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Commutative rings with finite quotient fields

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Abstract

We consider the class of all commutative reduced rings for which there exists a finite subset $T \subset A$ such that all projections on quotients by prime ideals of A are surjective when restricted to T. A complete structure theorem is given for this class of rings, and it is studied its relation with other finiteness conditions on the quotients of a ring over its prime ideals.

Introduction

Our aim is to study the structure of commutative rings that satisfy suitable finiteness conditions on its quotients by prime ideals. If A is a commutative ring and A/p represents the quotient ring of A by

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a prime ideal p, we will be interested in the following conditions on A:

1. All A/p are finite.

2. The cardinal of all A/p is bounded by some n.

3. There are $x_1, \dots, x_n \in A$ such that $A/p = \{x_1 + p, \dots, x_n + p\}$ for any prime ideal p of A.

Clearly, $3 \Rightarrow 2 \Rightarrow 1$ and at the end of this article examples can be found showing that no converse is true. Also, observe that the three conditions are the same for A as for the reduced ring $A/N(A)$, so we may restrict ourselves to reduced rings, that is, to commutative rings without nilpotent elements. One main result in this paper is that a complete structure theorem can be given for reduced rings satisfying condition (3). Namely, we associate to each such a ring A a tuple $(K_1, B_1, \dots, K_n, B_n)$, where the K_i's are non isomorphic finite fields and the B_i's are Boolean rings, in such a way that two of these rings are isomorphic if and only if the associated tuples are equal, up to isomorphism and order.

A reduced ring satisfying just condition (1) must be Von Neumann regular (absolutely flat, in other terminology), as it is any reduced ring in which all prime ideals are maximal [2, Theorem 1.16]. In this sense, our work continues that of N. Popescu and C. Vraciu in [3], where the structure of commutative Von Neumann regular rings is studied. Here it is shown that imposing these kind of finiteness conditions, strong structure results can be given. Let us note that C. Vraciu had already found in [4], in other direction, the following result concerning rings satisfying condition (1):

Theorem 1 *Let B be a Boolean ring and $k : Spec(B) \longrightarrow FFields$ a map from the Zariski spectrum $Spec(B)$ to the class of finite fields, such that for each $p \in Spec(B)$ there is a neighbourhood U of p such that $k(p) \subseteq k(p')$ for all $p' \in U$. Then, there is a commutative Von Neumann regular ring A such that $B(A) = B$ and for each*

$p \in Spec(B)$, A/pA is isomorphic to $k(p)$.

On the other hand, the class of CFG-rings introduced in [1] is exactly the class of reduced rings satisfying condition (3), so the structure theorem for this class presented here is a precise measure of the degree of generality of the results in [1] in terms of CFG-rings. Also, a result in [1] suggests the addition of the following condition to our list:

2.5. For any map $f : A \longrightarrow A$ the following are equivalent
   (i) There exists a polynomial $F \in A[X]$ such that $f(x) = F(x)$ for all $x \in A$.
   (ii) For all $x, y \in A$, $(f(x) - f(y))A \subseteq (x - y)A$.

Lemma 3.3 in [1] is just the assertion that (3) implies (2.5) and Proposition 13 below states that (2.5) implies (2). We give an example showing that the last converse is not true but we have been unable to decide about the first one.

Notations and terminology

In the sequel, all rings will be supposed to be commutative with identity. Letter A will always represent a ring. A reduced ring is a ring without nilpotent elements.

A polynomial map $f : A^n \longrightarrow A^m$ is a map for which all components $f_i : A^n \longrightarrow A$ are given by polynomials with coefficients in A.

A Boolean ring is a ring B such that $x^2 = x$ for all $x \in B$.

For a Boolean ring B and $a, b \in B$, the expression $a \leq b$ will mean $aB \subseteq bB$.

Let A be a ring. We will denote by $B(A)$ the set of all idempotent elements of A. Recall that the set $B(A)$ has a structure of Boolean ring with product inherited from A and with the sum $a\tilde{+}b = (a-b)^2$.

