THE PETTIS INTEGRAL FOR MULTI-VALUED FUNCTIONS VIA SINGLE-VALUED ONES

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ABSTRACT. We study the Pettis integral for multi-functions $F : \Omega \to cwk(X)$ defined on a complete probability space (Ω, Σ, μ) with values into the family $cwk(X)$ of all convex weakly compact non-empty subsets of a separable Banach space X. From the notion of Pettis integrability for such an F studied in the literature one readily infers that if we embed $cwk(X)$ into $\ell_\infty(B_{X^*})$ by means of the mapping $j : cwk(X) \to \ell_\infty(B_{X^*})$ defined by $j(C)(x^*) = \sup(x^*(C))$, then $j \circ F$ is integrable with respect to a norming subset of $B_{\ell_\infty(B_{X^*})^*}$. A natural question arises: When is $j \circ F$ Pettis integrable? In this paper we answer this question by proving that the Pettis integrability of any $cwk(X)$-valued function F is equivalent to the Pettis integrability of $j \circ F$ if and only if X has the Schur property that is shown to be equivalent to the fact that $cwk(X)$ is separable when endowed with the Hausdorff distance. We complete the paper with some sufficient conditions (involving stability in Talagrand's sense) that ensure the Pettis integrability of $j \circ F$ for a given Pettis integrable $cwk(X)$-valued function F.

1. INTRODUCTION

Since the pioneering papers by Aumann [2] and Debreu [5], several notions of integral for multi-valued functions (also called multi-functions) in Banach spaces have been developed. These notions have shown to be useful when modelling some theories in areas like Optimal Control and Mathematical Economics. For a detailed account on this subject we refer the reader to the monographs [4], [16] and the survey [13]. The Pettis integral for multi-functions was first considered by Castaing and Valadier [4, Chapter V, §4] and has been widely studied in recent years, see [1, 3, 6, 7, 10, 14, 18, 19].

Throughout this paper (Ω, Σ, μ) is a complete probability space, X a *separable* Banach space and $cwk(X)$ the family of all convex weakly compact non-empty subsets of X. Given $C \in cwk(X)$ and $x^* \in X^*$ we write

$$\delta^*(x^*, C) := \sup\{x^*(x) : \ x \in C\}.$$

A multi-function $F : \Omega \to cwk(X)$ is said to be *Pettis integrable* if

(i) For each $x^* \in X^*$, the function $\delta^*(x^*, F) : \Omega \to \mathbb{R}$ given by

$$\delta^*(x^*, F)(\omega) = \delta^*(x^*, F(\omega))$$

is μ-integrable.

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(ii) For each $A \in \Sigma$, there is $\int_A F \, d\mu \in cwk(X)$ such that

$$\delta^*(x^*, \int_A F \, d\mu) = \int_A \delta^*(x^*, F) \, d\mu \quad \text{for every } x^* \in X^*.$$

It is worth mentioning that the measurability of $\delta^*(x^*, F)$ for every $x^* \in X^*$ implies that such an F is Effros measurable, cf. [4, Theorem III.37], and therefore F admits strongly measurable selectors, cf. [4, Theorem III.6]. The following characterization shows the role played by these selectors in the multi-valued Pettis integral theory. The implication (iii)⇒(i) is essentially due to Castaing and Valadier [4, Chapter V, §4], whereas the other ones have been recently proved by El Amri and Hess [10] and Ziat [18] (see [19] for a corrected proof of (i)⇒(ii)).

Theorem A. *Let (Ω, Σ, μ) be a complete probability space, X a separable Banach space and $F : \Omega \to cwk(X)$ a multi-function. The following conditions are equivalent:*

(i) *F is Pettis integrable.*
(ii) *The family $W_F = \{\delta^*(x^*, F) : x^* \in B_{X^*}\}$ is uniformly integrable.*
(iii) *The family W_F is made up of measurable functions and any strongly measurable selector of F is Pettis integrable.*

In this case, for each $A \in \Sigma$ the Pettis integral $\int_A F \, d\mu$ coincides with the set of integrals over A of all Pettis integrable selectors of F.

