The Algebra of symmetric analytic functions of bounded type on the complex L_∞

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(joint work with Pablo Galindo and Andriy Zagorodnyuk)

- Galindo P., Vasylyshyn T., Zagorodnyuk A. *The algebra of symmetric analytic functions on L_∞*. Proceedings of the Royal Society of Edinburgh: Section A Mathematics 2017, **147A** 1–19. doi:10.1017/S0308210516000287
- Vasylyshyn T. *Topology on the spectrum of the algebra of entire symmetric functions of bounded type on the complex L_∞*. Carpathian Math. Publ. 2017, **9** (1), 22–27. doi:10.15330/cmp.9.1.22-27

Let L_∞ be the complex Banach space of all Lebesgue measurable essentially bounded complex-valued functions x on $[0,1]$ with norm

$$\|x\|_\infty = \operatorname{ess\,sup}_{t\in[0,1]}|x(t)|.$$

Let Ξ be the set of all measurable bijections of $[0,1]$ that preserve the measure.

A function $f : L_\infty \to \mathbb{C}$ is called symmetric if for every $x \in L_\infty$ and for every $\sigma \in \Xi$

$$f(x \circ \sigma) = f(x).$$

Theorem 1

Polynomials $R_n : L_\infty \to \mathbb{C}$, $R_n(x) = \int_{[0,1]} (x(t))^n \, dt$ for $n \in \mathbb{N}$, form an algebraic basis in the algebra of all symmetric continuous polynomials on L_∞.

Let $H_{bs}(L_\infty)$ be the Fréchet algebra of all entire symmetric functions $f : L_\infty \to \mathbb{C}$ which are bounded on bounded sets endowed with the topology of uniform convergence on bounded sets.

Since every $f \in H_{bs}(L_\infty)$ can be described by its Taylor series of continuous symmetric homogeneous polynomials, it follows that f can be uniquely represented as

$$f(x) = f(0) + \sum_{n=1}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1,\ldots,k_n} R_1^{k_1}(x) \cdots R_n^{k_n}(x).$$

Consequently, for every non-trivial continuous homomorphism $\varphi : H_{bs}(L_\infty) \to \mathbb{C}$, taking into account $\varphi(1) = 1$, we have

$$\varphi(f) = f(0) + \sum_{n=1}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1,\ldots,k_n} \varphi(R_1)^{k_1} \cdots \varphi(R_n)^{k_n}.$$

Therefore, φ is completely determined by the sequence of its values on R_n :

$$(\varphi(R_1), \varphi(R_2), \ldots).$$

We denote by M_{bs} the spectrum of $H_{bs}(L_\infty)$, that is, the set of all continuous homomorphisms $\varphi : H_{bs}(L_\infty) \to \mathbb{C}$.

Proposition 2

For every non-trivial continuous homomorphism $\varphi : H_{bs}(L_\infty) \to \mathbb{C}$, the sequence $\{\sqrt[n]{|\varphi(R_n)|}\}_{n=1}^\infty$ is bounded.

Theorem 3

For every sequence $\xi = \{\xi_n\}_{n=1}^\infty \subset \mathbb{C}$ such that $\sup_{n\in\mathbb{N}} \sqrt[n]{|\xi_n|} < +\infty$, there exists $x_\xi \in L_\infty$ such that $R_n(x_\xi) = \xi_n$ for every $n \in \mathbb{N}$.

Let $\varepsilon_n(t) = \operatorname{sign}\sin 2^n \pi t$

Theorem (J.-P. Kahane "Some random series of functions", 1968)

The series

$$\sum_{n=1}^{\infty} \varepsilon_n(t) u_n$$

is convergent almost everywhere on $[0,1]$ if and only if the series

$$\sum_{n=1}^{\infty} u_n^2$$

is convergent.

Corollary

The series

$$\sum_{n=1}^{\infty} \frac{\varepsilon_n(t)}{n+1}$$

converges almost everywhere on $[0,1]$.

