COUNTABLE PRODUCTS OF SPACES OF FINITE SETS

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Abstract. We consider the compact spaces $\sigma_n(\Gamma)$ of subsets of Γ of cardinality at most n and their countable products. We give a complete classification of their Banach spaces of continuous functions and a partial topological classification.

For an infinite set Γ and a natural number n, we consider the space

$$\sigma_n(\Gamma) = \{x \in \{0,1\}^\Gamma : |supp(x)| \leq n\}.$$

Here $supp(x) = \{\gamma \in \Gamma : x_\gamma \neq 0\}$. This is a closed, hence compact subset of $\{0,1\}^\Gamma$, which is identified with the family of all subsets of Γ of cardinality at most n. In this work we will study the spaces which are countable products of spaces $\sigma_n(\Gamma)$, mainly their topological classification as well as the classification of their Banach spaces of continuous functions.

Let T be the set of all sequences $(\tau_n)_{n=1}^\infty$ with $0 \leq \tau_n \leq \omega$. When τ runs over T, $\sigma_\tau(\Gamma) = \prod_1^\infty \sigma_n(\Gamma)^{\tau_n}$ runs over all finite and countable products of spaces $\sigma_k(\Gamma)$. For $\tau \in T$ we will call $j(\tau)$ to the supremum of all n with $\tau_n > 0$ and $i(\tau)$ to the supremum of all n with $\tau_n = \omega$. If $\tau_n < \omega$ for all $n \geq 1$, then $i(\tau) = 0$. Always $0 \leq i(\tau) \leq j(\tau) \leq \omega$. Theorem 1 below summarizes our knowledge about the topological classification and its proof consists of a number of lemmas along Section 2.

Theorem 1. *Let $\tau, \tau' \in T$ and Γ an uncountable set.*

1. *Suppose $j(\tau) < \omega$. In this case, $\sigma_{\tau'}(\Gamma)$ is homeomorphic to $\sigma_\tau(\Gamma)$ if and only if $i(\tau) = i(\tau')$ and $\tau_n = \tau'_n$ for all $n > i(\tau)$.*
2. *Suppose $i(\tau) = \omega$. In this case, if $i(\tau') = \omega$, then $\sigma_\tau(\Gamma)$ is homeomorphic to $\sigma_{\tau'}(\Gamma)$.*

This is not a complete classification and leaves the following question open:

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Problem 1. Let Γ be an uncountable set and $\tau, \tau' \in T$ such that $j(\tau') = j(\tau) = \omega$, $i(\tau) < \omega$ and there exists some $n \geq i(\tau)$ with $\tau_n \neq \tau'_n$. Is $\sigma_\tau(\Gamma)$ homeomorphic to $\sigma_{\tau'}(\Gamma)$?

For example, one particular instance of the problem is whether $\prod_{i=1}^\infty \sigma_i(\Gamma)$ is homeomorphic to $\prod_{i=2}^\infty \sigma_i(\Gamma)$.

About the spaces of continuous functions, it has been recently proved by Marciszewski [6] that a Banach space $C(K)$ is isomorphic to $c_0(\Gamma)$ if and only if $K \subset \sigma_n(\Gamma)$ for some $n < \omega$. This is the case of any compact of the form $K = \prod_{i=1}^n \sigma_{k_i}(\Gamma)$ which can be embedded into $\sigma_{\sum k_i}(\bigcup_1^n \Gamma \times \{i\})$ by $x \mapsto \bigcup_1^n x_i \times \{i\}$. Hence, it is a consequence of Marciszewski's result that the Banach spaces of continuous functions over finite products of spaces $\sigma_k(\Gamma)$ over a fixed Γ are all isomorphic. In Section 1 we prove a similar result for countable products:

Theorem 2. *Let Γ be an infinite set and (k_n) be any sequence of positive integers. Then the Banach spaces $C(\prod_{n<\omega} \sigma_{k_n}(\Gamma))$ and $C(\sigma_1(\Gamma)^\omega)$ are isomorphic.*

The techniques that we will use are based on the use of regular averaging operators and the so called Pełczyński's decomposition method, developed in [8] and [9] in order to achieve Miljutin's result that the spaces of continuous functions over uncountable metrizable compacta are all isomorphic.

Definition 3. Let $\phi : L \longrightarrow K$ be a continuous surjection between compact spaces. A regular averaging operator for ϕ is a bounded positive linear operator $T : C(L) \longrightarrow C(K)$ with $T(1_L) = 1_K$ and $T(x \circ \phi) = x$ for all $x \in C(K)$.

The countable products of spaces of the form $\sigma_n(\Gamma)$ are uniform Eberlein compact spaces, cf. [3]. This class consists of the weakly compact subsets of the Hilbert spaces, or equivalently of the compact subsets of the space

$$B(\Gamma) = \{x \in [-1,1]^\Gamma : \sum_{\gamma \in \Gamma} |x_\gamma| \leq 1\} \sim (B_{\ell_2(\Gamma)}, w)$$

for some set Γ. Indeed, $\sigma_n(\Gamma)$ is homeomorphic to $B(\Gamma) \cap \{0, \frac{1}{n}\}^\Gamma$. We establish the following result:

Theorem 4. *Let K be a uniform Eberlein compact of weight κ. Then there is a closed subspace L of $\sigma_1(\kappa)^\mathbb{N}$ and an onto continuous map $f : L \longrightarrow K$ which admits a regular averaging operator.*

This improves a result of Argyros and Arvanitakis [1] that for every uniform Eberlein compact space K there is a totally disconnected uniform Eberlein compact space L of the same weight and a continuous surjection

$f : L \longrightarrow K$ which admits a regular averaging operator, and also a result of Benyamini, Rudin and Wage [2] that every uniform Eberlein compact of weight κ is a continuous image of a closed subset of $\sigma_1(\kappa)^{\mathbb{N}}$. We note that there are many totally disconnected uniform Eberlein compact spaces which cannot be embedded into $\sigma_1(\kappa)^{\mathbb{N}}$, cf. Lemma 12 below.

