On mappings between Banach spaces preserving $\mathcal{A}$-compact sets

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(Joint work with Silvia Lassalle)

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Our interest

Fix a "Class of sets" defined on Banach spaces,

- Does polynomials or holomorphic mapping maps this class in the same class?

  If this is not the case,

- Which polynomials or holomorphic mappings does it?

Let E and F (complex) Banach, for $n \in \mathbb{N}$, $P \in \mathcal{P}(^n E; F)$ if there exists $A \in \mathcal{L}(^n E; F)$ such that $P(x) = A(x, x, \ldots, x)$.

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Examples: Compact and bounded sets

- Polynomials preserves compact sets.
- Polynomials preserves bounded sets.

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Example: Weakly compact sets

$P \in \mathcal{P}({}^2\ell_2;\ell_1), P(\sum_{n=1}^{\infty}\alpha_n e_n) = \sum_{n=1}^{\infty}\alpha_n^2 e_n$. Then $P(B_{\ell_2}) = B_{\ell_1}$.

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Thus,

- Polynomials do not preserves weakly compact sets.
- If F is reflexive, polynomials $P\colon E \to F$ preserves weakly compact sets.
- Weakly compact polynomials preserves weakly compact sets.

Our *"class of sets"* will be the $\mathcal{A}$-compact sets of Carl and Stephani (1984)

Definition (Carl and Stephani)

Fix a λ-Banach operator ideal $\mathcal{A}$. A subset $K \subset E$ is relatively $\mathcal{A}$-compact if there exist a Banach space F, an operator $S \in \mathcal{A}(F; E)$ and a compact set $L \subset B_F$ such that

$$K \subset S(L).$$

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For an $\mathcal{A}$-compact set $K \subset E$, $m_{\mathcal{A}}(K) = \inf\{\|S\|_{\mathcal{A}} \colon K \subset S(L)\}$.

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$$\mathcal{K}_{\mathcal{A}}(E;F) = \{T \colon E \to F : T(B_E) \text{ is rel. } \mathcal{A}\text{-compact}\}$$

$$\|T\|_{\mathcal{K}_{\mathcal{A}}} = m_{\mathcal{A}}(T(B_E))$$

Examples of $\mathcal{A}$-compact sets

We will use the ideals

- Π of absolutely summing operators.
- For $1 + \frac{1}{t} \geq \frac{1}{u} + \frac{1}{v}$, $\mathcal{N}_{(t,u,v)}$ of (t, u, v)-nuclear operators,
    - $\mathcal{N}_{(p,1,r')} \rightsquigarrow (p, r)$-compact (Ain, Lillemets and Oja).
    - $\mathcal{N}_{(p,1,p)} \rightsquigarrow p$-compact (Sinha and Karn).
    - $\mathcal{N}_{(\infty,p',r')} \rightsquigarrow \mathcal{U}$nconditionally (p, r)-compact (Ain and Oja).
    - $\mathcal{N}_{(\infty,p',p)} \rightsquigarrow \mathcal{U}$nconditionally p-compact (Kim).

First results

Every linear operator preserves $\mathcal{A}$-compact sets for any $\mathcal{A}$.

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For $1 \leq p \leq \infty$, every homogeneous polynomial preserves p-compact sets.

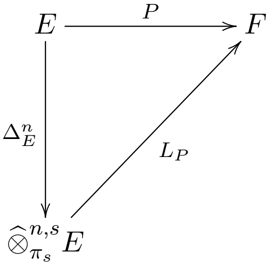
Example

For every $n \in \mathbb{N}$, there exists $P \in \mathcal{P}(^n\ell_2; \ell_1)$ which do not preserves Π-compact sets.

Example

For $1 \leq p < \infty$, $1 \leq r \leq p'$, and $n \in \mathbb{N}$ with $n \geq p$ there exists $P \in \mathcal{P}(^n\ell_p; \ell_1)$ which do not preserves $\mathcal{U}_{(p,r)}$-compact sets.