Recall that $cwk(X)$, equipped with the Hausdorff metric h, is a complete metric space that can be isometrically embedded into the Banach space $\ell_\infty(B_{X^*})$ by means of the mapping

$$j : cwk(X) \to \ell_\infty(B_{X^*}), \quad j(C)(x^*) := \delta^*(x^*, C),$$

see e.g. [4, Chapter II]. Thus any $cwk(X)$-valued function F can be looked at as a single-valued function $j \circ F$ taking values in $\ell_\infty(B_{X^*})$. Some stronger notions of integral for a multi-function $F : \Omega \to cwk(X)$, like the Debreu and Birkhoff integrals, see respectively [5] and [3], can be characterized in terms of the integrability properties of the composition $j \circ F : \Omega \to \ell_\infty(B_{X^*})$. To the best of our knowledge it has been [3] the first paper where the relationship between the Pettis integrability of F and the Pettis integrability of $j \circ F$ has been studied, namely the first and third named authors proved in [3, Proposition 3.5] the following results:

(i) if $j \circ F$ is Pettis integrable then F is Pettis integrable and $j(\int_A F \, d\mu)$ is the Pettis integral of $j \circ F$ over A for every $A \in \Sigma$;
(ii) the equivalence between F being Pettis integrable and that of $j \circ F$ holds true whenever F has essentially h-separable range (that is, there is $E \in \Sigma$ with $\mu(\Omega \setminus E) = 0$ such that $F(E)$ is h-separable).

Since the separability of X implies the separability for the Hausdorff distance h of the family $ck(X)$ of all convex *norm* compact non-empty subsets of X, cf. [4, Theorem II.8], statement (ii) implies, in particular, that when $F(\Omega) \subset ck(X)$ then F is Pettis integrable if and only if $j \circ F$ is Pettis integrable too. We note that this last equivalence has been recently rediscovered in [7, Lemma 1].

With this paper we aim a double target: (a) to characterize those Banach spaces for which the Pettis integrability of any $cwk(X)$-valued function F is equivalent to the Pettis integrability of $j \circ F$; (b) to provide sufficient conditions ensuring the Pettis integrability of $j \circ F$ for a standing alone Pettis integrable multi-function F.

Being more precise, in Section 2 we show that, in general, $j \circ F$ might be non Pettis integrable when F is. In fact, we prove that X has the *Schur property* (i.e. weakly convergent sequences are norm convergent) if and only if $j \circ F$ is Pettis integrable for every $cwk(X)$-valued Pettis integrable F, the latter also being equivalent to the fact that $(cwk(X), h)$ is separable, see Theorem 2.1. Our proof of Theorem 2.1 easily yields that a convex weakly compact set $K \subset X$ is norm compact if and only if $j \circ F$ is Pettis integrable for every Pettis integrable multi-function F taking values in $\{C \in cwk(X) : C \subset K\}$, see Proposition 2.3.

In Section 3 we exhibit some sufficient conditions for a Pettis integrable multi-function $F : \Omega \to cwk(X)$ that ensure the Pettis integrability of $j \circ F$, beyond the essential h-separability of the range of F. Theorem 3.3 shows that $j \circ F$ is Pettis integrable when every countable subset of $W_F = \{\delta^*(x^*, F) : x^* \in B_{X^*}\}$ is stable in Talagrand's sense, see Section 3 for the definition. In particular, under Martin's Axiom and dealing with a perfect probability μ, $j \circ F$ is Pettis integrable whenever it is scalarly measurable, see Corollary 3.6.