Notations

All topological spaces will be assumed to be completely regular. By identifying elements of $\{0,1\}^\Gamma$ with subsets of Γ, the space $\sigma_n(\Gamma) \subset \{0,1\}^\Gamma$ can be viewed as the family of all subsets of Γ of cardinality less than or equal to n, endowed with the topology which has a base the sets of the form

$$\Phi_F^G = \{y \in \sigma_n(\Gamma) : F \subset y \subset \Gamma \setminus G\}$$

for F and G finite subsets of Γ. We will denote by $p : \sigma_1(\Gamma)^k \longrightarrow \sigma_k(\Gamma)$ the continuous surjection given by $p(x_1, \ldots, x_k) = x_1 \cup \cdots \cup x_k$. Note that from the existence of such a function follows the fact that any countable product $\prod_{i<\omega} \sigma_{k_i}(\Gamma)$ is a continuous image of $\sigma_1(\Gamma)^\omega$. We will also denote $B^+(\Gamma) = B(\Gamma) \cap [0,1]^\Gamma$.

1. Banach space classification

The following Theorem 5 is the key result of this section. A somewhat similar fact can be found in [10], namely that the natural surjection $K^2 \longrightarrow exp_2(K) = \{\{x,y\} : x,y \in K\}$ given by $(x,y) \mapsto \{x,y\}$ has a regular averaging operator.

Theorem 5. *The map $p : \sigma_1(\Gamma)^k \longrightarrow \sigma_k(\Gamma)$ admits a regular averaging operator.*

Proof: For every $y \in \sigma_k(\Gamma)$ let us denote by $L(y)$ the subset of $p^{-1}(y)$ consisting of all $(x^1, \ldots, x^k) \in p^{-1}(y)$ such that $x^i \cap x^j = \emptyset$ for $i \neq j$ (that is, $L(y)$ consists of those tuples of $p^{-1}(y)$ in which no singleton appears twice).

The regular averaging operator $T : C(\sigma_1(\Gamma)^k) \longrightarrow C(\sigma_k(\Gamma))$ is defined as follows:

$$T(f)(y) = \frac{1}{|L(y)|} \sum_{x \in L(y)} f(x)$$

The only difficult point is in proving that $T(f)$ is a continuous function whenever f is continuous. So fix $f \in C(\sigma_1(\Gamma)^k)$ and a point $y \in \sigma_k(\Gamma)$ and $\varepsilon > 0$. For each $x = (x_1, \ldots, x_k) \in L(y)$, since f is continuous at x, there is a neighborhood U_x of x in $\sigma_1(\Gamma)^k$ in which $\sup_{x' \in U_x} |f(x) - f(x')| < \varepsilon$. The set U_x must contain a basic neighborhood of x of the form

$$\Phi_{x_1}^{G_1^x}\times\cdots\times\Phi_{x_k}^{G_k^x}\subset U_x$$

where G_i^x is a finite set of Γ disjoint with x_i. We define a neighborhood of y as

$$V=\Phi_y^{\bigcup_{x\in L(y)}\bigcup_{i=1}^k G_i^x\setminus y}$$

and we shall see that $|T(f)(y)-T(f)(y')|<\varepsilon$ for every $y'\in V$. So we fix $y'\in V$ (in particular $y\subset y'$). First, we define an onto map $r:L(y')\longrightarrow L(y)$ in the following way, if $(x_1,\ldots,x_k)\in L(y')$ then $r(x)=(r(x)_1,\ldots,r(x)_k)$ where $r(x)_i=x_i\cap y$. It is straightforward to check that all the fibers of r have the same cardinality, call $n=|r^{-1}(x)|$, so that $|L(y')|=n|L(y)|$. The key fact (used in the final inequality in the expression below) is that if $x\in L(y)$ and $x'\in r^{-1}(x)$, then $x'\in U_x$. To see this, take $x=(x_1,\ldots,x_k)\in L(y)$ and $x'=(x_1',\ldots,x_k')\in r^{-1}(x)$. We check that $x_i'\in\Phi_{x_i}^{G_i^x}$. If $x_i'\subset y$ then $x_i'=x_i$. If $x_i'=\{\gamma\}\subset y'\setminus y$ then $x_i=\emptyset$ and since $y'\in V$, $\gamma\not\in G_i^x$ and again $x_i'\in\Phi_{x_i}^{G_i^x}$. Finally,

$$\begin{aligned}
|T(f)(y')-T(f)(y)| &= \left|\frac{1}{|L(y')|}\sum_{x'\in L(y')}f(x')-\frac{1}{|L(y)|}\sum_{x\in L(y)}f(x)\right|\\
&= \left|\frac{1}{|L(y')|}\sum_{x\in L(y)}\sum_{x'\in r^{-1}(x)}f(x')-\frac{1}{|L(y)|}\sum_{x\in L(y)}f(x)\right|\\
&= \left|\frac{1}{n|L(y)|}\sum_{x\in L(y)}\sum_{x'\in r^{-1}(x)}f(x')-\frac{1}{|L(y)|}\sum_{x\in L(y)}f(x)\right|\\
&= \left|\frac{1}{|L(y)|}\sum_{x\in L(y)}\left(\left(\frac{1}{n}\sum_{x'\in r^{-1}(x)}f(x')\right)-f(x)\right)\right|\\
&= \left|\frac{1}{|L(y)|}\sum_{x\in L(y)}\left(\frac{1}{n}\sum_{x'\in r^{-1}(x)}(f(x')-f(x))\right)\right|\\
&\leq \frac{1}{|L(y)|}\sum_{x\in L(y)}\left(\frac{1}{n}\sum_{x'\in r^{-1}(x)}|f(x')-f(x)|\right)\\
&< \frac{1}{|L(y)|}\sum_{x\in L(y)}\left(\frac{1}{n}\sum_{x'\in r^{-1}(x)}\varepsilon\right)=\varepsilon
\end{aligned}$$

□

Lemma 6. (a) *Let* $g : L \longrightarrow K$ *be a continuous surjection between compact spaces which admits a regular averaging operator and let* M *be a closed subset of* K*. Then the restriction* $g : g^{-1}(M) \longrightarrow M$ *also admits a regular averaging operator* [1, Proposition 18].