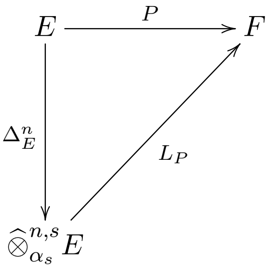
For $P \in \mathcal{P}({}^nE;F) = \mathcal{L}(\widehat{\otimes}^{n,s}_{\pi_s}E;F)$ isometrically.



$$\Delta^n_E(x) = \otimes^n x = x \otimes x \otimes \ldots \otimes x$$

For α_s a symmetric tensor norm of order n

$P \in \mathcal{P}_{\alpha_s}({}^n E; F) \Leftrightarrow L_P \in \mathcal{L}(\widehat{\otimes}^{n,s}_{\alpha_s} E; F)$; $\|P\|_{\alpha_s} = \|L_P \colon \widehat{\otimes}^{n,s}_{\alpha_s} E \to F\|$.



$$\Delta^n_E(x) = \otimes^n x = x \otimes x \otimes \ldots \otimes x$$

$$\mathcal{P}({}^n E; F) = \mathcal{P}_{\pi_s}({}^n E; F)$$

Proposition: Step 1

Let E be a Banach space, $\mathcal{A}$ a λ-Banach operator ideal and α_s a symmetric tensor norm of order n. Are equivalent

- For every Banach space F, every $P \in \mathcal{P}_{\alpha_s}(^n E; F)$ preserves $\mathcal{A}$-compact sets.
- $\Delta_E^n \colon E \to \widehat{\otimes}_{\alpha_s}^{n,s} E$ preserves $\mathcal{A}$-compact sets.

Moreover, there exists $C > 0$ such that, $\forall K \subset E$ $\mathcal{A}$-compact set

$$m_{\mathcal{A}}(P(K)) \leq C \|P\|_{\alpha_s} m_{\mathcal{A}}(K)^n$$

if and only if

$$m_{\mathcal{A}}(\Delta_E^n(K)) \leq C m_{\mathcal{A}}(K)^n.$$

Lemma

Let $\mathcal{A}$ be a λ-Banach operator ideal. A set $K \subset E$ is relatively $\mathcal{A}$-compact if and only if exists $T \in \mathcal{K}_{\mathcal{A}}(\ell_1; E)$ such that $K \subset T(B_{\ell_1})$.

For $T \in \mathcal{K}_{\mathcal{A}}(\ell_1; E)$, When $\Delta_E^n(T(B_{\ell_1}))$ is $\mathcal{A}$-compact?

Lemma

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For $T \in \mathcal{K}_{\mathcal{A}}(\ell_1; E)$, When $\Delta_E^n(T(B_{\ell_1}))$ is $\mathcal{A}$-compact?

$$\begin{array}{ccc} \ell_1 & \xrightarrow{\quad T \quad} & E \\ {\scriptstyle \Delta_{\ell_1}^n}\downarrow & & \downarrow{\scriptstyle \Delta_E^n} \\ \widehat{\otimes}_{\pi_s}^{n,s}\ell_1 & \xrightarrow{\otimes^n T} & \widehat{\otimes}_{\pi_s}^{n,s}E \end{array}$$

$$\overline{B_{\widehat{\otimes}_{\pi_s}^{n,s}\ell_1}} = \Gamma(\Delta_{\ell_1}^n(B_{\ell_1}))$$

$$\Delta_E^n(T(B_{\ell_1})) = \otimes^n T \circ \Delta_{\ell_1}^n(B_{\ell_1}) \subset \otimes^n T\Big(B_{\widehat{\otimes}_{\pi_s}^{n,s}\ell_1}\Big).$$

$$\otimes^n T\Big(B_{\widehat{\otimes}_{\pi_s}^{n,s}\ell_1}\Big) \subset \otimes^n T\Big(\Gamma(\Delta_{\ell_1}^n(B_{\ell_1}))\Big) = \Gamma\Big(\Delta_E^n(T(B_{\ell_1}))\Big).$$

Lemma

Let $\mathcal{A}$ be a λ-Banach operator ideal. A set $K \subset E$ is relatively $\mathcal{A}$-compact if and only if exists $T \in \mathcal{K}_{\mathcal{A}}(\ell_1; E)$ such that $K \subset T(B_{\ell_1})$.