A bit of terminology: our unexplained terminology can be found in our standard references for multi-valued functions [4, 16] and for vector integration [9, 17]. All vector spaces here are assumed to be real. Given a subset S of a vector space, we write $\mathrm{co}(S)$, $\mathrm{aco}(S)$ and $\mathrm{span}(S)$ to denote, respectively, the convex, absolutely convex and linear hull of S. Let Y be a Banach space. As usual, B_Y is the closed unit ball of Y and Y^* stands for the topological dual of Y. Given $y^* \in Y^*$ and $y \in Y$, we write $\langle y^*, y\rangle$ and $y^*(y)$ to denote the evaluation of y^* at y. A set $B \subset B_{Y^*}$ is said to be *norming* if $\|y\| = \sup\{|y^*(y)| : y^* \in B\}$ for every $y \in Y$. For the complete probability space (Ω, Σ, μ), a family $\mathcal{H}$ of real-valued μ-integrable functions defined on Ω is said to be *uniformly integrable* if it is bounded for $\|\cdot\|_1$ and for each $\varepsilon > 0$ there is $\delta > 0$ such that $\sup_{h\in\mathcal{H}} \int_E |h|\, d\mu \leq \varepsilon$ whenever $\mu(E) \leq \delta$. A function $f : \Omega \to Y$ is said to be *scalarly measurable* if the composition $\langle y^*, f\rangle : \Omega \to \mathbb{R}$, given by $\langle y^*, f\rangle(\omega) = \langle y^*, f(\omega)\rangle$, is measurable for every $y^* \in Y^*$. Recall that f is said to be *Pettis integrable* if

(i) $\langle y^*, f\rangle$ is μ-integrable for every $y^* \in Y^*$.
(ii) For each $A \in \Sigma$, there is $\nu_f(A) \in Y$ such that

$$\langle y^*, \nu_f(A)\rangle = \int_A \langle y^*, f\rangle \, d\mu \quad \text{for every } y^* \in Y^*.$$

In this case, the mapping $\nu_f : \Sigma \to Y$ is a countably additive measure. We note that a single-valued function $f : \Omega \to X$ is Pettis integrable if, and only if, the set-valued function $F : \Omega \to cwk(X)$ given by $F(\omega) = \{f(\omega)\}$ is Pettis integrable.

2. A CHARACTERIZATION OF THE SCHUR PROPERTY

If the Banach space X has the Schur property, the Eberlein-Smulyan theorem [8, p. 18] implies that every weakly compact set in X is norm compact, hence $cwk(X) = ck(X)$. Thus if X is separable and has the Schur property we know that $(cwk(X), h) = (ck(X), h)$ is separable, see [4, Theorem II.8], and consequently if $F : \Omega \to cwk(X)$ is Pettis integrable then $j \circ F$ is Pettis integrable by [3, Proposition 3.5]. In this section we prove that the converse also holds, that is, the Schur property of X is characterized by the fact that $j \circ F$ is Pettis integrable for each Pettis integrable $cwk(X)$-valued function F.

Theorem 2.1. *For a separable Banach space X the following statements are equivalent:*

(i) *X has the Schur property.*
(ii) *$(cwk(X), h)$ is separable.*
(iii) *For any complete probability space (Ω, Σ, μ) and any Pettis integrable multi-function $F : \Omega \to cwk(X)$ the composition $j \circ F$ is Pettis integrable.*
(iv) *For any Pettis integrable multi-function $F : [0, 1] \to cwk(X)$ the composition $j \circ F$ is Pettis integrable .*
(v) *For any h-bounded Pettis integrable multi-function $F : [0, 1] \to cwk(X)$ the composition $j \circ F$ is Pettis integrable.*

Proof. The comments just before the theorem thoroughly explain the implications (i)⇒(ii)⇒(iii). The implications (iii)⇒(iv) and (iv)⇒(v) are obvious.

To finish we prove the implication (v)⇒(i) by contradiction: we will show that if X fails the Schur property, then there is an h-bounded Pettis integrable multi-function $F : [0, 1] \to cwk(X)$ such that $j \circ F$ is not scalarly measurable (hence not Pettis integrable). Recall (cf. [11, 254K]) that $[0, 1]$, equipped with the Lebesgue measure on the σ-algebra of all Lebesgue measurable sets, is measure space isomorphic to the complete probability space $(\{0, 1\}^{\mathbb{N}}, \Sigma, \mu)$ obtained as the completion of the usual product probability measure on $\{0, 1\}^{\mathbb{N}}$. Therefore, for our purposes it suffices to work with $(\{0, 1\}^{\mathbb{N}}, \Sigma, \mu)$. We write $\pi_n : \{0, 1\}^{\mathbb{N}} \to \mathbb{R}$ to denote the n-th coordinate projection.