A complete family of orthogonal idempotents (shortly **c.f.o.i**) of A is a family $a_1, \ldots, a_n$ of elements of $B(A)$ such that $a_i a_j = 0$ for $i \neq j$ and $\sum_1^n a_i = 1$.

A convex combination of elements $x_1, \ldots, x_n \in A$ is a linear combination $\sum_1^n a_i x_i$ such that $a_1, \ldots, a_n$ is a c.f.o.i. of A. The set of all convex combinations of elements of a set $S \subseteq A$ will be denoted by $conv(S) \subseteq A$.

Regular ring will mean here commutative Von Neumann regular ring (also called commutative absolutely flat rings), i.e. a (commutative) ring for which any principal ideal is generated by an idempotent. If A is a regular ring, $e : A \longrightarrow B(A)$ will be the map that sends each $a \in A$ to the only idempotent $e(a) \in B(A)$ such that $aA = e(a)A$.

For a ring A, $Spec(A)$ will denote its Zariski spectrum, that is, the topological space whose underlying set is the set of all prime ideals of A and whose closed subsets are of the form $V(I) = \{p : I \subseteq p\}$ being I an ideal of A. If A is a regular ring, $B(A)$ can be identified with the ring of closed-open sets of $Spec(A)$ via the bijection $b \leftrightarrow O_b = \{p : b \notin p\}$. In fact, for all $a \in A$, $O_a = O_{e(a)}$ is a closed and open set since its complement is $O_{1-e(a)}$. Also, recall that the correspondence $p \mapsto p \cap B(A)$ induces a homeomorphism $Spec(A) \longrightarrow Spec(B(A))$, so that both spectra can be identified.

1. Characterizations of CFG-rings

Theorem 2 *Let A be a reduced ring and let $x_1, \ldots, x_n \in A$. The following are equivalent:*

1. $A = conv\{x_1, \ldots, x_n\}$.
2. $\prod_{i=1}^n (x - x_i) = 0$ *for all* $x \in A$.
3. $A/p = \{x_1 + p, \ldots, x_n + p\}$ *for every prime ideal* p *of* A.

Proof: Suppose $x \in conv\{x_1, \ldots, x_n\}$, so that $x = \sum_1^n a_i x_i$ and $\{a_i\}$ is a c.f.o.i. Then,

$$\prod_{i=1}^{n}(x - x_i) = \prod_{i=1}^{n}\sum_{j=1}^{n} a_j(x_j - x_i) = \sum_{j=1}^{n}\prod_{i=1}^{n} a_j(x_j - x_i) = 0$$

and this proves that (1) implies (2).

Clearly (2) implies (3) since the ideal p is prime. If (3) holds, then $\prod_{i=1}^{n}(x - x_i)$ belong to all prime ideals, so since A is reduced, this product is zero and (2) holds. So (2) and (3) are equivalent.

Suppose now that (2) holds. Let $x \in A$ and we show that $x \in conv\{x_1, \ldots, x_n\}$. We have the following equality in $B(A)$:

$$\prod_{i=1}^{n} e(x - x_i) = 0.$$

Let $b_i = e(x - x_i)$ and $a_i = (1 - b_i)\prod_{j=1}^{i-1} b_j$, then $\{a_i\}_{i=1}^{n}$ is a c.f.o.i and

$$\sum_{i=1}^{n} a_i e(x - x_i) = 0$$

so that

$$e(x - \sum_1^n a_i x_i) = e(\sum_1^n a_i(x - x_i)) = \sum_1^n a_i e(x - x_i) = 0.$$

and finally $x = \sum_1^n a_i x_i$. $\square$

Observe that condition (3) in Theorem 2 implies that each prime ideal is maximal, so A is regular by [2, Theorem 1.16]. Following [1], a CFG-ring is a regular ring satisfying condition (1) of this theorem for some $x_1, \ldots, x_n$. Equivalently, a CFG-ring is a reduced ring for which there are $x_1, \ldots, x_n$ satisfying any of the conditions of the theorem.

