Extensions of Boolean isometries

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Abstract

We study when a map between two subsets of a Boolean domain W can be extended to an automorphism of W. Under many hypotheses, if the underlying Boolean algebra is complete or if the sets are finite or Boolean domains, the necessary and sufficient condition is that it preserves the Boolean distance between every couple of points.

1 Introduction

Boolean domains and Boolean transformations are the Boolean analogues of algebraic varieties and morphisms of algebraic varieties. We fix once and for all a Boolean algebra B. A Boolean function $f : B^n \longrightarrow B$ is a function which admits a polynomial expression in terms of the operations and elements of B, such as for instance $f(x_1, x_2) = (x_1 \vee x_2) \triangle a$, where a is a fixed element of B. A Boolean domain (over B) is a subset $V \subset B^n$ which is the set of solutions to a Boolean equation, namely

$$V = \{(x_1, \dots, x_n) \in B^n : f(x_1, \dots, x_n) = 0\},$$

for some Boolean function $f : B^n \longrightarrow B$. If $U \subset B^n$ and $V \subset B^m$ are Boolean domains, a map $F : U \longrightarrow V$ is a Boolean transformation if there are Boolean functions $F_1, \dots, F_m : B^n \longrightarrow B$ such that

$$F(x) = (F_1(x), \dots, F_m(x))$$

for all $x \in U$. A Boolean isomorphism is a bijective Boolean transformation (its inverse map is, in fact, a Boolean transformation too). Two Boolean domains are isomorphic if there exists a Boolean isomorphism between them. We must

[1] Author supported by FPU grant of SEEU-MEC, Spain.

mention the books [8] and [9] as reference treaties about Boolean functions and equations.

In this paper, we consider the problem of when a given bijection between two subsets of a Boolean domain W can be extended to a Boolean isomorphism from the whole W onto itself. One main result is the following:

Theorem 1 *Let $U, V, W \subset B^n$ be Boolean domains with $U \cup V \subset W$ and let $F : U \longrightarrow V$ be a Boolean isomorphism. Then, F is the restriction of some Boolean isomorphism $F' : W \longrightarrow W$.*

A Boolean domain $U \subset B^n$ can always be considered as a Boolean metric space with the metric $d(x, y) = \bigvee_{i=1}^{n}(x_i \bigtriangleup y_i)$. A *Boolean metric space* (over B) is a set X together with a symmetric map $d : X \times X \longrightarrow B$ satisfying the following two properties: $d(x, y) = 0$ if and only if $x = y$, and $d(x, z) \leq d(x, y) \vee d(y, z)$ for all $x, y, z \in X$. This constitutes a category with maps $f : X \longrightarrow Y$ which are *contractive*, that is, $d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$. When this inequality is an equality and f is bijective, then f is called an *isometry*. This concept was early studied in a series of works like [2], [3], [4], [5] and [6]. In [1] the close relation between the metric and the algebraic structure of Boolean domains, in a more general context, is investigated. The Boolean transformations between Boolean domains coincide with the contractive maps and the Boolean isomorphisms with the isometries. Also, the category of Boolean domains and transformations is equivalent to the category of CFG-spaces (a subclass of Boolean metric spaces, whose definition is recalled below) and contractive maps and therefore Theorem 1 is equivalent to the following:

Theorem 2 *Let U, V, W be CFG-spaces with $U \cup V \subset W$ and let $F : U \longrightarrow V$ be an isometry. Then, F is the restriction of some isometry $F' : W \longrightarrow W$.*

A direct consequence of this theorem, together with [1, Theorem 1.15] is that the necessary and sufficient condition for a bijection between finite subsets of a Boolean domain W to be extended to a Boolean isomorphism of W is to be an isometry between these two finite sets.

It turns out in fact, that when B is a complete Boolean algebra, then U and V need not be assumed CFG-spaces:

Theorem 3 *Suppose that B is complete. Let W be a CFG-space, U, V subsets of W and $F : U \longrightarrow V$ an isometry. Then, F is the restriction of some isometry $F' : W \longrightarrow W$.*

If A is a p-ring for some prime number p (that is, a ring in which $x^p = x$ and $px = 0$ for all x) then A happens to be a Boolean metric space over its ring of idempotents with distance $d(x, y) = (x-y)^{p-1}$. These spaces were investigated in the papers [10] and [7] which study, among others, problems of extension of isometries. Namely, [10, theorem 5] is the same statement as our Theorem 3 but only for the particular case in which W is a p-ring.