Suppose that X fails the Schur property. Then there is a weakly null sequence (x_n) in X such that $\|x_n\| = 1$ for every $n \in \mathbb{N}$. According to the Bessaga-Pelczynski selection principle (cf. [8, p. 42]), by passing to a further subsequence we can and do assume that (x_n) is a basic sequence, i.e. it is a Schauder basis of $Y := \overline{\operatorname{span}\{x_n : n \in \mathbb{N}\}}$. Since the set $\{x_n : n \in \mathbb{N}\}$ is weakly relatively compact, the Krein-Smulyan theorem (cf. [9, Theorem 11, p. 51]) allows us to define the h-bounded multi-function $F : \{0, 1\}^{\mathbb{N}} \to cwk(X)$ by

$$F(\omega) := \overline{\operatorname{co}\{\pi_n(\omega)x_n : n \in \mathbb{N}\}}.$$

According to Theorem A, in order to prove that F is Pettis integrable it suffices to show that the family W_F is uniformly integrable. An easy computation yields that

$$\begin{aligned} &\delta^*(x^*, F)(\omega) = \delta^*(x^*, F(\omega)) = \\ &= \sup\Big\{x^*(x) : x \in \operatorname{co}\{\pi_n(\omega)x_n : n \in \mathbb{N}\}\Big\} = \sup_{n\in\mathbb{N}} \pi_n(\omega)x^*(x_n) \end{aligned} \tag{1}$$

for every $\omega \in \{0, 1\}^{\mathbb{N}}$ and $x^* \in X^*$. Observe that since each π_n is measurable $\delta^*(x^*, F)$ is measurable too. On the other hand, we have $|\delta^*(x^*, F)(\omega)| \leq 1$ for any $\omega \in \{0, 1\}^{\mathbb{N}}$ and $x^* \in B_{X^*}$. Therefore W_F is uniformly bounded and made up of measurable functions, hence uniformly integrable: Theorem A applies now to tell us that F is Pettis integrable.

We claim now that the composition $j \circ F : \Omega \to \ell_\infty(B_{X^*})$ is not scalarly measurable. Indeed, for each $n \in \mathbb{N}$ take $y_n^* \in Y^*$ the coefficient functional given by $y_n^* : \sum_m a_m x_m \to a_n$. Since (x_n) is a normalized Schauder basis of Y, there is $M > 0$ such that $\|y_n^*\| \leq M$ for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ use the Hahn-Banach theorem and fix $x_n^* \in X^*$ such that $y_n^* = x_n^*|_Y$ and $\|y_n^*\| = \|x_n^*\|$. Define $z_n^* := x_n^*/M \in B_{X^*}$ for every $n \in \mathbb{N}$. As usual, we identify $\{0, 1\}^{\mathbb{N}}$ with $\mathcal{P}(\mathbb{N})$ (the power set of $\mathbb{N}$) by means of the bijection $\psi : \{0, 1\}^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$ given by

$\psi(\omega) := \{n \in \mathbb{N} : \pi_n(\omega) = 1\}$. Fix a free ultrafilter $\mathcal{U} \subset \mathcal{P}(\mathbb{N})$. It is known that $\psi^{-1}(\mathcal{U}) \subset \{0,1\}^{\mathbb{N}}$ is not measurable (cf. [17, 13-1-1]). Define $\xi \in \ell_\infty(B_{X^*})^*$ by

$$\xi(h) := \mathcal{U} - \lim_m h(z_m^*), \qquad \text{for } h \in \ell_\infty(B_{X^*}),$$

where the symbol "$\mathcal{U} - \lim_m$" stands for the limit along the ultrafilter $\mathcal{U}$. Then the composition $\langle \xi, j \circ F\rangle$ is not measurable, because for each $\omega \in \{0,1\}^{\mathbb{N}}$ the equality (1) gives us

$$\langle \xi, j \circ F\rangle(\omega)$$
$$= \mathcal{U} - \lim_m \delta^*(z_m^*, F)(\omega) = \mathcal{U} - \lim_m \frac{\pi_m(\omega)}{M} = \begin{cases} \frac{1}{M} & \text{if } \omega \in \psi^{-1}(\mathcal{U}); \\ 0 & \text{if } \omega \notin \psi^{-1}(\mathcal{U}). \end{cases}$$

