THE UNIT BALL OF THE HILBERT SPACE IN ITS WEAK TOPOLOGY

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Abstract. We show that the unit ball of $\ell_p(\Gamma)$ in its weak topology is a continuous image of $\sigma_1(\Gamma)^{\mathbb{N}}$ and we deduce some combinatorial properties of its lattice of open sets which are not shared by the balls of other equivalent norms when Γ is uncountable.

For a set Γ and a real number $1 < p < \infty$, the Banach space $\ell_p(\Gamma)$ is a reflexive space, hence its unit ball is compact in the weak topology and in fact, it is homeomorphic to the following closed subset of the Tychonoff cube $[-1,1]^\Gamma$:

$$B(\Gamma) = \left\{ x \in [-1,1]^\Gamma : \sum_{\gamma\in\Gamma} |x_\gamma| \leq 1 \right\}.$$

The homeomorphism $h : B_{\ell_p(\Gamma)} \longrightarrow B(\Gamma)$ is given by $h(x)_\gamma = sign(x_\gamma) \cdot |x_\gamma|^p$. The spaces homeomorphic to closed subsets of some $B(\Gamma)$ constitute the class of uniform Eberlein compacta, introduced by Benyamini and Starbird [6]. The space $\sigma_k(\Gamma)$, the compact subset of $\{0,1\}^\Gamma$ which consists of the functions with at most k nonzero coordinates (k a positive integer) is an example of a uniform Eberlein compact. In fact, the following result was proven in [5]:

Theorem 1 (Benyamini, Rudin, Wage)**.** *Every uniform Eberlein compact of weight κ is a continuous image of a closed subset of $\sigma_1(\kappa)^{\mathbb{N}}$.*

In the same paper [5], it was posed the problem whether in fact, it was possible to get any uniform Eberlein compact as a continuous image of the full $\sigma_1(\Gamma)^{\mathbb{N}}$. This question was answered in the negative by Bell [2], by considering the following property:

A compact space K verifies property (Q) if for every uncountable regular cardinal λ and every family $\{U_\alpha, V_\alpha\}_{\alpha<\lambda}$ of open subsets of K with $\overline{U_\alpha} \subset V_\alpha$ one of the following two alternatives must hold:

(1) either there exists a set $A \subset \lambda$ with $|A| = \lambda$ such that $U_\alpha \cap U_\beta = \emptyset$ for every two different elements α and β in A,

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(2) or either there exists a set $A \subset \lambda$ with $|A| = \lambda$ such that $V_\alpha \cap V_\beta \neq \emptyset$ for every two different elements α and β in A.

Bell proved in [2] that property (Q) is satisfied by all polyadic spaces, that is, continuous images of $\sigma_1(\Gamma)^\Lambda$ for any sets Γ and Λ, (this concept was introduced in [8] and studied earlier by Gerlits [7]), but he constructed a uniform Eberlein compact without property (Q). Later, Bell [4] provided another example of a uniform Eberlein compact which is not a continuous image of any $\sigma_1(\Gamma)^\mathbb{N}$ but which is nevertheless polyadic. Our main result is the following:

Theorem 2. *$B(\Gamma)$ is a continuous image of $\sigma_1(\Gamma)^\mathbb{N}$.*

As a consequence, $B(\Gamma)$ satisfies property (Q) as well as other properties of the same type introduced by Bell in [4] and [3]. However, if Γ is uncountable, we show in Theorem 4 that a modification of one of the examples of Bell provides an equivalent norm on $\ell_p(\Gamma)$ whose unit ball is not a continuous image of $\sigma_1(\Gamma)^\mathbb{N}$, indeed not satisfying property (Q). In particular, we are showing the existence of equivalent norms in the nonseparable $\ell_p(\Gamma)$ whose closed unit balls are not homeomorphic in the weak topology. This contrasts with the separable case, since the balls of all separable reflexive Banach spaces are weakly homeomorphic [1, Theorem 1.1]. We refer to [1] for information about the problem whether the balls of equivalent norms in a Banach space are weakly homeomorphic in the separable case.

Proof of Theorem 2: For a set Δ we will use the notation $B^+(\Delta) = B(\Delta) \cap [0,1]^\Delta$. First, we point out that $B(\Gamma)$ is a continuous image of $B^+(\Gamma)$. Indeed, if we consider $\Gamma^o = \Gamma \times \{a,b\}$, we have a continuous surjection $\psi : B^+(\Gamma^o) \longrightarrow B(\Gamma)$ given by $\psi(x)_\gamma = x_{(\gamma,a)} - x_{(\gamma,b)}$.