The statement of Theorem 3 also holds for contractive maps instead of isometries:

Theorem 4 *Suppose that B is complete. Let W be a CFG-space, U, V subsets of W and $F : U \longrightarrow V$ a contractive map. Then, F is the restriction of some contractive map $F' : W \longrightarrow W$.*

We give examples that the hypothesis of completeness cannot be weakened in Theorems 3 and 4.

2 Notations

The operations in Boolean algebras will be denoted as $a \vee b$ and $a \wedge b$ for the supremum and infimum and $a \setminus b$ for the difference, 0 and 1 denote the lowest and greatest element, $\overline{a} = 1 \setminus a$ is the complement and $a \bigtriangleup b = (a \setminus b) \vee (b \setminus a)$ is the symmetric difference which allows to consider B as a ring with sum $\bigtriangleup$ and product $\wedge$. Elements $a_0, \ldots, a_n$ of B are *disjoint* if $a_i \wedge a_j = 0$ whenever $i \neq j$ and they are a *partition* if moreover $a_0 \vee \cdots \vee a_n = 1$. The lattice order of B is denoted as $a \leq b$.

With respect to Boolean metric spaces, the distance will be always denoted by d. The product space of the Boolean metric spaces X and Y is $X \times Y$ with the metric

$$d((x, y), (x', y')) = d(x, x') \vee d(y, y').$$

We will work in pointed Boolean metric spaces, that is, metric spaces X in which a point $0 \in X$ has been fixed. Formally,

Definition 5 *A pointed Boolean metric space is a couple $(X, 0)$ where X is a Boolean metric space with metric d and 0 is an element of X. A contractive map between two pointed spaces $f : (X, 0) \longrightarrow (X', 0')$ is a contractive map $f : X \longrightarrow X'$ such that $f(0) = 0'$.*

In such spaces we will also use the notation $|x| = d(x, 0)$. There is no deep difference in dealing with pointed spaces but it will be convenient for technical reasons. We shall make use of several tools in this context, as *convexity* and *orthogonality*, developed in [1], that are explained below.

Let $a_0, \ldots, a_n$ be a partition of B and $x_0, \ldots, x_n$ be elements of the metric space X. An element $x \in X$ is said to be a *convex combination* of $x_0, \ldots, x_n$

with coefficients $a_0, \ldots, a_n$ if $a_i \wedge d(x, x_i) = 0$ for all i. In this case we write $a_0x_0 + \cdots + a_nx_n = x$.

It turns out that X can be always embedded into a module over B considered as a ring (sending the fixed element 0 to the zero of the module) in such a way that these convex combinations correspond exactly with the usual linear combinations, cf. [1, Theorem 1.6] and [1, Proposition 1.11]. This means that the notation is coherent and all the usual properties of sum and multiplication by scalars apply. When $(X, 0)$ is a pointed metric space then we may suppress the term corresponding to 0 in notation $a_0 0 + a_1x_1 + \cdots + a_nx_n = a_1x_1 + \cdots + a_nx_n$, where $a_1, \ldots, a_n$ are just disjoint. We also recall that, in product spaces, convex combinations can be calculated coordinatewise.

A set $S \subset X$ is a *system of generators* of X, shortly $X = conv(S)$, if any element of X can be expressed as a convex combination of elements of S with some coefficients. We mention the fact that if two contractive maps coincide on a system of generators, then they are equal.

A metric space X is a *CFG-space* if it verifies the following two properties:

(1) It is *convex*, that is, for any $x_0, \ldots, x_n \in X$ and any partition $a_0, \ldots, a_n$ of B, the convex combination $x = a_0x_0 + \cdots + a_nx_n$ is an element of X.
(2) It is *finitely generated*, that is, there is a finite system of generators of X.

We also mention the fact that X is a CFG-space if and only if it is isometric to a Boolean domain, as it follows from [1, Theorem 3.8].

The elements x and y of the pointed space $(X, 0)$ are *orthogonal* ($x \perp y$) if $d(x, y) = |x| \vee |y|$. For a subset $U \subset (X, 0)$ with $0 \in U$ we set

$$U^\perp = \{y \in X : x \perp y \ \forall x \in U\}.$$

It turns out that $U^\perp$ is a CGF-space provided U is [1, Proposition 2.11]. The relation of this concept of orthogonality with the extension of isometries is the following statement:

Proposition 6 *Let* U, X, Y *be CFG-spaces with* $0 \in U \subset X$ *and* $f : (U, 0) \longrightarrow (Y, 0')$ *and* $g : (U^\perp, 0) \longrightarrow (Y, 0')$ *be isometries. Then, there is a unique isometry* $f \perp g : (X, 0) \longrightarrow (Y, 0')$ *which extends both* f *and* g.