2. Boolean envelopes of fields

In this section we define a family of CFG-rings for which we will prove in Theorem 9 that all CFG-rings are finite products of rings of this type in a unique way. Given a field K and a Boolean ring B, it is defined a K-algebra $K^{[B]}$ that will be a CFG-ring if K is a finite field. This algebras were firstly introduced by C. Vraciu [4]. Roughly speaking, $K^{[B]}$ will be the set of all *"formal convex combinations"* of elements of K with coefficients in B, where the sum and product are defined in order to extend the operations in K and to *"commute with convex combinations"*. More precisely:

Definition 3 *Let B be a Boolean ring and K a field. The K-algebra of all continuous functions from $Spec(B)$ to K (K is equipped with the discrete topology) will be denoted by $K^{[B]}$.*

We make some elementary remarks:

1. We identify K inside $K^{[B]}$, identifying each $x \in K$ with the corresponding constant map. In this way, $K^{[B]}$ is a K-algebra.

2. We identify B inside $K^{[B]}$, identifying each $b \in B$ with the map $Spec(B) \longrightarrow K$ that is constant equal 1 on b and vanishes outside b. In fact, this gives all idempotent elements of $K^{[B]}$. Shortly, $B(K^{[B]}) = B$.

3. The ring $K^{[B]}$ consists exactly of all functions $u : Spec(B) \longrightarrow K$ with finite image, such that $u^{-1}(x)$ is a closed-open set of $Spec(B)$ for all $x \in K$. Making the identifications above, this means that each element of $K^{[B]}$ has an expression like $u = \sum_1^n a_i x_i$ where $x_1, \ldots, x_n$ are elements of K (the image of u) and $a_1, \ldots, a_n$ is c.f.o.i of B ($a_i = u^{-1}(\{x_i\})$). So $conv(K) = K^{[B]}$.

4. $K^{[B]}$ is a regular ring. Clearly, each element is the product of a unit (a $(K \setminus \{0\})$-valued continuous map) and an idempotent (a $\{0, 1\}$-valued continuous map).

5. If K is finite, then $K^{[B]}$ is a CFG-ring.

Proposition 4 *Let K be a field, B a Boolean ring and A a Von Neumann regular commutative K-algebra such that $B(A)$ is isomorphic to B and $A = conv(K)$. Then, A and $K^{[B]}$ are isomorphic K-algebras.*

PROOF: We define a map $f : K^{[B]} \longrightarrow A$ in the following way: if $u = \sum_1^n a_i x_i \in K^{[B]}$ being $a_1, \ldots, a_n$ a c.f.o.i of B and $x_i \in K$, then $f(u) = \sum_1^n a_i x_i \in A$. First of all, we must check that this is a correct definition, that is, that $f(u)$ does not depend on the expression chosen for u as convex combination of elements of K. By elementary reduction arguments, it is enough to prove the following:

($\star$) Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be c.f.o.i. of B, and let $x_1, \ldots, x_n$ be n different elements of K. If

$$\sum_1^n a_i x_i = \sum_1^n b_i x_i \in K^{[B]}$$

then

$$\sum_1^n a_i x_i = \sum_1^n b_i x_i \in A.$$

Under those hypotheses $u = \sum_1^n a_i x_i \in K^{[B]}$ is the map $u : Spec(B) \longrightarrow K$ that takes the value x_i on a_i. Therefore

$$\sum_1^n a_i x_i = \sum_1^n b_i x_i \in K^{[B]} \Rightarrow a_i = b_i \text{for all } i$$

and assertion ($\star$) follows immediately.