In a second step, we apply the standard procedure to express the space $B^+(\Gamma)$ as a continuous image of a totally disconnected compact L_0. We fix a sequence $(r_n)_{n=0}^\infty$ of positive real numbers such that $\sum_{n=0}^\infty r_n = 1$ and such that the continuous map $\phi : \{0,1\}^\mathbb{N} \longrightarrow [0,1]$ given by $\phi(x) = \sum_{n=0}^\infty r_n x_n$ is surjective, for example $r_n = \frac{1}{2^{n+1}}$. We consider the power $\phi^\Gamma : \{0,1\}^{\Gamma\times\mathbb{N}} \longrightarrow [0,1]^\Gamma$ and then we set:

$$\begin{aligned} L_0 &= (\phi^\Gamma)^{-1}(B^+(\Gamma)), \\ f &= \phi^\Gamma|_{L_0}, \end{aligned}$$

so that $f : L_0 \longrightarrow B^+(\Gamma)$ is a continuous surjection. It will be convenient to have an explicit description of L_0. For $x \in \{0,1\}^{\Gamma\times\mathbb{N}}$ and $n \in \mathbb{N}$, we define $N_n(x) = |\{\gamma \in \Gamma : x_{(\gamma,n)} = 1\}|$.

$$\begin{aligned}
x \in L_0 &\iff \phi^\Gamma(x) \in B^+(\Gamma)\\
&\iff \sum_{\gamma\in\Gamma} \phi^\Gamma(x)_\gamma \leq 1\\
&\iff \sum_{\gamma\in\Gamma}\sum_{n=0}^{\infty} r_n x_{(\gamma,n)} \leq 1\\
&\iff \sum_{n=0}^{\infty} r_n N_n(x) \leq 1.
\end{aligned}$$

The compact space L_0 can be alternatively described as follows. Let Z be a compact subset of $\mathbb{N}^\mathbb{N}$ such that if $\sigma \in Z$ and $\tau_n \leq \sigma_n$ for all $n \in \mathbb{N}$, then $\tau \in Z$. Associated to such a set Z we construct the following space:

$$\mathcal{K}(Z,\Gamma) = \{x \in \{0,1\}^{\Gamma\times\mathbb{N}} : (N_n(x))_{n\in\mathbb{N}} \in Z\}.$$

We have that $L_0 = \mathcal{K}(Z_0,\Gamma)$ where $Z_0 = \left\{s \in \mathbb{N}^\mathbb{N} : \sum_{i\in\mathbb{N}} r_i s_i \leq 1\right\}$. Note that Z_0 is indeed compact since it is a closed subset of $\prod_{n\in\mathbb{N}}\{0,\ldots,M_n\}$ where M_n is the integer part of $\frac{1}{r_n}$. The proof will be complete after the following lemma:

Lemma 3. *Let Z be a compact subset of $\mathbb{N}^\mathbb{N}$ such that if $\sigma \in Z$ and $\tau_n \leq \sigma_n$ for all $n \in \mathbb{N}$, then $\tau \in Z$. Then $\mathcal{K}(Z,\Gamma)$ is a continuous image of $\sigma_1(\Gamma)^\mathbb{N}$.*

PROOF: First we check that $\mathcal{K}(Z,\Gamma)$ is a closed subset of $\{0,1\}^{\Gamma\times\mathbb{N}}$ and hence compact. Namely, if $x \in \{0,1\}^{\Gamma\times\mathbb{N}} \setminus \mathcal{K}(Z,\Gamma)$, then $(N_n(x))_{n\in\mathbb{N}} \notin Z$ and since Z is closed in $\mathbb{N}^\mathbb{N}$, there is a finite set $F \subset \mathbb{N}$ such that $\sigma \notin Z$ whenever $\sigma_n = N_n(x)$ for all $n \in F$. Indeed, by the definition of Z, if $\tau \in \mathbb{N}^\mathbb{N}$ and $\tau_n \geq \sigma_n$ of all $n \in F$, also $\tau \notin Z$. In this case,

$$W = \{y \in \{0,1\}^{\Gamma\times\mathbb{N}} : y_{\gamma,n} = 1 \text{ whenever } n \in F \text{ and } x_{\gamma,n} = 1\}$$

is a neighborhood which separates x from $\mathcal{K}(Z,\Gamma)$ and this finishes the proof that $\mathcal{K}(Z,\Gamma)$ is closed. Since Z is compact, for every $n \in \mathbb{N}$ there exists $M_n \in \mathbb{N}$ such that $\sigma_n \leq M_n$ for all $\sigma \in Z$. We define the following compact space:

$$L_1 = Z \times \prod_{m\in\mathbb{N}}\prod_{i=0}^{M_m} \sigma_i(\Gamma)$$

Note that L_1 is a continuous image of $\sigma_1(\Gamma)^\mathbb{N}$. On the one hand, since Z is a metrizable compact, it is a continuous image of $\{0,1\}^\mathbb{N}$ and in particular of $\sigma_1(\Gamma)^\mathbb{N}$. On the other hand, for any $i \in \mathbb{N}$, the space $\sigma_i(\Gamma)$ can be viewed as the family of all subsets of Γ of cardinality at most i. In this way, we consider the continuous surjection $p : \sigma_1(\Gamma)^i \longrightarrow \sigma_i(\Gamma)$ given by $p(x_1,\ldots,x_i) = x_1 \cup \cdots \cup x_i$. From the existence of such a function follows the fact that any countable product of spaces $\sigma_i(\Gamma)$ is a continuous image of $\sigma_1(\Gamma)^\mathbb{N}$, and in particular, the second factor in the expression of L_1 is such an image.