(b) *Let* $\{g_i : L_i \longrightarrow K_i\}$ *be a family of continuous surjections between compact spaces which admit regular averaging operators. Then the product map* $\prod g_i : \prod L_i \longrightarrow \prod K_i$ *admits a regular averaging operator too* [9, Proposition 4.7].

Proof of Theorem 4: We make the observation that the space $B(\Gamma)$ can be embedded into $B^+(\Gamma \times \{a, b\}) \sim B^+(\Gamma)$ by the map $u(x)_{\gamma,a} = \max(0, x_\gamma)$ and $u(x)_{\gamma,b} = \max(0, -x_\gamma)$. This observation allows to consider our K as a subset of $B^+(\Gamma)$ with $|\Gamma| = \kappa$. Let $\phi : \{0,1\}^\omega \longrightarrow [0,1]$ given by $\phi(x) = \sum r_i x_i$ where $r_i = \frac{1}{3}\left(\frac{2}{3}\right)^i$. It is proven in [1] that ϕ admits a regular averaging operator and hence by Lemma 6 also $\phi^\Gamma : \{0,1\}^{\omega\times\Gamma} \longrightarrow [0,1]^\Gamma$ and its restriction $\phi^\Gamma : L' = (\phi^\Gamma)^{-1}(K) \longrightarrow K$ admit a regular averaging operator. The space L' is a subspace of $L_0 = (\phi^\Gamma)^{-1}(B^+(\Gamma))$ for which we can give the following description:

$$\begin{aligned} x \in L_0 &\iff \phi^\Gamma(x) \in B^+(\Gamma) \\ &\iff \sum_{\gamma\in\Gamma} \phi^\Gamma(x)_\gamma \le 1 \\ &\iff \sum_{\gamma\in\Gamma}\sum_{n=0}^{\infty} r_n x_{(\gamma,n)} \le 1 \\ &\iff \sum_{n=0}^{\infty} r_n N_n(x) \le 1, \end{aligned}$$

where $N_n(x)$ is the cardinality of $supp(x|_{\Gamma\times\{n\}})$. From this description, if M_n denotes the integer part of r_n^{-1}, then $L' \subset L_0 \subset \prod_{n=1}^\infty \sigma_{M_n}(\Gamma)$. From Theorem 5 and part (b) of Lemma 6 follows the existence of a continuous surjection $g : \sigma_1(\Gamma)^\omega \longrightarrow \prod_{n=1}^\infty \sigma_{M_n}(\Gamma)$ which admits a regular averaging operator. Making use of part (a) of Lemma 6 we get a surjection $g : L = g^{-1}(L') \longrightarrow L'$ with regular averaging operator and the composition $L \longrightarrow L' \longrightarrow K$ is the desired map. □

We shall need now the so called Pełczyński's decomposition method, which is used to establish the existence of isomorphisms between Banach spaces. For Banach spaces X and Y we shall write $X|Y$ if there exists a Banach space Z such that $X \oplus Z$ is isomorphic to Y, shortly $X \oplus Z \sim Y$. Also,

$Y = (X_1 \oplus X_2 \oplus \cdots)_{c_0}$ denotes the c_0-sum of the Banach spaces $X_1, X_2, \ldots$,

$$Y = \{y = (x_n) \in \prod X_n : \lim \|x_n\| = 0\}, \quad \|y\| = \sup_n \|x_n\|.$$

Theorem 7 (cf. [9], §8)**.** *Let X and Y be Banach spaces such that $X|Y$, $Y|X$ and $(X \oplus X \oplus \cdots)_{c_0} \sim X$, then $X \sim Y$.*

If there exists a surjection $\phi : L \longrightarrow K$ with regular averaging operator, then $C(K)|C(L)$, cf. [9]. In particular if $L \subset K$ is a retract of K, since in this case the restriction operator is a regular averaging operator for the retraction. On the other hand, in order to guarantee the last hypothesis in Theorem 7 we shall use the criterion of Lemma 8. For topological spaces K_n, $K_1 \oplus K_2 \oplus \cdots$ denotes the discrete topological sum, while $\alpha(S)$ is the one point compactification of a locally compact space S.

Lemma 8. *Let K be a compact space which is homeomorphic to $\alpha(K \oplus K \oplus \cdots)$. Then $(C(K) \oplus C(K) \oplus \cdots)_{c_0} \sim C(K)$.*

Proof: We apply Theorem 7 to $X = (C(K) \oplus C(K) \oplus \cdots)_{c_0}$ and $Y = C(K)$. The only point is in checking that $X|Y$. Let ∞ denote the infinity point of $\alpha(K \oplus K \oplus \cdots) \sim K$. Then $X \sim Y' = \{f \in C(K) : f(\infty) = 0\}$ and $Y \sim Y' \oplus \mathbb{R}$. $\square$