We prove now that f is a K-algebra homomorphism. We check for instance that f commutes with addition (it is analogous for product). We take $x = \sum_i a_i x_i$ and $y = \sum_j b_j y_j$ in $K^{[B]}$ being $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ c.f.o.i. of B and $x_i, y_j \in K$. Note that the $c_{ij} = a_i b_j$

constitute a c.f.o.i. of B,

$$\begin{aligned} f(x+y) &= f\left(\sum_i a_i x_i + \sum_j b_j y_j\right) = f\left(\sum_{i,j} c_{ij} x_i + \sum_{i,j} c_{ij} y_j\right) \\ &= f\left(\sum_{i,j} c_{ij}(x_i + y_j)\right) = \sum_{i,j} c_{ij}(x_i + y_j) \\ &= \sum_{i,j} c_{ij} x_i + \sum_{i,j} c_{ij} y_j = \sum_i a_i x_i + \sum_j b_j y_j \\ &= f(x) + f(y) \end{aligned}$$

On the other hand, f is onto since $A = conv(K)$. We check that f is one to one. Suppose $f(u) = 0$ and $u = \sum_i a_i x_i$ with the a_i's a c.f.o.i of B and $x_1 = 0, x_2 \neq 0, \ldots, x_n \neq 0$ elements of K. Then, since $0 = \sum_i a_i x_i \in A$, by multiplication by a_i, $a_i x_i = 0$ for all i. Since $x_i \in K \setminus \{0\}$ is a unit for all $i > 1$, $a_i = 0$ for $i > 1$. This implies that $u = 0$. □

3. Structure theorem for CFG-rings

Lemma 5 *Let K be a finite field and B_1, B_2 Boolean rings. Then, $K^{[B_1 \times B_2]}$ and $K^{[B_1]} \times K^{[B_2]}$ are isomorphic K-algebras.*

Proof: The ring A on the right is a K-algebra (we identify K in A with the elements of the form (k,k) with $k \in K$) which is a regular ring, and

$$B(K^{[B_1]} \times K^{[B_2]}) \cong B(K^{[B_1]}) \times B(K^{[B_2]}) \cong B_1 \times B_2.$$

Making use of Proposition 4, we check that $A = conv(K)$. We know that

$$A = K^{[B_1]} \times K^{[B_2]} = conv(K) \times conv(K) = conv(K \times K)$$

so it suffices to see that any $(k, k') \in K \times K$ is in $conv(K)$, and this is trivial since $(k, k') = (1,0)(k,k) + (0,1)(k',k')$. □

Lemma 6 *Let A be a CFG-ring and $f : A \longrightarrow A$ a polynomial map. Then, the set of iterated maps $\{f^k : k \in \mathbf{N}\}$ is finite.*

PROOF: First, it is easy to check that a polynomial map commutes with convex combinations: the identity map and constant maps do and this property is preserved under sums and products. Suppose that $A = conv\{x_1, \ldots, x_n\}$. For $i = 1, \ldots, n$ we can express $f(x_i)$ like a convex combination $f(x_i) = a_{1i}x_1 + \cdots + a_{ni}x_n$. Call R the subring of $B(A)$ generated by the a_{ij}'s. The ring R is finite since it is a finitely generated subring of a Boolean ring. An induction argument shows that for all $k \in \mathbf{N}$, $f^k(x_i)$ can be expressed as a convex combination with coefficients in R: if $f^{k-1}(x_i) = b_1x_1 + \cdots + b_nx_n$, then

$$\begin{aligned} f^k(x_i) = f(f^{k-1}(x_i)) = f(\sum_{r=1}^{n} b_r x_r) &= \sum_{r=1}^{n} b_r f(x_r) \\ = \sum_{r=1}^{n} b_r \sum_{t=1}^{n} a_{tr} x_t &= \sum_{t=1}^{n} \left(\sum_{r=1}^{n} b_r a_{tr} \right) x_t. \end{aligned}$$

Therefore, for each $k \in \mathbf{N}$ there is a $n \times n$-matrix $M^k = \{a^k_{ij}\}$ of elements of R such that $f^k(x_j) = \sum_i a^k_{ij} x_i$. Since R is finite, the set of $n \times n$-matrices over R is finite too. □