It remains to define a continuous surjection $g : L_1 \longrightarrow \mathcal{K}(Z,\Gamma)$. We first fix some notation. An element of L_1 will be written as (z,x) where $z \in Z$ and $x \in$

$\prod_{m\in\mathbb{N}}\prod_{i=0}^{M_m}\sigma_i(\Gamma)$. At the same time, such an x is of the form $(x^m)_{m\in\mathbb{N}}$ with $x^m \in \prod_{i=0}^{M_m}\sigma_i(\Gamma)$ and again each x^m is $(x^{m,i})_{i=1}^{M_m}$ where $x^{m,i} \in \sigma_i(\Gamma)$. Finally $x^{m,i} = (x^{m,i}_\gamma)_{\gamma\in\Gamma} \in \sigma_i(\Gamma) \subset \{0,1\}^\Gamma$. The function $g : L_1 \longrightarrow \mathcal{K}(Z,\Gamma) \subset \{0,1\}^{\Gamma\times\mathbb{N}}$ is defined as follows:

$$g(z,x)_{\gamma,m} = x^{m,z(m)}_\gamma$$

Observe that $g(x,z)$ maps indeed L_1 onto $\mathcal{K}(Z,\Gamma)$ because for every m, $(x^{m,z(m)}_\gamma)_{\gamma\in\Gamma}$ is an arbitrary element of $\sigma_{z(m)}(\Gamma)$. $\square$

Theorem 4. *Let Γ be an uncountable set and $1 < p < \infty$. There exists an equivalent norm on $\ell_p(\Gamma)$ whose unit ball does not satisfy property (Q) and hence it is not polyadic.*

PROOF: This is a variation of an example of Bell [2], originally a scattered compact, so that to make it absolutely convex. We will consider ω_1 as a subset of Γ. Let $\phi : \omega_1 \longrightarrow \mathbb{R}$ be a one-to-one map and

$$G = \{(\alpha,\beta) \in \omega_1\times\omega_1 : \phi(\alpha) < \phi(\beta) \iff \alpha \preceq \beta\}.$$

We define an equivalent norm on $\ell_p(\Gamma)\times\ell_p(\Gamma) \sim \ell_p(\Gamma)$ by

$$\|(x,y)\|' = \sup\{\|x\|_p, \|y\|_p, |x_\alpha| + |y_\beta| : (\alpha,\beta) \in G\}.$$

and let K be its unit ball considered in its weak topology. Fix numbers $1 < \xi_1 < \xi_2 < 2^{1-\frac{1}{p}}$. The families of open sets

$$U_\alpha = \{(x,y) \in K : |x_\alpha| + |y_\alpha| > \xi_2\}, \quad \alpha < \omega_1$$

$$V_\alpha = \{(x,y) \in K : |x_\alpha| + |y_\alpha| > \xi_1\}, \quad \alpha < \omega_1$$

verify that $\overline{U_\alpha} \subset V_\alpha$ and that for any $\alpha,\beta < \omega_1$, $U_\alpha \cap U_\beta = \emptyset$ if and only if $(\alpha,\beta) \in G$ if and only if $V_\alpha \cap V_\beta = \emptyset$. Namely, if there is some $(x,y) \in V_\alpha \cap V_\beta$, then

$$|x_\alpha| + |y_\alpha| + |x_\beta| + |y_\beta| > \xi_1 + \xi_1 > 2$$

and therefore either $|x_\alpha|+|x_\beta| > 1$ or $|y_\alpha|+|y_\beta| > 1$ and this implies that $(\alpha,\beta) \notin G$ since $(x,y) \in K$. On the other hand, if $(\alpha,\beta) \notin G$ then the element $(x,y) \in \ell_p(\Gamma)\times\ell_p(\Gamma)$ which has all coordinates zero except $x_\alpha = x_\beta = y_\alpha = y_\beta = 2^{-\frac{1}{p}}$ lies in $U_\alpha \cap U_\beta$. Since there is no uncountable well ordered (or inversely well ordered) subset of $\mathbb{R}$ there is no uncountable subset A of ω_1 such that $A\times A \subset G$ or $(A\times A)\cap G = \emptyset$. Therefore, the families $\{U_\alpha\}$ and $\{V_\alpha\}$ witness the fact that K does not have property (Q) and hence, it is not polyadic. $\square$

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