For $T \in \mathcal{K}_{\mathcal{A}}(\ell_1; E)$, When $\Delta^n_E(T(B_{\ell_1}))$ is $\mathcal{A}$-compact?

$$\begin{array}{ccc} \ell_1 & \xrightarrow{T} & E \\ \downarrow{\scriptstyle \Delta^n_{\ell_1}} & & \downarrow{\scriptstyle \Delta^n_E} \\ \widehat{\otimes}^{n,s}_{\pi_s} \ell_1 & \xrightarrow{\otimes^n T} & \widehat{\otimes}^{n,s}_{\alpha_s} E \end{array}$$

$$\overline{B_{\widehat{\otimes}^{n,s}_{\pi_s} \ell_1}} = \Gamma(\Delta^n_{\ell_1}(B_{\ell_1}))$$

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Proposition: Step 2

Let E be a Banach space, $\mathcal{A}$ a λ-Banach operator ideal and α_s a symmetric tensor norm of order n. Are equivalent

- For every Banach space F, every $P \in \mathcal{P}_{\alpha_s}({}^n E; F)$ preserves $\mathcal{A}$-compact sets.
- For every $T \in \mathcal{K}_{\mathcal{A}}(\ell_1; E)$; $\otimes^n T \in \mathcal{K}_{\mathcal{A}}(\widehat{\otimes}^{n,s}_{\pi_s} \ell_1; \widehat{\otimes}^{n,s}_{\alpha_s} E)$.

Moreover, there exists $C > 0$ such that, $\forall K \subset E$ $\mathcal{A}$-compact set

$$m_{\mathcal{A}}(P(K)) \leq C \|P\|_{\alpha_s} m_{\mathcal{A}}(K)^n$$

if and only if

$$\| \otimes^n T\|_{\mathcal{K}_{\mathcal{A}}} \leq C \|T\|^n_{\mathcal{K}_{\mathcal{A}}}$$

If $T \in \mathcal{K}_{\mathcal{A}}(\ell_1; E)$, When $\otimes^n T \in \mathcal{K}_{\mathcal{A}}(\widehat{\otimes}^{n,s}_{\pi_s} \ell_1; \widehat{\otimes}^{n,s}_{\pi_s} E)$?

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For $n = 2$

$$
\begin{array}{ccc}
\widehat{\otimes}^{2,s}_{\pi_s} \ell_1 & \xrightarrow{\otimes^2 T} & \widehat{\otimes}^{2,s}_{\pi_s} E \\
\iota \big\downarrow & & \big\uparrow \sigma_2 \\
\widehat{\otimes}^{2}_{\pi} \ell_1 & \xrightarrow[(\otimes T)^2]{} & \widehat{\otimes}^{2}_{\pi} E \\
\Big\| & & \Big\| \\
\ell_1 \widehat{\otimes}_\pi \ell_1 & \xrightarrow[T \otimes T]{} & E \widehat{\otimes}_\pi E
\end{array}
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$$\otimes^2 T = \sigma_2 \circ (\otimes T)^2 \circ \iota$$

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For $n = 3$

$$
\begin{array}{ccc}
\widehat{\otimes}^{3,s}_{\pi_s} \ell_1 & \xrightarrow[\otimes^3 T]{} & \widehat{\otimes}^{3,s}_{\pi_s} E \\
\downarrow \iota & & \uparrow \sigma_3 \\
\widehat{\otimes}^{3}_{\pi} \ell_1 & \xrightarrow[(\otimes T)^3]{} & \widehat{\otimes}^{3}_{\pi} E \\
\Vert & & \Vert \\
\ell_1 \widehat{\otimes}_\pi \left(\widehat{\otimes}^{2}_{\pi} \ell_1 \right) & \xrightarrow[T \otimes (\otimes T)^2]{} & E \widehat{\otimes}_\pi \left(\widehat{\otimes}^{2}_{\pi} E \right)
\end{array}
$$