Lemma 7 *Let A be a CFG-ring and R a finitely generated subring of A. Then R is finite.*

PROOF: We proceed by induction on n, the number of generators. For $n = 0$, R is the prime ring of A, that is finite, by applying Lemma 6 to the map $x \mapsto x + 1$. Suppose the assertion of the lemma for n, and we will prove it for $n + 1$. Any subring generated by $n + 1$ elements is of the form $T[x]$ with T generated by n elements and hence, by the induction hypothesis, finite. $T[x] = \{\sum_{i=1}^{k} a_i x^i : a_i \in T\}$. Applying again Lemma 6 to $a \mapsto ax$, we deduce that $\{x^k : k \in \mathbf{N}\}$ is finite, so $T[x]$ is finite. □

Lemma 8 *Let K be a field, B a Boolean ring, and let p be a prime ideal of $A = K^{[B]}$. Then A/p is isomorphic to K.*

Proof: Let us check that the composition $f : K \hookrightarrow A \longrightarrow A/p$ is an isomorphism. It is injective, since K is a field. It is surjective: for all $x \in A$, express x as a convex combination of elements of K, $x = \sum_{i=1}^{n} a_i k_i$. When mapping to the quotient ring, all idempotents a_i map into idempotents of the domain A/p, that is, 0 or 1. So $x + p \in K/p = Im(f)$. $\square$

Theorem 9 *Let A be a CFG-ring. Then, there are Boolean rings $B_1, \ldots, B_n$ and non-isomorphic finite fields $K_1, \ldots, K_n$ such that*

$$A \cong K_1^{[B_1]} \times \cdots \times K_n^{[B_n]}$$

Furthermore, this decomposition is unique, up to isomorphism and order.

Proof: Existence of such a decomposition: By Lemma 5, it suffices to find a decomposition where there may be isomorphic finite fields. Suppose $A = conv(H)$ where $H = \{x_1, \ldots, x_n\}$. Call T the subring generated by H, that is finite, by Lemma 7. Therefore, T is a finite reduced ring, so it is a product of finite fields (just apply the Chinese Remainder Theorem to the set of prime ideals of T). Take an isomorphism $h : K_1 \times \cdots \times K_m \longrightarrow T \hookrightarrow A$. Call $e_i = (\delta_{ij})_{j=1}^{m}$ ($\delta_i j$ is the Kronecker delta) and $\varepsilon_i = h(e_i)$. We have a ring decomposition $A \cong \prod_{i=1}^{m} A\varepsilon_i$ since the ε_i's constitute a c.f.o.i. of A. The restriction $h : K_i = Ke_i \longrightarrow A\varepsilon_i$ provides a ring homomorphism, so we can view $A\varepsilon_i$ as a K_i-algebra, that is regular, since it is a factor of a regular ring. If we prove that $conv(h(K_i)) = A\varepsilon_i =: A_i$ we will deduce, by Proposition 4, that $A_i \cong K_i^{[B(A_i)]}$. We have to see that any element of A_i is a convex combination of elements of $h(K_i)$ with scalars in $B(A_i)$. Take $x \in A_i$. There is a convex combination in A, $x = \sum_j b_j r_j$ with $r_j = h(k_j) \in T$. Then, $x = \varepsilon_i^2 x = \sum_j (\varepsilon_i b_j)(\varepsilon_i r_j) = \sum_j (\varepsilon_i b_j) h(e_i k_j)$ provides us the expression desired.