Therefore, $j \circ F$ is not scalarly measurable, as claimed and the proof of the theorem is over. □

Remark 2.2. We notice that in fact the proof of Theorem 2.1 shows that conditions (i)–(v) are equivalent to the following one:

(vi) *For any h-bounded Pettis integrable multi-function $F : [0,1] \to cwk(X)$ the composition $j \circ F$ is scalarly measurable.*

In Corollary 3.6 we will see that, at least under Martin's Axiom, given a Pettis integrable multi-function $F : [0,1] \to cwk(X)$, the scalar measurability of $j \circ F$ is sufficient to ensure that $j \circ F$ is Pettis integrable.

Similar ideas to those in the proof of Theorem 2.1 allow us to prove:

Proposition 2.3. *Let X be a separable Banach space and $K \in cwk(X)$. The following statements are equivalent:*

(i) *K is norm compact.*
(ii) *The family $cwk(K) := \{C \in cwk(X) : C \subset K\}$ is h-separable.*
(iii) *For any complete probability space (Ω, Σ, μ) and any Pettis integrable multi-function $F : \Omega \to cwk(X)$ such that $F(\Omega) \subset cwk(K)$, the composition $j \circ F$ is Pettis integrable.*
(iv) *For any Pettis integrable multi-function $F : [0,1] \to cwk(X)$ such that $F(\Omega) \subset cwk(K)$, the composition $j \circ F$ is Pettis integrable.*
(v) *For any Pettis integrable multi-function $F : [0,1] \to cwk(X)$ such that $F(\Omega) \subset cwk(K)$, the composition $j \circ F$ is scalarly measurable.*

Proof. The implication (i)⇒(ii) is as follows: the norm compactness of K says that $cwk(K) \subset ck(X)$. Since $ck(X)$ is h-separable, the same holds for $cwk(K)$.

Implication (ii)⇒(iii) again follows from [3, Proposition 3.5].

The implications (iii)⇒(iv)⇒(v) are immediate.

To finish we prove that (v)⇒(i). Suppose that K is not norm compact. We shall construct a Pettis integrable multi-function $F : \{0,1\}^{\mathbb{N}} \to cwk(X)$ such that $F(\Omega) \subset cwk(K)$ and $j \circ F$ is not scalarly measurable. Since K is weakly but not norm compact, there is a sequence (y_n) in K converging weakly to some $y \in K$ such that $\inf_{n\in\mathbb{N}} \|y_n - y\| > 0$. Define $x_n := y_n - y \in X$ for every $n \in \mathbb{N}$. Then (x_n) is weakly null and, by the Bessaga-Pelczynski selection principle (cf. [8, p. 42]), we can assume further that (x_n) is a basic sequence. Then, as in the proof of Theorem 2.1, the multi-function

$$G : \{0,1\}^{\mathbb{N}} \to cwk(X), \quad G(\omega) := \overline{\text{co}\{\pi_n(\omega)x_n : \, n \in \mathbb{N}\}},$$

is Pettis integrable but $j \circ G$ is not scalarly measurable. Now it is not difficult to check that the multi-function $F : \{0,1\}^{\mathbb{N}} \to cwk(X)$ defined by $F(\omega) := G(\omega) + y$ satisfies the required properties. □

3. STABILITY AND THE PETTIS INTEGRAL FOR MULTI-FUNCTIONS

We begin this section by recalling the notion of stable family of real-valued functions and its relationship with the Pettis integral theory. For detailed information on this subject we refer the reader to [17] and [12, Chapter 46].

