Random unconditional convergence of vector-valued Dirichlet series

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Joint work with D. Carando, F. Marceca y P. Tradacete

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Unconditional basis

Definition

A basic sequence $(x_n)_n$ in a Banach space X is **unconditional** if for every $x \in \overline{\text{span}(x_n)_n}$ its expansion $\sum_n a_n x_n$ converges unconditionally.

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- There is a constant $C > 0$ such that for every choice of signs $\varepsilon = \pm 1$,

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- Considering each ε_n a Rademacher random variable,

$$C^{-1} \Big\| \sum_{n=1}^{N} a_n x_n \Big\| \le \mathbb{E} \Big\| \sum_{n=1}^{N} \varepsilon_n a_n x_n \Big\| \le C \Big\| \sum_{n=1}^{N} a_n x_n \Big\|.$$

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- Considering each ε_n a Rademacher random variable,

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Proposition (Carando-Defant-Sevilla)

Let $1 \leq p < \infty$. For every $N \in \mathbb{N}$ and every sequence of scalars $(a_n)_{n=1}^N$ we have

$$\mathbb{E}\Big\| \sum_{n=1}^{N} \varepsilon_n a_n n^{-s} \Big\|_{\mathcal{H}_p(\mathbb{C})} \sim \left(\sum_{n=1}^{N} |a_n|^2 \right)^{1/2} = \Big\| \sum_{n=1}^{N} a_n n^{-s} \Big\|_{\mathcal{H}_2(\mathbb{C})}$$

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RUC and RUD sequences

Definition (Billard-Kwapien-Pełczynski-Samuel, 1985)

A basic sequence $(x_n)_n$ in a Banach space X is **Random Unconditionally Convergent** (RUC), when there is a constant $C \geq 1$ such that for every $N \in \mathbb{N}$ and every sequence of scalars $(a_n)_{n=1}^N$ we have

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Definition

A basic sequence $(x_n)_n$ is **Random Unconditionally Divergent** (RUD), when there is a constant $C \geq 1$ such that for every $N \in \mathbb{N}$ and every sequence of scalars $(a_n)_{n=1}^N$ we have

$$\Big\| \sum_{n=1}^{N} a_n x_n \Big\| \leq C\, \mathbb{E}\Big\| \sum_{n=1}^{N} \varepsilon_n a_n x_n \Big\|.$$

- If a basic sequence is RUC and RUD, then it is unconditional.

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Summarizing

- $(n^{-s})_n$ is RUC in $\mathcal{H}_p(\mathbb{C})$ if and only if $2 \leq p < \infty$.
- $(n^{-s})_n$ is RUD in $\mathcal{H}_p(\mathbb{C})$ if and only if $1 < p \leq 2$.
- $(n^{-s})_n$ is unconditional in $\mathcal{H}_p(\mathbb{C})$ if and only if $p = 2$.

Dirichlet series

A Dirichlet series with values in E is a functions $D = D(s)$ of the form

$$D = \sum_n a_n \frac{1}{n^s}$$

with coefficients $a_n \in E$ and variable s in some region of $\mathbb{C}$.

Vector-valued Dirichlet series and Hardy spaces

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For $\alpha = (\alpha_1, \dots, \alpha_N, 0, \dots) \in \mathbb{N}_0^{(\mathbb{N})}$ we set

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Notation:

If $z = (z_1, z_2, z_3, \dots)$ is any sequence of complex numbers we write

$$z^\alpha = z_1^{\alpha_1} \times \cdots \times z_N^{\alpha_N}.$$

Hardy spaces of Dirichlet series

We have the one to one correspondence:

formal power series in E		Dirichlet series in E
$\mathfrak{P}$	$\longrightarrow$	$\mathfrak{D}$
$\sum_\alpha c_\alpha z^\alpha$	$\xrightarrow{c_\alpha = a_{p^\alpha}}$	$\sum_n a_n \frac{1}{n^s}$
$\cup$		
$H_p(\mathbb{T}^{\mathbb{N}}, E)$	$\longrightarrow$	

$$H_p(\mathbb{T}^{\mathbb{N}}; E) := \{f \in L_p(\mathbb{T}^{\mathbb{N}}; E) : \hat{f}(\alpha) \neq 0 \text{ only if } \alpha \in \mathbb{N}_0^{(\mathbb{N})}\}.$$

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Define $\mathcal{H}_p(E)$ as the image of $H_p(\mathbb{T}^{\mathbb{N}}, E)$ under Bohr's transform, equipped with the norm

$$\Big\| \sum_{n \in \mathbb{N}} x_n n^{-s} \Big\|_{\mathcal{H}_p(E)} = \Big\| \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} x_{n(\alpha)} z^\alpha \Big\|_{H_p(\mathbb{T}^{\mathbb{N}}, E)}.$$

The Banach space $\mathcal{H}_p^{rad}(E)$

Given $1 \leq p \leq \infty$ and a Banach space E, we define

$$\mathcal{H}_p^{rad}(E) := \{\sum a_n n^{-s} : \sum r_n a_n n^{-s} \in L_1([0,1], \mathcal{H}_p(E))\}.$$

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- If E has the $\mathcal{H}_p$ random convergence property then it has the $\mathcal{H}_q$ random convergence property for every $q \geq p$.

Theorem

If E has type 2, then E has the $\mathcal{H}_p$ random convergence property for $p \geq 2$. In other words, for each $p \geq 2$ we have

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Example

The space $L_r\left(\mathbb{T}^{\mathbb{N}}\right)$ has the $\mathcal{H}_p$ random convergence property for some $2 \leq p < \infty$ if and only if $2 \leq r < \infty$.

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Theorem

If E has the $\mathcal{H}_p$ random convergence property for some $2 \leq p < \infty$, then

$$\sup\{r : E \textit{ has type } r\} = 2.$$

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Sketch of the proof:

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- As $L_s\left(\mathbb{T}^{\mathbb{N}}\right)$ fails to have the $\mathcal{H}_p$ random convergence property for $2 \leq p < \infty$ and this is a local property, the result follows.

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Take $1 \le p_n \nearrow 2$ such that $\left(\frac{1}{p_n} - \frac{1}{2}\right)\log n \to \infty$. Define $X = \bigoplus_{n=1}^{\infty} \ell_{p_n}^n$ with the ℓ_2−norm for the sum. Then:

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1) X only has type p for $p < 2$;

2) X has cotype 2;

3) X fails to have the $\mathcal{H}_2$ random convergence property.

¡Muchas gracias!