Uniqueness: Suppose given one such a decomposition $A \cong A_1 \times \cdots \times A_n$. Each $A_i = K_i^{[B_i]}$ can be seen as a principal ideal of A. Any prime ideal of A is of the form $p_i^e = A_1 \times \cdots \times A_{i-1} \times p_i \times A_{i+1} \times \cdots \times A_n$ for some prime ideal p_i of A_i, and by Lemma 8, $K_i \cong A_i/p_i \cong A/p_i^e$. Therefore, the fields K_i are uniquely determined, up to isomorphism, by A, since they are those that appear as quotients by prime ideals. Furthermore, since A_i is regular, the intersection of all prime ideals of A_i is 0. Hence, the intersection of all prime ideals of A whose quotients are isomorphic to K_i is $A_1 \times \cdots \times A_{i-1} \times 0 \times A_{i+1} \times \cdots \times A_n$. Then, the intersection of all prime ideals of A whose quotients are not isomorphic to K_j is $0 \times \cdots \times 0 \times A_j \times \cdots \times 0 \equiv A_j$. Therefore, the factor ring of the decomposition corresponding to K_j is also uniquely determined by A, and also the Boolean ring B_j because it must be (isomorphic to) the ring of idempotents of that factor ring. $\square$

4. Other finiteness conditions

During the proof of Lemma 6, it was observed that any polynomial map $f : A \longrightarrow A$ commutes with convex combinations. Maps with this property and their relation with polynomials are object of study in [1]. We summarize in the following proposition the information that we need about this:

Proposition 10 *Let A be a regular ring. For a function $f : A \longrightarrow A$ the following are equivalent:*

1. *The map f commutes with convex combinations, that is, for any $x_1, \ldots x_n \in A$ and any c.f.o.i of A, $a_1, \ldots, a_n$, $f(\sum_1^n a_i x_i) = \sum_1^n a_i f(x_i)$.*
2. *$e(f(x) - f(y)) \leq e(x - y)$ for all $x, y \in A$.*

Furthermore, if A is a CFG-ring, then the following condition is also equivalent:

3. *f is polynomial map.*

A map verifying conditions (1) and/or (2) above will be called a contractive map. We observe that the map $e : A \longrightarrow A$ is always contractive. The equivalence of (3) with the others is the statement of [1, Lema 3.3].

We recall the four finiteness conditions exposed in the introduction:

(1) All quotients by prime ideals are finite.

(2) The cardinality of the quotients by prime ideals is bounded by an integer.

(2.5) All contractive maps $f : A \longrightarrow A$ are polynomial maps.

(3) A is a CFG-ring.

We complete the diagram of implications, except for (2.5) $\Rightarrow$ (3), which we do not know whether it is true or not. We begin with a characterization of condition (2) which shows that (2.5) implies (2):

Proposition 11 *For a regular ring A, the following are equivalent:*

i. *The map $e : A \longrightarrow A$ is a polynomial map.*

ii. *There exists a natural number n such that $|A/p| < n$ for all prime ideals p of A.*

PROOF: Suppose that $e(x) = a_k x^k + \cdots + a_0$. Take a prime ideal p of A and the natural projection $\pi : A \longrightarrow A/p$. Since $e(x)$ is idempotent, $\pi(e(x))$ is an idempotent of the field A/p, so $\pi(e(x)) \in \{0, 1\}$ and

$$\begin{aligned} 0 &= (\pi(e(x)) - 1)\pi(e(x)) \\ &= (\pi(a_k)\pi(x)^k + \cdots + \pi(a_0) - 1)(\pi(a_k)\pi(x)^k + \cdots + \pi(a_0)) \end{aligned}$$

for all $x \in A$. Therefore, all elements in the field A/p are roots of a polynomial of degree $2k$ and $|A/p| < 2k$.

Conversely, suppose that $\{n_1, \ldots, n_r\}$ is the set of cardinalities of quotients of A by prime ideals and call $m_i = n_i - 1$, $m = m_1 \cdots m_r$, $k_i = m/m_i$. We prove that $e(x) = x^m$. Since the intersection of all prime ideals is zero, it is sufficient to check that $\pi(e(x)) = \pi(x)^m$ for every projection $\pi : A \longrightarrow A/p$. If $|A/p| = n_i$, then

$$\pi(x)^m = \pi(x)^{m_i k_i} = (\pi(x)^{n_i - 1})^{k_i} = (1 - \delta_{\pi(x)0})^{k_i} = 1 - \delta_{\pi(x)0}$$

where $\delta_{\pi(x)0}$ is the Kronecker delta. On the other hand, since $e(x)$ is an idempotent associated to x, $\pi(e(x))$ is an idempotent associated to $\pi(x)$ in the field A/p, so $\pi(e(x)) = 1 - \delta_{\pi(x)0}$.