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In general...

$$
\begin{array}{ccc}
\widehat{\bigotimes}_{\pi_s}^{n,s}\ell_1 & \xrightarrow[\otimes^n T]{} & \widehat{\bigotimes}_{\pi_s}^{n,s}E \\
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\widehat{\bigotimes}_{\pi}^{n}\ell_1 & \xrightarrow[(\otimes T)^n]{} & \widehat{\bigotimes}_{\pi}^{n}E \\
\| & & \| \\
\ell_1 \widehat{\otimes}_\pi \left(\widehat{\bigotimes}_{\pi}^{n-1}\ell_1\right) & \xrightarrow[T \otimes \left(\otimes T\right)^{n-1}]{} & E \widehat{\otimes}_\pi \left(\widehat{\bigotimes}_{\pi}^{n-1}E\right)
\end{array}
$$

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In general...

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Definition

Let $\mathcal{A}$ be a λ-Banach operator ideal and α, β tensor norms. Then $\mathcal{A}$ is said to be (α, β)-tensorstable with constant $C \geq 1$ if for every Banach spaces E, F, X, Y, any $S \in \mathcal{A}(E; F)$ and $T \in \mathcal{A}(X; Y)$,

$$S \otimes T \in \mathcal{A}(E \widehat{\otimes}_\alpha X; F \widehat{\otimes}_\beta Y) \quad \text{and} \quad \|S \otimes T\|_{\mathcal{A}} \leq C \|S\|_{\mathcal{A}} \|T\|_{\mathcal{A}}$$

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And when we fix the spaces E and F, we say that $\mathcal{A}$ is (α, β)-tensorstable for $(E; F)$ with constant $C \geq 1$.

Proposition: Step 3

Let E be a Banach space and $\mathcal{A}$ a λ-Banach operator ideal.
If $\mathcal{K}_{\mathcal{A}}$ is (π, π)-tensorstable for $(\ell_1; E)$ with constant $C \geq 1$, then for every Banach space F, every $P \in \mathcal{P}(^n E; F)$ preserves $\mathcal{A}$-compact sets.
Moreover, for every $\mathcal{A}$-compact set $K \subset E$,

$$m_{\mathcal{A}}(P(K)) \leq C^{n-1} e^n \|P\| m_{\mathcal{A}}(K)^n.$$

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If $\mathcal{K}_{\mathcal{A}}$ is (π, ε)-tensorstable for $(\ell_1; E)$ with constant $C \geq 1$, then for every Banach space F, every $P \in \mathcal{P}_{\varepsilon_s}(^n E; F)$ preserves $\mathcal{A}$-compact sets.
Moreover, for every $\mathcal{A}$-compact set $K \subset E$,

$$m_{\mathcal{A}}(P(K)) \leq C^{n-1} \|P\|_{\varepsilon_s} m_{\mathcal{A}}(K)^n.$$

Lemma

Let E be a Banach space, $\mathcal{A}$ a λ-Banach operator ideal and β a tensor norm. If $\mathcal{A}$ is (π, β)-tensorstable for $(\ell_1; E)$ with constant $C \geq 1$, then If $\mathcal{K}_{\mathcal{A}}$ is (π, β)-tensorstable for $(\ell_1; E)$ with constant $C \geq 1$.

Proposition: Step 4

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Examples

Proposition

For $1 \leq p < \infty$ and $1 \leq r \leq p'$, the ideal $\mathcal{N}_{(p,1,r')}$ is (π, π)-tensorstable with constant $C = 1$.
(The case $r = p'$ is due Carl, Defant and Ramanujan (1989))

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$$m_{(p,r)}(P(K)) \leq e^n \|P\| m_{(p,r)}(K)^n.$$

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Example

For $1 \leq p < \infty$ and $1 \leq r \leq p'$, every $P \in \mathcal{P}_{\varepsilon_s}(^n E; F)$ preserves $\mathcal{U}_{(p,r)}$-compact sets.
Moreover, for every $\mathcal{U}_{(p,r)}$-compact set $K \subset E$

$$m_{\mathcal{U}_{(p,r)}}(P(K)) \leq \|P\|_\varepsilon m_{\mathcal{U}_{(p,r)}}(K)^n.$$

Examples

Proposition (Holub 1974)

The ideal Π is (π, π)-tensorstable for $(\ell_1; L_1(\mu))$ with constant $C = 1$.

