(p)$ such that

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This can be generalized to any norm $\|\cdot\|$ on $\mathbb{C}^n$ and any m. We have:

$$\mu^m_{\|\cdot\|}(D^{u,v}) \leq 1\,,$$

$$\mu^m_{\|\cdot\|}(T^{u,v}) \leq \log_2(2n)\,,$$

$$\mu^m_{\|\cdot\|_p}(T^{u,v}) \leq c_3\,.$$

Decomposing $\mathfrak{A}^k$ (cont.)

Using that

$$\mu^m_{\|\cdot\|}(A * B) \leq \mu^m_{\|\cdot\|}(A) \cdot \mu^m_{\|\cdot\|}(B)$$

and that $\mu^m_{\|\cdot\|}$ is a norm, we are able to use our decomposition and the norm estimates on the previous slide to estimate the Schur norm of $\mathfrak{A}^k$.

We obtain

$$\mu^m_{\|\cdot\|}(\mathfrak{A}^k) \leq k3^k \big(\mu^m_{\|\cdot\|}(T^{u,k})\big)^{k-1} \leq (c_1 \log n)^k \,,$$

respectively for $1 \leq p < 2$

$$\mu^m_{\|\cdot\|_p}(\mathfrak{A}^k) \leq k3^k \big(\mu^m_{\|\cdot\|_p}(T^{u,k})\big)^{k-1} \leq c_2^k \,.$$

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$$\begin{aligned}\|L_P\|_\infty = \|\mathcal{S}_1 L_P\|_\infty \leq c^2 \cdot \|\mathcal{S}_2 L_P\|_\infty \leq c^2 c^3 \cdot \|\mathcal{S}_3 L_P\|_\infty \\ \leq \ldots \leq c^{2+3+\ldots+m} \cdot \|\mathcal{S}_m L_P\|_\infty\end{aligned}$$

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Simple improvement

We established the identity

$$\mathcal{S}_{k-1} L_P = \mathfrak{A}^k * \mathcal{S}_k L_P \,,$$

thus

$$L_P = \left(\mathop{*}_{k=2}^{m} \mathfrak{A}^k \right) * \mathcal{S} L_P \,.$$

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In the general case, we get an $\log n$ factor out of every occurrence of $T^{u,k}$ in $\mathfrak{A}^k$. In the product $\mathop{*}_{k=2}^{m} \mathfrak{A}^k$ we have the factor

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This gives, using the naive approach, a factor of $(\log n)^{m^2}$.

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$$L_P = \left(\mathop{*}_{k=2}^{m} \mathfrak{A}^k \right) * \mathcal{S} L_P \,.$$

In the general case, we get an $\log n$ factor out of every occurrence of $T^{u,k}$ in $\mathfrak{A}^k$. In the product $\mathop{*}_{k=2}^{m} \mathfrak{A}^k$ we have the factor

$$\mathop{*}_{k=2}^{m} \left(\mathop{*}_{u=1}^{k-1} T^{u,k} \right) .$$

This gives, using the naive approach, a factor of $(\log n)^{m^2}$. However, we can improve the result by checking that

$$\mathop{*}_{k=2}^{m} \left(\mathop{*}_{u=1}^{k-1} T^{u,k} \right) = \mathop{*}_{k=2}^{m} T^{k-1,k} \,,$$

Simple improvement

We established the identity

$$\mathcal{S}_{k-1} L_P = \mathfrak{A}^k * \mathcal{S}_k L_P \,,$$

thus

$$L_P = \left(\mathop{\ast}_{k=2}^{m} \mathfrak{A}^k \right) * \mathcal{S} L_P \,.$$

In the general case, we get an $\log n$ factor out of every occurrence of $T^{u,k}$ in $\mathfrak{A}^k$. In the product $\mathop{\ast}\limits_{k=2}^{m} \mathfrak{A}^k$ we have the factor

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$$\mathop{\ast}_{k=2}^{m} \left(\mathop{\ast}_{u=1}^{k-1} T^{u,k} \right) = \mathop{\ast}_{k=2}^{m} T^{k-1,k} \,,$$

and we obtain $\mu^m_{\|\cdot\|}(\mathop{\ast}\limits_{k=2}^{m} \mathfrak{A}^k) \leq c^{m^2} (\log n)^{m-1}$ in the general case.

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