□

A counterexample for $(1 \not\Rightarrow 2)$: Let B be a Boolean ring and

$$I_1 \supset I_2 \supset I_3 \supset \cdots$$

a descending chain of ideals of B such that for any prime ideal p of B there is some n with $p \supset I_n$. Fix also a prime number $q \in \mathbf{N}$ and a tower of finite fields of characteristic q,

$$F_1 \subset F_2 \subset F_3 \subset \cdots$$

and call $|F_i| = q^{n_i}$, $F = \bigcup_1^\infty F_i$. From these data, we will construct a ring $A = F_*^{[I_*]}$ that verifies (1), and that if all ideals I_i are nonzero, it does not verify (2). For instance, we can take $B = \{0,1\}^{\mathbf{N}}$ and

$$I_n = \{x \in B : \{j : x_j = 1\} \ is\ finite,\ x_i = 0\ for\ i < n\}.$$

(If p is a prime ideal of B, either it contains I_1 or it is of the form $p_i = \{x : x_i = 0\}$).

We define A to be the subset of $F^{[B]}$ formed by the convex combinations of elements of F for which the coefficients of elements that are not in F_i are in I_i.

First we check that A is a subring. It clearly contains 0, 1 and -1. Suppose $x, y \in A$. We express x and y as convex combinations of elements of F like $x = \sum_1^n a_i x_i$ and $y = \sum_1^m b_j y_j$ in such a way that if $x_i \notin F_k$ then $a_i \in I_k$ and the same for y. Then $x + y = \sum_{i,j} a_i b_j (x_i + y_j)$. Now, if $x_i + y_j \notin F_k$ then either $x_i \notin F_k$ or $y_j \notin F_k$. In any case, $a_i b_j \in I_k$. So $x + y \in A$. The same reasoning leads to $xy \in A$.

Since $F^{[B]}$ is regular, A is reduced. Also,

$$B \cong B(F^{[B]}) = conv\{0,1\} \subseteq A.$$

We will see that $|A/p|$ is finite for all prime ideals of A. That will prove also that A is regular, by [2, Theorem 1.16]. Take p a prime ideal of A. Then, $p \cap B(A)$ is a prime ideal of $B(A) \cong B$, so $p \cap B(A) \supset I_k$ for some k. Take $x \in A$ and express it as a convex combination of elements of F, $x = \sum_1^n a_i x_i$ where $a_i \in I_k \subset p$ whenever $x_i \notin F_k$. Hence, the class of any x modulo p is a linear combination of classes of elements of F_k with coefficients idempotents of the domain A/p and therefore $|A/p| \leq |F_k| < \infty$.

Finally, supposing all I_i are nonzero, we will prove that for any $k > 0$ there is a prime ideal p of A such that $|F_k| \leq |A/p|$. Since A is reduced, the intersection of all prime ideals of A is zero and there exists a prime ideal p of A such that $0 \neq I_k A \not\subseteq p$. Take $a \in I_k$, $a \notin p$. We claim that the elements $\{ax : x \in F_k\}$ are in A and are all distinct modulo p. First, $ax = ax + (1-a)0$ and provided $x \in F_k$, if $x \notin F_i$ then $i > k$ and $a \in I_k \subset I_i$. This shows the $ax \in A$ whenever $x \in F_k$. Now, suppose ax equals ay modulo p for $x, y \in F_k$, $x \neq y$. Then $a(x-y) \in p$, and since $(x-y)^{-1} \in F_k$, $a(x-y)^{-1} \in A$ and on the other hand, p is an ideal in A, so $a(x-y)a(x-y)^{-1} = a \in p$ and this leads to a contradiction.