This is the content of Proposition 2.12 in [1] except that there it is written contractive map instead of isometry. However, it is straightforward to check in that proof, that if f and g are assumed to be isometries, then $f \perp g$ that is obtained is again an isometry.

3 The first extension theorem

In this section we will prove Theorem 2. What we will really prove instead of it will be the following statement about orthogonal spaces:

Theorem 7 *Let $(X,0)$ be a pointed CFG-space and U_1, U_2 CFG-subspaces of X with $0 \in U_1 \cap U_2$. If U_1 is isometric to U_2, then $U_1^{\perp}$ is isometric to $U_2^{\perp}$.*

Let us see, first, that Theorem 2 follows from Theorem 7. For this, apart from Proposition 6, we need another result [1, Theorem 4.6], that CFG-spaces are homogeneous, that is, if X is a CFG-space and $x, y \in X$, there is an isometry $\phi : X \longrightarrow X$ such that $\phi(x) = y$. Let U, V, W and F be as in the hypotheses of Theorem 2 and, by homogeneity, fix $0 \in U$ and an isometry $\phi : W \longrightarrow W$ such that $\phi(F(0)) = 0$. We apply Theorem 7 to $X = W$, $U_1 = U$, $U_2 = \phi(V)$ and we obtain that $U^{\perp}$ and $\phi(V)^{\perp}$ are isometric. Again, by homogeneity, we find an isometry $g : (U^{\perp}, 0) \longrightarrow (\phi(V)^{\perp}, 0)$. Finally, the map $F' = \phi^{-1} \circ ((\phi \circ F) \perp g)$ is the desired isometry.

Before passing to the proof of Theorem 7, we must recall the criteria of isometry and the concept of base developed in [1].

For a space X and an integer $k > 0$, we define an element

$$\alpha_k(X) = \sup\{ \bigwedge_{0 \leq i < j \leq k} d(u_i, u_j) : u_0, \ldots, u_k \in X \}$$

This supremum exists and is indeed attained whenever X is either finite or a CFG-space. In the latter case in addition, there exists k_0 with $\alpha_k(X) = 0$ for all $k > k_0$ and $\alpha_k(X) \geq \alpha_{k+1}(X)$ for all k. Another property is that if A is a system of generators of X, $X = conv(A)$, then $\alpha_k(A) = \alpha_k(X)$ for all k. The importance of these functions is that they determine the isometry classes of CFG-spaces: two CFG-spaces X and Y are isometric if and only if $\alpha_k(X) = \alpha_k(Y)$ for all k, cf. [1, §4].

Another result that we need is the existence of *bases*: Any pointed CFG-space $(X, 0)$ has a base, that is, a set $\{x_1, \ldots, x_n\}$ such that

(1) $X = conv(0, x_1, \ldots, x_n)$,
(2) $x_i \perp x_j$ for any $i \neq j$,
(3) $\alpha_i(X) = |x_i| > 0$ for $i = 1, \ldots, n$.

We point out that condition (1) above implies that $\alpha_i(X) = 0$ for $i > n$. The following lemma investigates the relation between the functions $\alpha_k(U)$, $\alpha_k(U^{\perp})$ and $\alpha_k(X)$ when U is a CFG-subspace of X. It will be useful now to convene that $\alpha_0(Y) = 1$ for any space Y.

Lemma 8 *Let $(X,0)$ be a CFG-space and U a CFG-subspace with $0 \in U$. Then, for all $n \in \mathbb{N}$,*

$$\alpha_n(X) = \bigvee_{i=0}^{n} \alpha_i(U) \wedge \alpha_{n-i}(U^\perp).$$

PROOF: Take bases $B_1 = \{x_1, \ldots, x_r\}$ and $B_2 = \{y_1, \ldots, y_s\}$ of $(U,0)$ and $(U^\perp, 0)$, respectively and define $B = B_1 \cup B_2 \cup \{0\}$. From [1, Proposition 2.11] we have $X = conv(U \cup U^\perp)$ and hence $X = conv(B)$ and $\alpha_n(X) = \alpha_n(B)$. Now the result follows by applying the definition of the function α_n to that set, having in mind the relations

$$\begin{aligned} |x_1| \geq |x_2| \geq \cdots , & \qquad (1)\\ |y_1| \geq |y_2| \geq \cdots , & \qquad (2)\\ d(x_i, x_j) = |x_i| \vee |x_j| = |x_{\min(i,j)}|, & \qquad (3)\\ d(y_i, y_j) = |y_i| \vee |y_j| = |y_{\min(i,j)}|. & \qquad (4) \end{aligned}$$