Proof of Theorem 2: Set $K = \sigma_1(\Gamma)^\omega$ and $L = \prod \sigma_{k_n}(\Gamma)$. We apply Theorem 7 to $X = C(K)$ and $Y = C(L)$. First, we already observed that from Theorem 5 and Lemma 6(b) follows the existence of a surjection with regular averaging operator $f : K \longrightarrow L$ and hence $C(L)|C(K)$. On the other hand, K is a retract of L because for any k, $\sigma_1(\Gamma)$ is homeomorphic to a clopen subset of $\sigma_k(\Gamma)$: the family of all subsets which contain fixed elements $\gamma_1, \ldots, \gamma_{k-1}$. Therefore $C(K)|C(L)$. By Lemma 8, it only remains to show that $\alpha(K \oplus K \oplus \cdots) \sim K$. For this, fix $\gamma \in \Gamma$ and set for $n = 1, 2, \ldots$

$$K_n = \{x \in K = \sigma_1(\Gamma)^\omega : \gamma \in x_1 \cap \cdots \cap x_{n-1} \setminus x_n\}.$$

The sets K_n are disjoint clopen sets homeomorphic to K and K is the one point compactification of their union with point of infinity $(\{\gamma\}, \{\gamma\}, \ldots)$. $\square$

2. Topological classification

This section is devoted to the proof of Theorem 1. Before entering this, we point out why we assume Γ to be uncountable. The reasonings below do not apply in the countable case and the situation is indeed completely different. All perfect totally disconnected metrizable compact spaces are homeomorphic [5, Theorem 7.4] and this implies that all countable products of spaces $\sigma_k(\omega)$ are homeomorphic. The finite products are countable compacta, whose topological classification is also well known after the classical paper [7]: two of them are homeomorphic if and only if they have same Cantor-Bendixson derivation index and the same cardinality of the last nonempty Cantor-Bendixson derivative. Straightforward computations give that these two invariants for a finite product $\prod_{i=1}^n \sigma_{k_i}(\omega)$ take the values $1+\sum_1^n k_i$ and 1 respectively. From now on, Γ will be always an uncountable set.

Lemma 9. *If $m < n$ then $\sigma_m(\Gamma) \times \sigma_n(\Gamma)^\omega$ is homeomorphic to $\sigma_n(\Gamma)^\omega$.*

Proof: We denote again by $(X_1 \oplus X_2 \oplus \cdots)$ the discrete topological sum of the spaces $X_1, X_2, \ldots$ and by αX the one-point compactification of the locally compact space X. Fix $\gamma_0, \ldots, \gamma_{n-1} \in \Gamma$. We consider the set $L = \omega \times \{0, \ldots, n-1\}$ endowed with the lexicographical order: $(k,i) < (k',i')$ whenever either $k < k'$ or $k = k'$ and $i < i'$. For every $(k,i) \in L$ we define a clopen set of $\sigma_n(\Gamma)^\omega$ as

$$\begin{aligned} A_{(k,i)} &= \{x \in \sigma_n(\Gamma)^\omega : \gamma_i \notin x_k, \gamma_{i'} \in x_{k'} \forall (k',i') < (k,i)\} \\ &= \{x \in \sigma_n(\Gamma)^\omega : \gamma_i \notin x_k \supset \{\gamma_0, \ldots, \gamma_{i-1}\}, x_j = \{\gamma_0, \ldots, \gamma_{n-1}\} \forall j < k\}. \end{aligned}$$

Notice that $A_{(k,i)}$ is homeomorphic to $\sigma_{n-i}(\Gamma) \times \sigma_n(\Gamma)^\omega$ and that $\{A_l : l \in L\}$ constitutes a disjoint sequence of clopen subsets of $\sigma_n(\Gamma)^\omega$ with only limit point the sequence $\xi \in \sigma_n(\Gamma)^\omega$ constantly equal to $\{\gamma_0, \ldots, \gamma_{n-1}\}$. Hence,

$$\sigma_n(\Gamma)^\omega \approx \alpha \left(\bigoplus_{l \in L} A_l \right) \approx \alpha \left(\bigoplus_{i=0}^{n-1} \bigoplus_{j<\omega} (\sigma_{n-i}(\Gamma) \times \sigma_n(\Gamma)^\omega) \right).$$

On the other hand, we can perform a similar decomposition in $\sigma_m(\Gamma) \times \sigma_n(\Gamma)^\omega$ defining, for $j < m$ and $(k,i) \in L$:

$$\begin{aligned} B'_j &= \{(y,x) \in \sigma_m(\Gamma) \times \sigma_n(\Gamma)^\omega : \gamma_j \notin y, \{\gamma_0, \ldots, \gamma_{j-1}\} \subset y\} \\ B_{(k,i)} &= \{(y,x) \in \sigma_m(\Gamma) \times \sigma_n(\Gamma)^\omega : \gamma_i \notin x_k, \gamma_{i'} \in x_{k'} \forall (k',i') < (k,i), \\ &\quad \{\gamma_0, \ldots, \gamma_{m-1}\} \subset y\} \end{aligned}$$

Again B'_j is homeomorphic to $\sigma_{m-j}(\Gamma) \times \sigma_n(\Gamma)^\omega$, $B_{(k,i)}$ is homeomorphic to $\sigma_{n-i}(\Gamma) \times \sigma_n(\Gamma)^\omega$ and altogether they constitute a disjoint sequence of clopen sets with a single limit point $(\{\gamma_0, \ldots, \gamma_{m-1}\}, \xi)$ out of them, so

$$\sigma_m(\Gamma) \times \sigma_n(\Gamma)^\omega \approx \alpha \left(\bigoplus_{l \in L} B_l \oplus \bigoplus_{j=0}^{m-1} B'_j \right) \approx \alpha \left(\bigoplus_{i=0}^{n-1} \bigoplus_{j<\omega} (\sigma_{n-i}(\Gamma) \times \sigma_n(\Gamma)^\omega) \right).$$

Lemma 10. *If $m < n < \omega$ then $\sigma_m(\Gamma)^\omega \times \sigma_n(\Gamma)^\omega$ is homeomorphic to $\sigma_n(\Gamma)^\omega$.*