A family $\mathcal{H} \subset \mathbb{R}^{\Omega}$ – (Ω, Σ, μ) complete probability space – is said to be *stable* (in Talagrand's sense) if, for each $A \in \Sigma$ with $\mu(A) > 0$ and each pair of real numbers $\alpha < \beta$, there are $k, l \in \mathbb{N}$ such that

$$\mu^*_{k+l}(D_{k,l}(\mathcal{H}, A, \alpha, \beta)) < \mu(A)^{k+l},$$

where μ_{k+l} denotes the product of $k+l$ copies of μ and

$$D_{k,l}(\mathcal{H}, A, \alpha, \beta) := \bigcup_{h \in \mathcal{H}} \{(\omega_i)_{i=1}^{k+l} \in A^{k+l} : \ h(\omega_i) < \alpha \text{ for all } 1 \le i \le k, \\ h(\omega_i) > \beta \text{ for all } k+1 \le i \le k+l\}.$$

If $\mathcal{H}$ is stable then it is made up of measurable functions and $\overline{\mathcal{H}}^{\mathfrak{T}_p(\Omega)}$ is also stable, where $\mathfrak{T}_p(\Omega)$ stands for the pointwise convergence topology (i.e. the product topology) on $\mathbb{R}^{\Omega}$, [17, Section 9-1].

Stability and Pettis integration are related as follows. Given a Banach space Y, a norming set $B \subset B_{Y^*}$ and a function $f : \Omega \to Y$, we can consider the family

$$Z_{f,B} = \{\langle y^*, f\rangle : \ y^* \in B\} \subset \mathbb{R}^{\Omega}.$$

Notice that in the particular case when $B = B_{Y^*}$ the set $Z_f := Z_{f,B_{Y^*}}$ is $\mathfrak{T}_p(\Omega)$-compact after Alaoglu's theorem. A well known result of Talagrand states that f is Pettis integrable provided that it is *properly measurable*, i.e. Z_f is stable, and Z_f is uniformly integrable, [17, Theorem 6-1-2].

Lemma 3.1. *Let (Ω, Σ, μ) be a complete probability space and Y a Banach space. Let $f : \Omega \to Y$ be a function such that:*

(i) *There is a countable partition (A_n) of Ω in Σ such that the restriction $f|_{A_n}$ is Pettis integrable for every $n \in \mathbb{N}$.*
(ii) *There is a norming set $B \subset B_{Y^*}$ such that the family $Z_{f,B}$ is uniformly integrable.*

Then f is Pettis integrable.

Proof. We begin by proving that for each $E \in \Sigma$ the series $\sum_n \nu_{f|_{A_n}}(E \cap A_n)$ is unconditionally convergent. Indeed, observe that for every finite set $Q \subset \mathbb{N}$ we have

$$\Big\|\sum_{n\in Q} \nu_{f|_{A_n}}(E \cap A_n)\Big\| = \\ = \sup_{y^* \in B} \Big|\sum_{n \in Q} \langle y^*, \nu_{f|_{A_n}}(E \cap A_n)\rangle\Big| = \sup_{y^*\in B} \Big|\int_{E \cap (\bigcup_{n\in Q} A_n)} \langle y^*, f\rangle \, d\mu\Big|.$$

The unconditional convergence of $\sum_n \nu_{f|_{A_n}}(E \cap A_n)$ now follows from the uniform integrability of the family $Z_{f,B}$.

We next show that f is Pettis integrable and that for every $E \in \Sigma$ we have $\nu_f(E) = \sum_n \nu_{f|_{A_n}}(E \cap A_n)$. To this end, observe first that f is scalarly measurable. We check now that f is scalarly integrable (i.e. Dunford integrable). For fixed $y_0^* \in B_{Y^*}$ and every $n \in \mathbb{N}$ we have

$$\frac{1}{2}\Big(\int_{\bigcup_{k=1}^n A_k} |\langle y_0^*, f\rangle|\, d\mu\Big) \leq$$

$$\leq \sup_{\substack{E\in\Sigma \\ E\subset\bigcup_{k=1}^n A_k}} \Big|\int_E \langle y_0^*, f\rangle\, d\mu\Big| = \sup_{E\in\Sigma}\Big|\sum_{k=1}^n \int_{E\cap A_k} \langle y_0^*, f\rangle\, d\mu\Big| \leq$$