Example

For every Banach space F, every $P \in \mathcal{P}(^n L_1(\mu); F)$ preserves Π-compact set.
Moreover, for every Π-compact set $K \subset L_1(\mu)$,

$$m_\Pi(P(K)) \leq e^n \|P\| m_\Pi(K)^n.$$

- For $1 \leq p < \infty$ and $1 \leq r \leq p'$, every $P \in \mathcal{P}(^n E; F)$ preserves (p,r)-compact sets and $m_{(p,r)}(P(K)) \leq e^n \|P\| m_{(p,r)}(K)^n$.

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- For $1 \leq p < \infty$ and $1 \leq r \leq p'$, every $P \in \mathcal{P}_{\varepsilon_s}(^n E; F)$ preserves $\mathcal{U}_{(p,r)}$-compact sets and $m_{\mathcal{U}_{(p,r)}}(P(K)) \leq \|P\| m_{\mathcal{U}_{(p,r)}}(K)^n$.

Analytic Functions

For Banach spaces E and F, $U \subset E$ and open set, $f \in \mathcal{H}(U;F)$ if for each $x_0 \in U$ there exists a sequence of homogeneous polynomials $P_n f(x_0) \in P(^n E;F)$ such that

$$f(x) = \sum_{n=0}^{\infty} P_n f(x_0)(x - x_0),$$

uniformly for all x in a neighborhood of x_0. And

$$r(f;x_0) = \frac{1}{\limsup \|P_n f(x_0)\|^{1/n}} = \sup\{r \colon \sup_{x \in B(x_0,r)} \|f(x)\| < \infty\}.$$

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Let E and F, $U \subset E$ an open balance set and $f \in \mathcal{H}(U;F)$. For $1 \leq p < \infty$, consider a sequence $(x_j)_j \in \ell_p(E)$ such that $(x_j)_j \subset U$. Then for the p-compact set $K = \{\sum_{j=1}^{\infty} \alpha_j x_j \colon (\alpha_j)_j \in B_{\ell_{p'}}\}$, $f(K)$ is p-compact.

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Let E and F, $U \subset E$ an open balance set and $f \in \mathcal{H}(U; F)$. For $1 \leq p < \infty$, consider a sequence $(x_j)_j \in \ell_p(E)$ such that $(x_j)_j \subset U$. Then for the p-compact set $K = \{\sum_{j=1}^{\infty} \alpha_j x_j \colon (\alpha_j)_j \in B_{\ell_{p'}}\}$, $f(K)$ is p-compact.

For every $n \in \mathbb{N}$, there exists $P \in \mathcal{P}(^n\ell_2; \ell_1)$ which do not preserves Π-compact sets.

For $1 \leq p < \infty$, $1 \leq r \leq p'$, and $n \in \mathbb{N}$ with $n \geq p$ there exists $P \in \mathcal{P}(^n\ell_p; \ell_1)$ which do not preserves $\mathcal{U}_{(p,r)}$-compact sets.

For $f \in \mathcal{H}(U;F)$, if $x_0 \in U$, and $K \subset U$ $\mathcal{A}$-compact,

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Lemma

Let E be a Banach space, $\mathcal{A}$ a λ-Banach operator ideal and let $K_1, K_2, \ldots$ be $\mathcal{A}$-compact sets such that

$$\sum_{n=1}^{\infty} m_{\mathcal{A}}(K_j)^{\lambda} < \infty.$$

Then the set $K = \{\sum_{n=1}^{\infty} x_j : x_j \in K_j\}$ is $\mathcal{A}$-compact and $m_{\mathcal{A}}(K)^{\lambda} < \sum_{n=1}^{\infty} m_{\mathcal{A}}(K_j)^{\lambda}$.