Proof: $\sigma_m(\Gamma)^\omega \times \sigma_n(\Gamma)^\omega \approx (\sigma_m(\Gamma) \times \sigma_n(\Gamma)^\omega)^\omega \approx (\sigma_n(\Gamma)^\omega)^\omega \approx \sigma_n(\Gamma)^\omega$. $\square$

Lemma 11. *Let $m_1,\ldots,m_r < n < \omega$ and $e_1,\ldots,e_r \leq \omega$. Then the space $\prod_{i=1}^r \sigma_{m_i}(\Gamma)^{e_i} \times \sigma_n(\Gamma)^\omega$ is homeomorphic to $\sigma_n(\Gamma)^\omega$.*

Proof: Follows from repeated application of Lemmas 9 and 10 above. □

From Lemma 11 it follows that any space $\sigma_\tau(\Gamma)$ with $i(\tau) = \omega$ is homeomorphic to $\sigma_{(\omega,\omega,\ldots)}(\Gamma)$ (because we can substitute each factor $\sigma_n(\Gamma)^\omega$ of $\sigma_\tau(\Gamma)$ by the homeomorphic $\prod_{i\leq n}\sigma_i(\Gamma)^\omega$) and this proves part (2) of Theorem 1. Lemma 11 also shows that it is irrelevant in determining the homeomorphism class of $\sigma_\tau(\Gamma)$ which are the values τ_n for $n < i(\tau)$. Hence, in order to prove part(1) of Theorem 1 it remains to show that if $j(\tau) < \omega$ and $\sigma_\tau(\Gamma)$ is homeomorphic to $\sigma_{\tau'}(\Gamma)$ then $\tau_n = \tau'_n$ for all $n > i(\tau)$.

We recall that a family $\{S_\eta\}_{\eta\in H}$ of sets is a Δ-system if there is a set S (called the root of the Δ-system) such that $S_\eta \cap S_{\eta'} = S$ for all $\eta \neq \eta'$. We will make use of the fact that any uncountable family of finite sets has an uncountable subfamily which is a Δ-system, cf.[4, Theorem 1.4] for $\kappa = \omega$ and $\alpha = \omega_1$.

The following lemma includes as a particular case that $\sigma_{n+1}(\Lambda)$ does not embed into $\sigma_n(\Gamma)^\omega$. This fact, whose proof corresponds to Steps 1 - 3 below was shown to us by Witold Marciszewski, and it seems that it was known to several people before.

Lemma 12. *If $|\Lambda| > \omega$, $n \geq 0$, $k \geq 0$, then the space $\sigma_{n+1}(\Lambda)^{k+1}$ does not embed into $\sigma_n(\Gamma)^\omega \times \sigma_{n+1}(\Gamma)^k$.*

Proof: Suppose that there exists such an embedding.

Step 1. Passing to a suitable uncountable subset of Λ, we can suppose that there is an embedding

$$\phi : \sigma_{n+1}(\Lambda)^{k+1} \longrightarrow \sigma_n(\Gamma)^m \times \sigma_{n+1}(\Gamma)^k$$

for some $m < \omega$. To see this, take $\varphi : \sigma_{n+1}(\Lambda)^{k+1} \longrightarrow \sigma_n(\Gamma)^\omega \times \sigma_{n+1}(\Gamma)^k$ our original embedding. In this step, we shall denote an element $x \in \sigma_{n+1}(\Lambda)^{k+1}$ as $x = (x_0,\ldots,x_k)$. For each $\lambda \in \Lambda$ and every $i \in \{0,\ldots,k\}$ we find a clopen set A^i_λ of $\sigma_n(\Gamma)^\omega \times \sigma_{n+1}(\Gamma)^k$ which separates the disjoint compact sets $\varphi(\{x : \lambda \in x_i\})$ and $\varphi(\{x : \lambda \not\in x_i\})$. Associated to A^i_λ we have a finite subset $F^i_\lambda \subset \omega$ such that $A^i_\lambda = \sigma_n(\Gamma)^{\omega\setminus F^i_\lambda} \times B^i_\lambda$ with B^i_λ a clopen subset of $\sigma_n(\Gamma)^{F^i_\lambda} \times \sigma_{n+1}(\Gamma)^k$. We choose Λ' to be an uncountable subset of Λ such that $\bigcup_{i=0}^k F^i_\lambda = \bigcup_{i=0}^k F^i_{\lambda'} = F$ for all $\lambda,\lambda' \in \Lambda'$ and in this case the composition

$$\sigma_{n+1}(\Lambda')^{k+1} \hookrightarrow \sigma_{n+1}(\Lambda)^{k+1} \longrightarrow \sigma_n(\Gamma)^\omega \times \sigma_{n+1}(\Gamma)^k \longrightarrow \sigma_n(\Gamma)^F \times \sigma_{n+1}(\Gamma)^k$$

is one-to-one. The reason is that if $x, y \in \sigma_{n+1}(\Lambda')^{k+1}$ are different then there exists $i \in \{0, \ldots, k\}$ and $\lambda \in \Lambda'$ such that $\lambda \in x_i$ but $\lambda \notin y_i$ (or viceversa). Then $\phi(x) \in A^i_\lambda$ and $\phi(y) \notin A^i_\lambda$ so either the coordinate of $\sigma_{n+1}(\Gamma)^k$ or some coordinate of $F^\lambda_i \subset F$ must be different for $\phi(x)$ and $\phi(y)$.