Namely, for a subset A of B we define

$$\phi(A) = \bigwedge_{u,v \in A, u \neq v} d(u,v),$$

so that $\alpha_n(B)$ is the supremum of all $\phi(A)$ when A runs over all subsets of B of cardinality $n+1$. Whenever $n - s \leq i \leq r$, we can consider the set

$$A_i = \{0, x_1, \ldots, x_i, y_1, \ldots, y_{n-i}\}$$

of cardinality $n+1$, so that $\alpha_n(X) \geq \phi(A_i)$ and by the relations mentioned above, it is easily calculated that $\phi(A_i) = |x_i| \wedge |y_{n-i}| = \alpha_i(U) \wedge \alpha_{n-i}(U^\perp)$. When $n - s \leq i \leq r$ does not hold, then $\alpha_i(U) \wedge \alpha_{n-i}(U^\perp) = 0$. This proves that $\alpha_n(X) \geq \bigvee_{i=0}^{n} \alpha_i(U) \wedge \alpha_{n-i}(U^\perp)$. For the other inequality, we take an arbitrary subset A of B of cardinality $n+1$ and we shall prove that $\phi(A) \leq \bigvee_{i=0}^{n} \alpha_i(U) \wedge \alpha_{n-i}(U^\perp)$. For such an A, we find $i_1 < \cdots < i_t$ and $j_1 < \cdots < j_u$ such that

$$\begin{aligned} A \cap B_1 &= \{x_{i_1}, \ldots, x_{i_t}\},\\ A \cap B_2 &= \{y_{j_1}, \ldots, y_{j_u}\}. \end{aligned}$$

Now, if $0 \in A$ then $t + u = n$ and using relations $(1) - (4)$ above

$$\phi(A) \leq d(0, x_{i_t}) \wedge d(0, y_{j_u}) = |x_{i_t}| \wedge |y_{j_u}| \leq |x_t| \wedge |y_u| = \alpha_t(U) \wedge \alpha_u(U^\perp).$$

On the other hand, if $0 \notin A$, then $u + t = n + 1$ and calculating again,

$$\begin{aligned}\text{if } t,u \geq 2, \quad \phi(A) &\leq d(x_{i_t}, y_{j_u}) \wedge d(x_{i_t}, x_{i_{t-1}}) \wedge d(y_{j_u}, y_{j_{u-1}}) \\ &= (|x_{i_t}| \vee |y_{j_u}|) \wedge |x_{i_{t-1}}| \wedge |y_{j_{u-1}}| \\ &= (|x_{i_t}| \wedge |y_{j_{u-1}}|) \vee (|x_{i_{t-1}}| \wedge |y_{j_u}|) \\ &\leq (|x_t| \wedge |y_{u-1}|) \vee (|x_{t-1}| \wedge |y_u|),\end{aligned}$$

$$\begin{aligned}\text{if } t=1, u>1, \quad \phi(A) &\leq d(x_{i_1}, y_{j_u}) \wedge d(y_{j_u}, y_{j_{u-1}}) \\ &= (|x_{i_1}| \vee |y_{j_u}|) \wedge |y_{j_{u-1}}| \\ &= (|x_{i_1}| \wedge |y_{j_{u-1}}|) \vee |y_{j_u}|) \\ &\leq (|x_1| \wedge |y_{u-1}|) \vee |y_u| \\ &= (\alpha_1(U) \wedge \alpha_{u-1}(U^\perp)) \vee (\alpha_0(U) \wedge \alpha_u(U^\perp)),\end{aligned}$$

and the other cases are checked similarly. $\square$

PROOF OF THEOREM 7: For every $i \in \mathbb{N}$, we set

$$a_i = \alpha_i(U_1) = \alpha_i(U_2), \;\; b_i = \alpha_i(X), \;\; r_i = \alpha_i(U_1^\perp), \;\; s_i = \alpha_i(U_2^\perp).$$

What we must prove is that $r_i = s_i$ for every i. Let d be the greatest integer with $\alpha_d(X) > 0$. Clearly, $r_i = s_i = 0$ for all $i > d$ and by Lemma 8 both $(r_i)_{i=1}^d$ and $(s_i)_{i=1}^d$ are solutions to the following system of equations in the variables $x_1, \ldots, x_d$:

$$x_1 \geq \cdots \geq x_d, \tag{5}$$

$$\bigvee_{i=0}^{n} (a_{n-i} \wedge x_i) = b_n \qquad n = 1, \ldots, d+1, \tag{6}$$

where $x_0 = a_0 = 1$ and $x_{d+1} = 0$ are constants.