A counterexample for $(2 \not\Rightarrow 2.5)$: Let $K = \{0, 1, a, b\}$ be a field with four elements, and A the subring of $K^{\mathbf{N}}$ formed by the sequences in which only a finite number of terms are different from 0 and 1 (This example appears also in [2] for other purposes).

It is plain that A is a reduced ring.

Since $x(x+1)(x^2+x+1) = 0$ for all $x \in K$, the same relation holds for all $x \in A$. If p is a prime ideal of A, then A/p is a domain and this formula implies that $|A/p| \leq 4$. In particular, every A/p is a field, so A is a regular ring, by [2, Theorem 1.16].

Finally, we find a contractive map $f : A \longrightarrow A$ that is not polynomial. Consider $\hat{a} \in K_4^{\mathbf{N}}$ the sequence constant equal to a, and $f : A \longrightarrow A$ given by $f(x) = x(x+1)(x+\hat{a})$. The map f is contractive, since it is the restriction of a polynomial in $K_4^{\mathbf{N}}$. Let us see now that f is not a polynomial map in A. Suppose it is: $f(x) =$

$\sum_{i=0}^{m} c^{(i)} x^i$ for some $c^{(i)} \in A$. Just by definition of A, there exists an index k such that $c_k^{(i)} \in \{0,1\}$ for all i. Then, for all $x \in A$, we have $x_k(x_k+1)(x_k+a) = f(x)_k = \sum_{i=0}^{m} c_k^{(i)} x_k^i$ with $c_k^{(i)} \in \{0,1\} = \mathbf{Z}_2$. This is a contradiction because the map $h : K_4 \longrightarrow K_4$ given by $h(t) = t(t+1)(t+a)$ cannot be given by a polynomial with coefficients in $\mathbf{Z}_2$ since $h(a) = 0$ but $h(b) \neq 0$.

One natural question is up to which point the strong structure theorem we have obtained for CFG-rings can be generalized to larger classes of reduced rings with finite quotient fields. We finish by making a remark and posing a problem in that direction.

Proposition 12 *Let A be a reduced ring with all quotient fields finite. Then, $char(A) = q_1 \cdots q_n \neq 0$ with $q_1, \ldots, a_n$ different prime numbers and there is a canonical decomposition $A \cong A_1 \times \cdots \times A_n$ with $char(A_i) = q_i$.*

Proof: For each prime number $q > 0$, since A is a regular ring, $V(q) = \{p \in Spec(A) : q \in p\}$ is a closed open set. Since all quotient fields are finite, and hence have prime characteristic, the set of all $V(q)$ is an open covering of $Spec(A)$. By compactness $Spec(A) = V(q_1) \cup \cdots \cup V(q_m)$. Therefore $q_1 \cdots q_m$ is in all prime ideals of A and $char(A)$ divides $q_1 \cdots q_m$.

Since $V(q) \cap V(q') = \emptyset$ for different prime numbers q, q', also $e(q) \vee e(q') = 1 \in A$, and $(1 - e(q))(1 - e(q')) = 0$. On the other hand, since $Spec(A) = V(q_1) \cup \cdots \cup V(q_m)$,

$$e(q_1) \cdots e(q_n) = 0 \quad \text{and} \quad \bigvee_{1}^{m} (1 - e(q_i)) = 1.$$

All this gives

$$A = (1 - e(q_1))A \oplus \cdots \oplus (1 - e(q_n))A$$

what induces the desired decomposition. □

Problem 1 *Are all reduced rings with finite quotient fields of prime characteristic isomorphic to rings of type $F_*^{[I_*]}$ as constructed in the* *counterexample for* $(1 \not\Rightarrow 2)$*?*

If the answer were positive, there would be a complete structure theorem, not only for CFG-rings but for reduced rings verifying condition (2).

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