Step 2. For $i = 0, \ldots, k$ and $\lambda \in \Lambda$ we define $e^\lambda_i \in \sigma_{n+1}(\Lambda)^{k+1}$ to be the element which has $\{\lambda\}$ in coordinate i and $\emptyset$ in all other coordinates. Each $\phi(e^\lambda_i)$ will be of the form

$$\phi(e^\lambda_i) = (x^\lambda_i[1], \ldots, x^\lambda_i[m], x^\lambda_i[m+1], \ldots, x^\lambda_i[m+k])$$

with $x^\lambda_i[j] \in \sigma_n(\Gamma)$ if $j \leq m$ and $x^\lambda_i[j] \in \sigma_{n+1}(\Gamma)$ if $m < j \leq m+k$. Passing to a suitable uncountable subset of Λ, we can assume that for every fixed $i \in \{0, \ldots, k\}$ and $j \in \{1, \ldots, m+k\}$ the family $\{x^\lambda_i[j] : \lambda \in \Lambda\}$ is a Δ-system of root $R_i[j]$ formed by sets of the same cardinality $c_i[j]$.

Step 3. We claim that for $i = 0, \ldots, n$ and $j = 1, \ldots, m$, the Δ-system $\{x^\lambda_i[j] : \lambda \in \Lambda\}$ is constant. Suppose the contrary for some fixed $i \leq n$ and $j \leq m$. Then $x^\lambda_i[j] = R \cup S^\lambda \in \sigma_n(\Gamma)$ where $R \cap S^\lambda = \emptyset$, $S^\lambda \neq \emptyset$, and $S^\lambda \cap S^{\lambda'} = \emptyset$ for $\lambda \neq \lambda'$. We consider the sets

$$A_\lambda = \{y = (y[1], \ldots, y[m+k]) \in \sigma_n(\Gamma)^m \times \sigma_{n+1}(\Gamma)^k : y[j] \supset S^\lambda\}.$$

The A_λ's are neighborhoods of the $\phi(e^\lambda_i)$'s with the property that for every $F \subset \Lambda$ with $|F| > n$, $\bigcap_{\lambda \in F} A_\lambda = \emptyset$ (because for y in that intersection, $|y[j]| > n$ and $y[j] \in \sigma_n(\Gamma)$). Let $\psi : \sigma_{n+1}(\Lambda) \longrightarrow \sigma_{n+1}(\Lambda)^{k+1}$ be the map defined by $\psi(x)_i = x$ and $\psi(x)_{i'}(x) = \emptyset$ if $i' \neq i$. Then the $(\phi\psi)^{-1}(A_\lambda)$'s are neighborhoods of the $\{\lambda\}$'s in $\sigma_{n+1}(\Lambda)$ with the property that for every $F \subset \Lambda$ with $|F| > n$, $\bigcap_{\lambda \in F} (\phi\psi)^{-1}(A_\lambda) = \emptyset$. This is a contradiction since such a family of neighborhoods cannot be found. Namely, take basic neighborhoods with $\{\lambda\} \in \Phi^{G_\lambda}_{\{\lambda\}} \subset (\phi\psi)^{-1}(A_\lambda)$ and take $\Lambda' \subset \Lambda$ uncountable with $\{G_\lambda : \lambda \in \Lambda'\}$ a Δ-system of root R'. Then construct inductively a finite sequence $F = \{\lambda_1, \ldots, \lambda_{n+1}\} \subset \Lambda' \setminus R'$ such that $\lambda_p \notin \bigcup_{q<p} G_{\lambda_q}$ and $G_{\lambda_p} \cap \{\lambda_1, \ldots, \lambda_{p-1}\} = \emptyset$ (notice that it is possible to choose such a λ_p because $\{\lambda_1, \ldots, \lambda_{p-1}\} \cap R' = \emptyset$ and hence there are only finitely many G_λ's with $\lambda \in \Lambda'$ and $G_\lambda \cap \{\lambda_1, \ldots, \lambda_{p-1}\} \neq \emptyset$). In this case we have $F \in \bigcap_{\lambda \in F} (\phi\psi)^{-1}(A_\lambda)$.

Step 4. Notice that, in the case when $k = 0$ we already arrived to a contradiction and the proof is complete. When $k > 0$ we need some extra work. From step 3, we deduce that for each $i \in \{0, \ldots, k\}$ there must exist $j \in \{m+1, \ldots, m+k\}$ such that the family $\{x^\lambda_i[j] : \lambda \in \Lambda\}$ is a nonconstant Δ-system. Since i runs in a set of $k+1$ elements and j in a set of k elements, there must exist two different $i, i' \in \{0, \ldots, k\}$ such that for the same j, $\{x_i^\lambda[j] : \lambda \in \Lambda\}$ and $\{x_{i'}^\lambda[j] : \lambda \in \Lambda\}$ are nonconstant Δ-systems. We assume that $c_i[j] \geq c_{i'}[j]$ (these numbers are defined in step 2). Again, for $\lambda \in \Lambda$ we consider the sets

$$A_\lambda = \{(y[1], \dots, y[m+k]) \in \sigma_n(\Gamma)^m \times \sigma_{n+1}(\Gamma)^k : y[j] \supset x_i^\lambda[j]\},$$

$$A'_\lambda = \{(y[1], \dots, y[m+k]) \in \sigma_n(\Gamma)^m \times \sigma_{n+1}(\Gamma)^k : y[j] \supset x_{i'}^\lambda[j]\}.$$

The A_λ's and the A'_λ's are neighborhoods of the $\phi(e_i^\lambda)$'s and the $\phi(e_{i'}^\lambda)$'s respectively with the property that

$$(*)\ \forall \lambda \in \Lambda \forall F \subset \Lambda \left(|F| > n \wedge x_i^\lambda[j] \not\subseteq \bigcup_{\mu \in F} x_{i'}^\mu[j] \right) \Rightarrow A_\lambda \cap \bigcap_{\mu \in F} A'_\mu = \emptyset.$$