Proposition: Step 1

Let $\mathcal{A}$ be a λ-Banach operator ideal and let E, F Banach spaces, $U \subset E$ an open set. Take $f \in \mathcal{H}(U; F)$ such that, for $x_0 \in U$

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- There exist C_n such that for all $\mathcal{A}$-compact sets $K \subset U$

$$m_{\mathcal{A}}(P_n f(x_0)(K)) \leq C_n \|P_n f(x_0)\| m_{\mathcal{A}}(K)^n.$$

If $K \subset U$ is $\mathcal{A}$-compact such that

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What we do with “bigger” $\mathcal{A}$-compact sets?

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Proposition

Let $\mathcal{A}$ be a λ-Banach operator ideal E a Banach space and $K \subset E$ an $\mathcal{A}$-compact set with $0 \in K$. Then, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$m_{\mathcal{A}}(K \cap B(0;\delta)) \leq \epsilon$$

Example

Let E and F be Banach spaces and $U \subset E$ an open set. For $1 \leq p < \infty$ and $1 \leq r \leq p'$, every $f \in \mathcal{H}(U; F)$ preserves (p, r)-compact sets.

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Proof.

- Every homogeneous polynomial preserves (p, r)-compact sets and, if $P \in \mathcal{P}(^n E; F)$, and $K \subset E$ is (p, r)-compact, then $m_{(p,r)}(P(K)) \leq e^n \|P\| m_{(p,r)}(K)$.

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Take $K \subset U$ (p,r)-compact and for each $x \in K$ take $\delta_x > 0$ such that $m_{\mathcal{A}}\big((K-x) \cap B(0,\delta_x)\big) < \frac{r(f,x)}{e}$.

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Take $K \subset U$ (p,r)-compact and for each $x \in K$ take $\delta_x > 0$ such that $m_{\mathcal{A}}\big((K-x) \cap B(0,\delta_x)\big) < \frac{r(f,x)}{e}$. There exist $x_1, \ldots, x_k \in K$ such that $K = \bigcup_{j=1}^{k} K \cap B(x_j, \delta_{x_j})$. Then $f(K) = \bigcup_{j=1}^{k} f(K \cap B(x_j, \delta_{x_j}))$ and each $K \cap B(x_j, \delta_{x_j})$ satisfies the hypothesis of Step 1. □

For every Banach space F, every $P \in \mathcal{P}({}^n L_1(\mu); F)$ preserves Π-compact set. Moreover if $K \subset L_1(\mu)$ Π-compact, then $m_\Pi(P(K)) \leq e^n \|P\| m_\Pi(K)^n$.

Example

Let F be a Banach space and $U \subset L_1(\mu)$ an open set. Every $f \in \mathcal{H}(U; F)$ preserves Π-compact sets.

For $1 \leq p < \infty$ and $1 \leq r \leq p'$, every homogeneous polynomial preserves in $\mathcal{P}_{\varepsilon_s}$ preserves $\mathcal{U}_{(p,r)}$-compact sets. Moreover, if $P \in \mathcal{P}_{\varepsilon_s}({}^n E; F)$, and $K \subset E$ is $\mathcal{U}_{(p,r)}$-compact, then $m_{\mathcal{U}_{(p,r)}}(P(K)) \leq \|P\|_{\varepsilon_s} m_{\mathcal{U}_{(p,r)}}(K)^n$.

Example

Let E and F be a Banach space and $U \subset E$ an open set. For $1 \leq p < \infty$ and $1 \leq r \leq p'$, every $f \in \mathcal{H}(U; F)$ such that for every $x_0 \in U$, $P_n f(x_0) \in \mathcal{P}_{\varepsilon_s}({}^n E; F)$ for all $n \in \mathbb{N}$ and $\limsup \|P_n f(x_0)\|_{\varepsilon_s} < \infty$, then f preserves $\mathcal{U}_{(p,r)}$-compact sets.

THANK YOU!
¡GRACIAS!