Let

$$p_n(t) = \exp\left(\frac{i\pi}{2n} \sum_{k=1}^{\infty} \frac{\varepsilon_k(t)}{k+1}\right).$$

By the Corollary, $p_n \in L_\infty$.

Note that

$$R_m(p_n) = \prod_{k=1}^{\infty} \cos\left(\frac{\pi m}{2n} \frac{1}{k+1}\right).$$

Let us define functions $y_n : [0,1] \to \mathbb{C}$. For $t \in \left[\frac{k-1}{n}, \frac{k}{n}\right)$ let

$$y_n(t) = \alpha_k p_n(nt - k + 1),$$

where

$$\alpha_k = \exp\left(\frac{2\pi i k}{n}\right), \quad (k = 1, 2, \ldots, n).$$

Note that $\|y_n\|_\infty = \operatorname{ess\,sup}_{t\in[0,1]} |y_n(t)| = 1$.

We have

$$R_m(y_n) = \int_{[0,1]} (y_n(t))^m \, dt = \sum_{k=1}^{n} \int_{\left[\frac{k-1}{n}, \frac{k}{n}\right)} (\alpha_k p_n(nt - k + 1))^m \, dt =$$

$$= \left(\frac{1}{n} \sum_{k=1}^{n} \alpha_k^m\right) \int_{[0,1]} (p_n(t))^m \, dt.$$

Since

$$\frac{1}{n}\sum_{k=1}^{n}\alpha_k^m = \begin{cases} 1, & \text{if } m \text{ is a multiple of } n, \\ 0, & \text{otherwise.} \end{cases} ,$$

it follows that $R_m(y_n) = 0$ if m is not a multiple of n.

If m is a multiple of n, i.e. $m = k_0 n$ for some $k_0 \in \mathbb{N}$, we have

$$R_m(y_n) = \int_{[0,1]} (p_n(t))^m \, dt = \prod_{k=1}^{\infty} \cos\left(\frac{\pi}{2} \cdot \frac{k_0}{k+1}\right).$$

If $m \neq n$, then $k_0 > 1$ and one of the factors is equal to $\cos \frac{1}{2}\pi$, and, therefore, $R_m(y_n) = 0$. In the $m = n$ case we have

$$R_n(y_n) = \prod_{k=1}^{\infty} \cos\left(\frac{\pi}{2}\frac{1}{k+1}\right).$$

Let $M = R_n(y_n)$. Note that $0 < M < 1$.

Let

$$x_n(t) = \frac{1}{\sqrt[n]{M}} y_n(t).$$

Note that

$$R_m(x_n) = \begin{cases} 1, & \text{if } m = n, \\ 0, & \text{otherwise.} \end{cases}$$

Let us define $x_\xi(t)$.

For $t \in \left[\frac{2^{n-1}-1}{2^{n-1}}, \frac{2^n-1}{2^n}\right)$ let

$$x_\xi(t) = 2\sqrt[n]{\xi_n}\, x_n(2^n t - 2^n + 2).$$

Corollary 4

Every $\varphi \in M_{bs}$ is a point-evaluation functional.

Corollary 5

The spectrum M_{bs} can be identified with the set of all sequences $\xi = \{\xi_n\}_{n=1}^{\infty} \subset \mathbb{C}$ such that $\{\sqrt[n]{|\xi_n|}\}_{n=1}^{\infty}$ is bounded.

Let $\nu : L_\infty \to M_{bs}$ be defined by $\nu(x) = (R_1(x), R_2(x), \ldots)$.

Let τ_∞ be the topology on L_∞, generated by $\|\cdot\|_\infty$.

Let us define an equivalence relation on L_∞ by $x \sim y \Leftrightarrow \nu(x) = \nu(y)$.

Let τ be the quotient topology on M_{bs} :

$$\tau = \{\nu(V) : \ V \in \tau_\infty\}.$$

Theorem 6

$(M_{bs}, +, \tau)$ is an abelian topological group, where "+" is an operation of coordinate-wise addition.