Hence, we must see that this system of equations has a unique solution, under the hypotheses that $b_1 \geq \cdots \geq b_{d+1} = 0$, $a_1 \geq \cdots \geq a_{d+1} = 0$ and $a_i \leq b_i$ for all i. We need, therefore, a criterion to ensure the uniqueness of solutions of a certain system of Boolean equations, which is provided by the following lemma:

Lemma 9 *Let Y be a CFG-space and $\{y_0, \ldots, y_n\}$ a system of generators of Y such that $d(y_i, y_j) = 1$ for all $i \neq j$. Let $f : Y \longrightarrow B$ be a contractive function such that $f(y_i) \vee f(y_j) = 1$ for all $i \neq j$. If the equation $f(x) = 0$ has a solution for $x \in Y$, then this solution is unique.*

PROOF: Notice that, even if $i = j$ we always have $d(y_i, y_j) \leq f(y_i) \vee f(y_j)$ for all $i, j = 0, \ldots, n$. The set of all couples (y_i, y_j) is a system of generators of the product space $Y \times Y$. We consider the function $h(x, y) =$

$d(x,y)\setminus(f(x)\vee f(y))$ on $Y\times Y$. First, we notice that h is contractive. The map $(x,y)\mapsto f(x)\vee f(y)$ is contractive since it is the composition of contractive maps $(x,y)\mapsto(f(x),f(y))$ and $(a,b)\mapsto a\vee b$. The map $(x,y)\mapsto d(x,y)$ is also contractive, cf. property (3') after [1, Definition 1.1]. Hence h is contractive since it is a Boolean operation of two contractive maps. On the other hand, h is equal to zero on the system of generators $\{(y_i,y_j)\}$ and therefore, it is constant equal to zero on all $Y\times Y$. Hence, if $f(x)=f(y)=0$, then $d(x,y)=0$ and $x=y$. $\square$

Back to the proof of Theorem 7, we shall apply Lemma 9 to

$$Y=\{(x_1,\ldots,x_d)\in B^d : x_1\geq\cdots\geq x_d\},$$

which is a metric space with the usual metric $d(x,x')=\bigvee_{i=1}^d(x_i\bigtriangleup x_i')$. It is checked in [1] that in these metric spaces, convex combinations are calculated simply coordinatewise in the natural way. It is straightforward to check that in fact, Y is a CFG-space with the set of generators

$$\begin{aligned}
y_0 &= (0,0,\ldots,0,0),\\
y_1 &= (1,0,\ldots,0,0),\\
&\ \ \ldots\\
y_{d-1} &= (1,1,\ldots,1,0),\\
y_d &= (1,1,\ldots,1,1).
\end{aligned}$$

Namely, if $c=(c_1,\cdots,c_d)$ then $c=(c_1\setminus c_2)y_1+(c_2\setminus c_3)y_2+\cdots+c_dy_d+(1\setminus\bigvee c_i)y_0$. After [1, Theorem 3.8], the contractive functions from Y to B are exactly the Boolean functions. We will finish the proof provided we can apply Lemma 9 to the Boolean function $f(x)=\bigvee_{n=1}^{d+1}f_n(x)$, where

$$f_n(x_1,\ldots,x_d)=b_n\bigtriangleup\left(\bigvee_{i=0}^{n}a_{n-i}\wedge x_i\right).$$

It remains to check that $f(y_j)\vee f(y_k)=1$ whenever $j,k=0,\ldots,d$, $j\neq k$. First, we calculate the value of the $f_n(y_j)$'s. For notational simplicity we convene that $(y_j)_0=1$.

$$
\begin{aligned}
f_n(y_j) &= b_n \bigtriangleup \bigvee_{i=0}^{n} a_{n-i} \wedge (y_j)_i = b_n \bigtriangleup (a_{n-0} \vee a_{n-1} \vee \cdots \vee a_{n-j}) \\
&= a_{n-j} \bigtriangleup b_n \text{ if } j < n;
\end{aligned}
$$

$$
\begin{aligned}
f_n(y_j) &= b_n \bigtriangleup \bigvee_{i=0}^{n} a_{n-i} \wedge (y_j)_i = b_n \bigtriangleup a_0 = b_n \bigtriangleup 1 \\
&= \overline{b_n} \text{ if } j \geq n.
\end{aligned}
$$