$$\leq \sup_{E\in\Sigma}\Big\|\sum_{k=1}^n \nu_{f|_{A_k}}(E\cap A_k)\Big\| = \sup_{E\in\Sigma}\sup_{y^*\in B}\Big|\sum_{k=1}^n \int_{E\cap A_k} \langle y^*, f\rangle\, d\mu\Big| \leq$$

$$\leq \sup_{y^*\in B}\sup_{E\in\Sigma}\int_E |\langle y^*, f\rangle|\, d\mu \leq \sup_{y^*\in B}\int_\Omega |\langle y^*, f\rangle|\, d\mu < \infty.$$

Therefore f is scalarly integrable. Finally, notice that

$$\int_E \langle y^*, f\rangle\, d\mu = \sum_n \langle y^*, \nu_{f|_{A_n}}(E\cap A_n)\rangle = \langle y^*, \sum_n \nu_{f|_{A_n}}(E\cap A_n)\rangle$$

for every $y^* \in Y^*$ and every $E \in \Sigma$. The proof is complete. □

Lemma 3.2. *Let (Ω, Σ, μ) be a complete probability space and Y a Banach space. Let $f : \Omega \to Y$ be a function such that:*

(i) *There is a countable partition (A_n) of Ω in Σ such that the restriction $f|_{A_n}$ is bounded for every $n \in \mathbb{N}$.*

(ii) *There is a norming set $B \subset B_{Y^*}$ such that the family $Z_{f,B}$ is uniformly integrable and stable.*

Then f is Pettis integrable and properly measurable.

Proof. Fix $n \in \mathbb{N}$. Since the family $Z_{f|_{A_n},B}$ is uniformly bounded and stable, a result of Talagrand [17, Theorem 11-2-1] (cf. [12, 465N]) ensures that $\mathrm{aco}(Z_{f|_{A_n},B})$ is also stable and, therefore, the same holds for $\overline{\mathrm{aco}(Z_{f|_{A_n},B})}^{\mathfrak{T}_p(\Omega)} = \overline{Z_{f|_{A_n},\mathrm{aco}(B)}}^{\mathfrak{T}_p(\Omega)}$. On the other hand, bearing in mind that B is norming, the Hahn-Banach theorem yields the equality $B_{Y^*} = \overline{\mathrm{aco}(B)}^{w^*}$, hence $\overline{Z_{f|_{A_n},\mathrm{aco}(B)}}^{\mathfrak{T}_p(\Omega)} = Z_{f|_{A_n}}$. Since $Z_{f|_{A_n}}$ is stable and $f|_{A_n}$ is bounded, we can apply [17, Theorem 6-1-2] to conclude that $f|_{A_n}$ is Pettis integrable.

The Pettis integrability of f now follows from Lemma 3.1. Finally, bearing in mind that $\Omega = \bigcup_n A_n$ and that $Z_{f|_{A_n}}$ is stable for every $n \in \mathbb{N}$, it is easy to check that Z_f is stable, as required. □

We are now ready to prove the main result of this section, Theorem 3.3 below. We first need to introduce some terminology. Given a Banach space X, we write τ_M to denote the Mackey topology on X^*, that is, the topology of uniform convergence on weakly compact subsets of X. Recall that, by the Mackey-Arens theorem, τ_M is the finest locally convex topology on X^* whose topological dual is X, hence $\overline{C}^{w^*} = \overline{C}^{\tau_M}$ for every convex set $C \subset X^*$ (cf. [15, Chapter 8]).