That intersection is empty because if y belongs to it, then

$$x_i^\lambda[j] \cup \bigcup_{\mu \in F} x_{i'}^\mu[j] \subset y[j] \in \sigma_{n+1}(\Gamma)$$

and the set in the left, if $x_i^\lambda[j] \not\subseteq \bigcup_{\mu \in F} x_{i'}^\mu[j]$, has cardinality greater than $n+1$, a contradiction. Since the Δ-systems are not constant and $c_i[j] \geq c_{i'}[j]$, if $x_i^\lambda[j] \subseteq \bigcup_{\mu \in F} x_{i'}^\mu[j]$ holds, there must be some $\mu \in F$ and some $\gamma \in x_i^\lambda[j]$ such that $\gamma \in x_{i'}^\mu[j] \setminus R_{i'}[j]$. For a fixed λ there are only finitely many μ's with $(x_{i'}^\mu[j] \setminus R_{i'}[j]) \cap x_i^\lambda[j] \neq \emptyset$. Hence for every λ, we can find a cofinite subset Λ_λ of Λ such that the hypothesis $x_i^\lambda[j] \not\subseteq \bigcup_{\mu \in F} x_{i'}^\mu[j]$ of statement $(*)$ holds whenever $F \subset \Lambda_\lambda$. For short, we know that for every $\lambda \in \Lambda$ there exists a cofinite subset Λ_λ of Λ such that

$$\forall F \subset \Lambda_\lambda \quad |F| > n \Rightarrow A_\lambda \cap \bigcap_{\mu \in F} A'_\mu = \emptyset.$$

This contradicts the following lemma for $B_\lambda = \phi^{-1}(A_\lambda)$ and $B'_\lambda = \phi^{-1}(A'_\lambda)$:

Lemma 13. *For every $\lambda \in \Lambda$, let B_λ and B'_λ be neighborhoods of e_i^λ and $e_{i'}^\lambda$ respectively in $\sigma_{n+1}(\Lambda)^{k+1}$. Then there exists $\lambda_0 \in \Lambda$ and an infinite set $S \subset \Lambda$ such that for every $F \subset S$ with $|F| = n+1$,*

$$B_{\lambda_0} \cap \bigcap_{\mu \in F} B'_\mu \neq \emptyset$$

Proof: For a simpler notation, we will assume that $i = 0$ and $i' = 1$. Notice that a basic clopen set Φ_F^G of $\sigma_{n+1}(\Lambda)$ is nonempty if and only if $F \cap G = \emptyset$ and $|F| \leq n+1$. Each B_λ and each B'_μ contain basic clopen sets of the form

$$\Phi_{\{\lambda\}}^{G_0^\lambda} \times \Phi_\emptyset^{G_1^\lambda} \times \Phi_\emptyset^{G_2^\lambda} \times \cdots \times \Phi_\emptyset^{G_k^\lambda} \subseteq B_\lambda$$

$$\Phi_\emptyset^{H_0^\mu} \times \Phi_{\{\mu\}}^{H_1^\mu} \times \Phi_\emptyset^{H_2^\mu} \times \cdots \times \Phi_\emptyset^{H_k^\mu} \subseteq B'_\mu$$

with all G_l^λ and H_l^μ finite subsets of Λ and $\lambda \notin G_0^\lambda$ and $\mu \notin H_1^\mu$. First, we find $M \subset \Lambda$ a countably infinite set such that $\mu' \notin H_1^\mu$ for every $\mu, \mu' \in M$. This can be done as follows. We begin with an infinite $M_1 \subset \Lambda$ such that the family $\{H_1^\mu : \mu \in M_1\}$ is a Δ-system of root R, and we set $M_2 = M_1 \setminus R$. Then we can find recursively a sequence $(\mu_p)_{p<\omega} \subset M_2$ such that $\mu_p \notin \bigcup_{q<p} H_1^{\mu_q}$ and $H_1^{\mu_p} \cap \{\mu_1, \dots, \mu_{p-1}\} = \emptyset$. After this, we set $M = \{\mu_p : p < \omega\}$. Now, we choose $\lambda_0 \notin \bigcup_{\mu \in M} H_0^\mu$. Taking $S = \{\mu \in M : \mu \notin G_1^{\lambda_0}\}$, then λ_0 and S are as desired. Namely, take $F \subset S$ with $|F| = n+1$, and for every $j = 0, \dots, k$ call $I_j = G_j^{\lambda_0} \bigcup_{\mu \in F} H_j^\mu$ so that

$$B_{\lambda_0} \cap \bigcap_{\mu \in F} B'_\mu \supset \Phi_{\{\lambda_0\}}^{I_0^\mu} \times \Phi_F^{I_1^\mu} \times \prod_{j=2}^{k} \Phi_\emptyset^{I_j^\mu}.$$

On the one hand, $\Phi_{\{\lambda_0\}}^{I_0^\mu} \neq \emptyset$ because we chose $\lambda_0 \notin \bigcup_{\mu \in M} H_0^\mu$, so $\lambda_0 \notin I_0^\mu$. On the other hand, $\Phi_F^{I_1^\mu} \neq \emptyset$ because, first, since $F \subset M$ and $\mu' \notin H_1^\mu$ for every $\mu, \mu' \in M$, it follows that $F \cap \bigcup_{\mu \in F} H_1^\mu = \emptyset$ and second, since $F \subset S$, just by the definition of S, $F \cap G_1^{\lambda_0} = \emptyset$. $\square$