The value of the $f(y_j)$'s is then

$$
\begin{aligned}
f(y_0) &= (a_1 \bigtriangleup b_1) \vee (a_2 \bigtriangleup b_2) \vee \cdots \vee (a_d \bigtriangleup b_d) \vee 0; \\
f(y_1) &= \overline{b_1} \vee (a_1 \bigtriangleup b_2) \vee \cdots \vee (a_{d-1} \bigtriangleup b_d) \vee a_d; \\
f(y_2) &= \overline{b_1} \vee \overline{b_2} \vee (a_1 \bigtriangleup b_3) \vee \cdots \vee (a_{d-2} \bigtriangleup b_d) \vee a_{d-1}; \\
&\cdots \\
f(y_j) &= \overline{b_1} \vee \cdots \overline{b_j} \vee (a_1 \bigtriangleup b_{j+1}) \vee \cdots \vee (a_{d-j} \bigtriangleup b_d) \vee a_{d-j+1}; \\
&\cdots \\
f(y_{d-1}) &= \overline{b_1} \vee \cdots \vee \overline{b_{d-1}} \vee (a_1 \bigtriangleup b_d) \vee a_2; \\
f(y_d) &= \overline{b_1} \vee \cdots \vee \overline{b_d} \vee a_1.
\end{aligned}
$$

We can simplify since $b_1 \geq b_2 \geq \cdots \geq b_d$:

$$
\begin{aligned}
f(y_0) &= (a_1 \bigtriangleup b_1) \vee (a_2 \bigtriangleup b_2) \vee \cdots \vee (a_d \bigtriangleup b_d); \\
f(y_1) &= \overline{b_1} \vee (a_1 \bigtriangleup b_2) \vee \cdots \vee (a_{d-1} \bigtriangleup b_d) \vee a_d; \\
f(y_2) &= \overline{b_2} \vee (a_1 \bigtriangleup b_3) \vee \cdots \vee (a_{d-2} \bigtriangleup b_d) \vee a_{d-1}; \\
&\cdots \\
f(y_j) &= \overline{b_j} \vee (a_1 \bigtriangleup b_{j+1}) \vee \cdots \vee (a_{d-j} \bigtriangleup b_d) \vee a_{d-j+1}; \\
&\cdots \\
f(y_{d-1}) &= \overline{b_{d-1}} \vee (a_1 \bigtriangleup b_d) \vee a_2; \\
f(y_d) &= \overline{b_d} \vee a_1.
\end{aligned}
$$

Now, we fix i, j and $a_1 \geq \cdots \geq a_d$. We must see that for any $(b_1 \ldots, b_d) \in Y$, $f(y_i) \vee f(y_j) = 1$. Again, the function $\phi(b)$ which associates to each $b = (b_1, \ldots, b_d) \in Y$ the corresponding value of $\phi(b) = f(y_i) \vee f(y_j)$ is a Boolean function, and in order to see that ϕ is constant equal to one on Y it is enough to check that $\phi(y_k) = 1$ for $k = 0, \ldots, d$. For $(b_1, \ldots, b_d) = y_k$ we obtain:

$$
\begin{aligned}
f(y_0) &= (a_1 \triangle 1) \vee \cdots \vee (a_k \triangle 1) \vee a_{k+1} \vee \cdots \vee a_d; \\
f(y_1) &= (a_1 \triangle 1) \vee \cdots \vee (a_{k-1} \triangle 1) \vee a_k \vee \cdots \vee a_d; \\
&\cdots \\
f(y_j) &= (a_1 \triangle 1) \vee \cdots \vee (a_{k-j} \triangle 1) \vee a_{k-j+1} \vee \cdots \vee a_{d-j+1}; \\
&\cdots \\
f(y_{k-1}) &= (a_1 \triangle 1) \vee a_2 \vee \cdots \vee a_{d-k+2}; \\
f(y_k) &= a_1 \vee a_2 \vee \cdots \vee a_{d-k+1}; \\
f(y_{k+1}) &= f(y_{k+2}) = \cdots = f(y_d) = 1.
\end{aligned}
$$

Now, it is clear that $f(y_i) \vee f(y_j) = 1$ for $i \neq j$ because if $i < j$ then $a_{k-j+1} \leq f(y_j)$ and $a_{k-j+1} \triangle 1 \leq f(y_i)$. This finishes the proof of Theorem 7 and hence, also the proofs of Theorems 2 and 1. $\square$

4 The second extension theorem

In this section we prove Theorems 3 and 4. Hence, we assume from now on that our fixed Boolean algebra B is complete, that is, that whenever S is a subset of B there exists $s = \bigvee S \in B$ the supremum of S. We recall that the distributivity law still holds in the infinite case: $x \wedge \bigvee\{y_i : i \in I\} = \bigvee\{x \wedge y_i : i \in I\}$ whenever $x \in B$ and $y_i \in B$ for all $i \in I$.