Theorem 3.3. *Let (Ω, Σ, μ) be a complete probability space and X a separable Banach space. Let $F : \Omega \to cwk(X)$ be a Pettis integrable multi-function such* *that every countable subset of W_F is stable. Then $j \circ F$ is Pettis integrable and properly measurable.*

Proof. Notice first that $j : cwk(X) \to \ell_\infty(B_{X^*})$ actually takes its values in the Banach space $Y := C_b(B_{X^*}, \tau_M)$ of all real-valued, bounded and τ_M-continuous functions on B_{X^*}, equipped with the supremum norm. Since X is separable, B_{X^*} contains a countable w^*-dense subset C. In view of the comments just before the theorem, we can assume without loss of generality that C is even τ_M-dense. For each $x^* \in B_{X^*}$, define $e_{x^*} \in B_{\ell_\infty(B_{X^*})^*}$ by $\langle e_{x^*}, h\rangle := h(x^*)$. Then the set

$$B := \{e_{x^*}|_Y : x^* \in C\} \subset B_{Y^*}$$

is norming and $\|(j \circ F)(\omega)\|_\infty = \sup_{x^* \in C} |\delta^*(x^*, F)(\omega)|$ for every $\omega \in \Omega$. Since the family W_F is made up of measurable functions, we conclude that the mapping $\omega \mapsto \|(j \circ F)(\omega)\|_\infty$ is measurable. In particular, there is a countable partition (A_n) of Ω in Σ such that the restriction $j \circ F|_{A_n}$ is bounded for every $n \in \mathbb{N}$. On the other hand, the family

$$Z_{j\circ F,B} = \{\delta^*(x^*, F) : x^* \in C\}$$

is uniformly integrable after Theorem A and stable since C is countable. It follows from Lemma 3.2 that $j \circ F$ is Pettis integrable and properly measurable. The proof of the theorem is complete. □

Remark 3.4. Our Theorem 3.3 generalizes the fact that $j \circ F$ *is Pettis integrable for any Pettis integrable multi-function F with essentially h-separable range*, [3, Proposition 3.5]. Indeed, for such an F the single-valued function $j \circ F$ has essentially separable range and $\langle e_{x^*}, j \circ F\rangle$ is measurable for every $x^* \in B_{X^*}$. Since $\{e_{x^*} : x^* \in B_{X^*}\} \subset B_{\ell_\infty(B_{X^*})^*}$ is norming, an appeal to Pettis' measurability theorem (cf. [9, Corollary 4, p. 42]) ensures that $j \circ F$ is strongly measurable. It follows that $j \circ F$ is properly measurable (this can be deduced easily from [9, Corollary 3, p. 42]) and, in particular, the family W_F is stable.

Remark 3.5. We also mention that there are Pettis integrable multi-functions F for which the family W_F is stable but the range of F is not essentially h-separable, see [3, Example 3.10].

Recall that the complete probability space (Ω, Σ, μ) is said to be perfect (cf. [17, 1-3-1]) if for every measurable function $h : \Omega \to \mathbb{R}$ and every $E \subset \mathbb{R}$ with $f^{-1}(E) \in \Sigma$, there is a Borel set $B \subset E$ such that $\mu(f^{-1}(B)) = \mu(f^{-1}(E))$. For instance, every Radon topological probability space is perfect (cf. [17, 1-3-2]). Perfect probability spaces play a relevant role in the study of pointwise compact sets of measurable functions, see [17]. Under Martin's Axiom (or even weaker axioms), every pointwise compact separable family of real-valued measurable functions defined on a perfect complete probability space is stable, see [17, Section 9-3]. As a consequence we obtain the following result.

Corollary 3.6 (Martin's Axiom)**.** *Let (Ω, Σ, μ) be a perfect complete probability space and X a separable Banach space. Let $F : \Omega \to cwk(X)$ be a multi-function. The following conditions are equivalent:*

(i) *F is Pettis integrable and $j \circ F$ is scalarly measurable.*
(ii) *$j \circ F$ is Pettis integrable.*

Proof. We only have to give a proof for the implication (i)⇒(ii). Notice that, for each countable set $C \subset B_{X^*}$, the family

$$\overline{\{\delta^*(x^*, F) :\ x^* \in C\}}^{\mathfrak{T}_p(\Omega)} \subset Z_{j\circ F}$$

is $\mathfrak{T}_p(\Omega)$-compact separable and is made up of measurable functions, therefore it is stable. The result now follows from Theorem 3.3. □

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