It is a consequence of Lemma 12 that $j(\tau) = j(\tau')$ whenever $\sigma_\tau(\Gamma) = \sigma_{\tau'}(\Gamma)$, since it shows that $j(\tau) = \omega$ if and only if $\sigma_n(\Gamma)$ can be embedded into $\sigma_\tau(\Gamma)$ for all $n < \omega$ and, if it is not the case, $j(\tau)$ is the greatest integer n for which $\sigma_n(\Gamma)$ embeds into $\sigma_\tau(\Gamma)$. Hence, in the situation of part (1) of Theorem 1, it happens that $j(\tau) = j(\tau') = j$ and moreover that $\tau_n = \tau'_n$ for all $n \geq j$ since, by Lemma 12 again, $\tau_j = \tau'_j$ is the greatest integer k such that $\sigma_j(\Gamma)^k$ embeds into $\sigma_\tau(\Gamma)$ and of course, $\tau_n = \tau'_n = 0$ for all $n > j$. In order to finish the proof of this part (1), we must check that $i(\tau) = i(\tau') = i$ and that $\tau_k = \tau'_k$ for $i < k < j$. In order to get this, we shall look at embeddability of spaces $\sigma_n(\Gamma)^k$ into the clopen sets of $\sigma_\tau(\Gamma)$. For this purpose, we observe that it is enough to look at some basic family of clopen sets, if the others are union of them:

Lemma 14. *Let X be a compact space and $C_1, \dots, C_t$ open subsets of X. If $\sigma_n(\Lambda)^k$ embeds into $\bigcup_1^t C_i$, then there exists $i \leq t$ such that $\sigma_n(\Lambda)^k$ embeds into C_i.*

Proof: It reduces to prove that whenever we express $\sigma_n(\Lambda)^k$ as a union of open sets as

$$\sigma_n(\Lambda)^k = C_1 \cup \cdots \cup C_t$$

then some C_i must contain a copy of $\sigma_n(\Lambda)^k$. Pick $i \in \{1, \dots, t\}$ such that $x_0 = (\emptyset, \dots, \emptyset) \in C_i$. There are finite sets $G^1, \dots, G^k$ of Λ such that

$$x_0 \in \Phi_\emptyset^{G^1} \times \cdots \times \Phi_\emptyset^{G^k} \subset C_i$$

This finishes the proof because $\Phi_\emptyset^{G^1} \times \cdots \times \Phi_\emptyset^{G^k}$ is homeomorphic to $\sigma_n(\Lambda)^k$. $\square$

Let us denote now by $K = \prod_{s\in S}\sigma_{n_s}(\Gamma)$ any finite or countable product of spaces of type $\sigma_n(\Gamma)$. Any clopen set of K is a finite union of basic clopen sets of the form

$$C = \prod_{s\in A}\Phi_{F_s}^{G_s} \times \prod_{s\not\in A}\sigma_{n_s}(\Gamma)$$

where A is a finite subset of S and $\Phi_{F_s}^{G_s}$ a basic clopen set of $\sigma_{n_s}(\Gamma)$. Such a basic clopen set is homeomorphic to

$$(\star)\ C \sim \prod_{s\in A}\sigma_{n_s-|F_s|}(\Gamma) \times \prod_{s\not\in A}\sigma_{n_s}(\Gamma)$$

Now, after Lemma 12, Lemma 14 and the topological description $(\star)$ of the basic clopen sets given above, we are in a position to state that, in the situation of part (1) of Theorem 1, the following hold:

(A) $i(\tau) = i(\tau') = i$ is the greatest integer n such that $\sigma_n(\Gamma)$ embeds into any clopen set of $\sigma_\tau(\Gamma)$.

(B) For $n = j, j-1, j-2, \ldots, i+1$, $\tau_n = \tau'_n$ is the greatest integer k such that there is a clopen set C of $\sigma_\tau(\Gamma)$ in which $\sigma_{n+1}(\Gamma)$ cannot be embedded, but in which nevertheless $\sigma_n(\Gamma)^{k+\sum_{r>n}\tau_r}$ does embed.

This finishes the proof of Theorem 1. For statement (A), since $\sigma_{i(\tau)}(\Gamma)^\omega$ is one of the factors of $\sigma_\tau(\Gamma)$, it is clear that still $\sigma_{i(\tau)}(\Gamma)^\omega$ is a factor of any clopen set like in $(\star)$. On the other hand, there are only finitely many factors of type $\sigma_m(\Gamma)$, $m > i(\tau)$ in $\sigma_\tau(\Gamma)$, hence a clopen set like in $(\star)$ can be obtained so that all factors in $\prod_{s\in A}\sigma_{n_s-|F_s|}(\Gamma) \times \prod_{s\not\in A}\sigma_{n_s}(\Gamma)$ are of the form $\sigma_m(\Gamma)$ with $m \leq i(\tau)$. By Lemma 12, $\sigma_k(\Gamma)$ does not embed in such C if $k > i(\tau)$.

Statement (B) is proved by "downwards induction" starting in j and finishing with $i+1$. We know, by Lemma 11, that

$$\sigma_\tau(\Gamma) \sim \sigma_i(\Gamma)^\omega \times \prod_{m=i+1}^{j}\sigma_m(\Gamma)^{\tau_m}$$

Now statement (B) for $n = j$ is a direct consequence of Lemma 12 since no clopen can contain $\sigma_{j+1}(\Gamma)$ and the maximal exponent of $\sigma_j(\Gamma)$ inside $\sigma_\tau(\Gamma)$ is τ_j. We pass to the case when $i < n < j$. The "biggest" possible basic clopen set C of $\sigma_\tau(\Gamma)$ not containing $\sigma_{n+1}(\Gamma)$ is obtained by reducing as necessary the factors $\sigma_m(\Gamma)$ with $m > n$:

$$C \sim \sigma_i(\Gamma)^\omega \times \prod_{m=i+1}^{n} \sigma_m(\Gamma)^{\tau_m} \times \prod_{m=n+1}^{j} \sigma_n(\Gamma)^{\tau_m}$$

The maximal exponent of $\sigma_n(\Gamma)$ in such a C is $\sum_{m=n}^{j} \sigma_{\tau_m}$. $\square$

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