Lemma 10 *Let X be a metric space over B and $\{f_i : X \longrightarrow B\}_{i \in I}$ a family of contractive maps. Then, the pointwise supremum $\bigvee f_i$ is again a contractive map.*

PROOF: Recall that the metric on B is given by $d(x,y) = x \triangle y$ and hence $f : X \longrightarrow B$ is contractive if and only if $f(x) \triangle f(y) \leq d(x,y)$ for all $x, y \in X$. Moreover, this can be rewritten as

$$\overline{d(x,y)} \wedge f(y) \leq f(x) \leq f(y) \vee d(x,y)$$

for all $x, y \in X$. With this characterization and using the infinite distributivity law, the proof of the lemma becomes apparent. $\square$

Lemma 11 *Let X be a CFG space over the complete Boolean algebra B and let $\{K_i\}_{i \in I}$ be a family of CFG-subspaces of X. Then $\bigcap_I K_i$ is a CFG-space.*

PROOF: By [1, Lemma 3.5] a subspace $K \subset X$ is a CFG-space if and only if there exists $f : X \longrightarrow B$ contractive with $K = f^{-1}(\{0\})$. This together with Lemma 10 proves the Lemma. $\square$

By Lemma 11, given a subset U of a CFG-space X, we can consider $Conv(U)$ the least CFG-space that contains U, obtained as the intersection of all CFG-subspaces that contain U. Any nonprincipal I ideal of B is an example in which $I = conv(I) \neq Conv(I)$ since I is convex but not a CFG-space.

Theorem 12 *Let X and Y be CFG-spaces over the complete Boolean algebra B and let $f : U \longrightarrow V$ be a contractive map between two arbitrary subsets $U \subset X$ and $V \subset Y$. Then there is a unique contractive map $Conv(f) : Conv(U) \longrightarrow Conv(V)$ that extends f. In addition, if f is an isometry, so is $Conv(f)$.*

Notice that Theorem 3 is a direct consequence of Theorem 12 above together with Theorem 2, while Theorem 4 follows from Theorem 12 and [1, Proposition 2.12].

PROOF OF THEOREM 12: First, we check that $Conv(f)$, provided it exists, is uniquely determined. Suppose that $g, h : Conv(U) \longrightarrow Conv(V)$ are two contractive extensions of f. Then the set

$$K = \{x \in Conv(U) : d(g(x), h(x)) = 0\}$$

is, by [1, Lemma 3.5] a CFG-space which contains U, hence $Conv(U) \subset K$ and $g = h$.

For the existence of $Conv(f)$, we prove first a particular case, namely, that any contractive function $f : U \longrightarrow B$ extends to a contractive map $G : Conv(U) \longrightarrow B$. For every $u \in U$ we consider the contractive map $g_u : Conv(U) \longrightarrow B$ given by

$$g_u(x) = f(u) \setminus d(u, x)$$

and we set $G = \bigvee\{g_u : u \in U\}$. On the one hand, for any $u \in U$, $f(u) = g_u(u) \leq G(u)$. On the other hand for any $u, v \in U$, $f(u) \bigtriangleup f(v) \leq d(u, v)$ and hence $f(v) \geq f(u) \setminus d(u, v) = g_u(v)$, so taking suprema over U, also $f(v) \geq G(v)$. Now we pass to the general case and we use the fact that Y can be viewed as a subspace of B^n for some natural number n. Extending coordinate by coordinate, we know that there is a contractive map $h : Conv(U) \longrightarrow B^n$ which extends f. It remains to show that the range of h verifies $h(Conv(U)) \subset Conv(V) \subset Y$. Again, by [1, Lemma 3.5] there is a contractive map $s : B^n \longrightarrow B$ such that $Conv(V) = s^{-1}(\{0\})$. Notice that for every $u \in U$, $h(u) \in V \subset Conv(V) = s^{-1}(\{0\})$ so $s(h(u)) = 0$. Therefore the composed map $s \circ h : Conv(U) \longrightarrow B$ is a contractive map which extends the constant map $c : U \longrightarrow B$, $c(u) = 0$. By the uniqueness of extensions to $Conv(U)$ that we have already proved, we obtain that $s \circ h = 0$, so $h(Conv(U)) \subset s^{-1}(\{0\}) = Conv(V)$.

With respect to the last assertion of the theorem, if f is an isometry then $f^{-1} : V \longrightarrow U$ is a contractive map and $Conv(f^{-1})$ must be a contractive inverse map for $Conv(f)$ (since the compositions in both senses are contractive extensions of the identity maps in $Conv(U)$ and $Conv(V)$). This implies that $Conv(f)$ is an isometry. $\square$

We finish by presenting an example which shows that the hypotheses of Theorems 2 and 3 cannot be essentially weakened.

Assuming that B is not complete we construct a CFG space X and an isometry $f : U \longrightarrow V$ between subsets of X which cannot be extended to any contractive map $F : X \longrightarrow X$. Take S a subset of B which does not have a supremum and set

$$I = \{a \in B : \exists a_1, \dots, a_n \in S : a \leq a_1 \vee \cdots \vee a_n\};$$

the ideal generated by S which neither has a supremum. Namely, if x were the supremum of I, then it would be also the supremum of S because S and I have the same upper bounds: if y is an upper bound of S and $a \in I$, then $a \leq a_1 \vee \cdots \vee a_n$ for some elements $a_i \in S$, so that $a_i \leq y$ for all i and finally $a \leq y$. Set

$$\begin{aligned} J &= \{a \in B : a \wedge x = 0 \ \forall x \in I\}, \\ I + J &= \{a \bigtriangleup b : a \in I, b \in J\}, \\ X &= \{(x, y) \in B^2 : x \wedge y = 0\}, \\ V &= \{(x, y) \in X : x \in I, y \in J\}, \\ U &= \{(z, 0) \in X : z \in I + J\}. \end{aligned}$$

Observe that X is a CFG-space since it is a Boolean domain, in fact $X = conv\{(0,0), (0,1), (1,0)\}$. The isometry is $f = g^{-1}$, the inverse map of $g : V \longrightarrow U$ given by $g(x, y) = (x \bigtriangleup y, 0)$. Namely g is an isometry because it is clearly onto and for any $x, x' \in I$ and $y, y' \in J$,

$$\begin{aligned} d(g(x, y), g(x', y')) &= x \bigtriangleup y \bigtriangleup x' \bigtriangleup y' = (x \bigtriangleup x') \bigtriangleup (y \bigtriangleup y'); \\ d((x, y), (x', y')) &= (x \bigtriangleup x') \vee (y \bigtriangleup y') \end{aligned}$$

and the two expressions are equal because $x \bigtriangleup x' \in I$ and $y \bigtriangleup y' \in J$, so they are disjoint.

Suppose that we could extend f to some contractive map $F : X \longrightarrow X$. We claim that if $F(1, 0) = (a, b)$ then a is the supremum of I, which is a contradiction. Namely, for every $x \in I$,

$$(x \bigtriangleup a) \vee b = d((x, 0), (a, b)) = d(F(x, 0), F(1, 0)) \leq d((x, 0), (1, 0)) = \overline{x}$$

so that $x \leq a$ and analogously for every $y \in J$,

$$(y \bigtriangleup b) \vee a = d((0,y),(a,b)) = d(F(y,0),F(1,0)) \leq d((y,0),(1,0)) = \overline{y}$$

and $y \leq b$. This means that a is an upper bound of I and b an upper bound of J. If c is now an arbitrary upper bound of I then $\overline{c} \in J$, so $\overline{c} \leq b$, so $a \wedge \overline{c} \leq a \wedge b = 0$ and $a \leq c$.

Observe that the space X in the example is "two-dimensional". In fact the case $X = B$ is special and even if B is not complete, arbitrary isometries between subsets can be always extended. This is because if $f : U \longrightarrow V$ is an isometry between $U, V \subset B$ then $f(x) \bigtriangleup f(y) = x \bigtriangleup y$ for all $x, y \in U$ and this implies that the function $x \bigtriangleup f(x)$ is constant equal to some $a \in B$, and then $F(x) = a \bigtriangleup x$ is an isometry of B that extends F. However, this particularity does not apply when we consider extensions of contractive maps instead of isometries. Take for instance two infinite sets $M \subset \Omega$ and B the Boolean algebra of the finite or cofinite subsets of Ω and $U \subset B$ the family of the finite subsets of Ω. Then the contractive map $f : U \longrightarrow U$ given by $f(x) = M \cap x$ cannot be contractively extended to B.

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