RADON-NIKODÝM COMPACT SPACES OF LOW WEIGHT AND BANACH SPACES

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Abstract. We prove that a continuous image of a Radon-Nikodým compact of weight less than **b** is Radon-Nikodým compact. As a Banach space counterpart, subspaces of Asplund generated Banach spaces of density character less than **b** are Asplund generated. In this case, in addition, there exists a subspace of an Asplund generated space which is not Asplund generated which has density character exactly **b**.

The concept of Radon-Nikodým compact, due to Reynov [12], has its origin in Banach space theory, and it is defined as a topological space which is homeomorphic to a weak* compact subset of the dual of an Asplund space, that is, a dual Banach space with the Radon-Nikodým property (topological spaces will be here assumed to be Hausdorff). In [9], the following characterization of this class is given:

Theorem 1. *A compact space K is Radon-Nikodým compact if and only if there is a lower semicontinuous metric d on K which fragments K.*

Recall that a map $f : X \times X \longrightarrow \mathbb{R}$ on a topological space X is said to *fragment* X if for each (closed) subset L of X and each $\varepsilon > 0$ there is a nonempty relative open subset U of L of f-diameter less than ε, i.e. $\sup\{f(x,y) : x,y \in U\} < \varepsilon$. Also, a map $g : Y \longrightarrow \mathbb{R}$ from a topological space to the real line is *lower semicontinuous* if $\{y : g(y) \leq r\}$ is closed in Y for every real number r.

It is an open problem whether a continuous image of a Radon-Nikodým compact is Radon-Nikodým. Arvanitakis [2] has made the following approach to this problem: if K is a Radon-Nikodým compact and $\pi : K \longrightarrow L$ is a continuous surjection, then we have a lower semicontinuous fragmenting metric d on K, and if we want to prove that L is Radon-Nikodým compact, we should find such a metric on L. A natural candidate is:

$$d_1(x,y) \quad = \quad d(\pi^{-1}(x), \pi^{-1}(y)) = \inf\{d(t,s) : \pi(t) = x,\ \pi(s) = y\}.$$

The map d_1 is lower semicontinuous and fragments L and it is a *quasi metric*, that is, it is symmetric and vanishes only if $x = y$. But it is not a metric because, in general, it lacks triangle inequality. Consequently, Arvanitakis [2] introduced the

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following concept:

Definition 2. A compact space L is said to be *quasi Radon-Nikodým* if there exists a lower semicontinuous quasi metric which fragments L.

The class of quasi Radon-Nikodým compacta is closed under continuous images but it is unknown whether it is the same class as that of Radon-Nikodým compacta or even the class of their continuous images. At least other two superclasses of continuous images of Radon-Nikodým compacta appear in the literature. Reznichenko [1, p. 104] defined a compact space L to be *strongly fragmentable* if there is a metric d which fragments L such that each pair of different points of L possess disjoint neighbourhoods at a positive d-distance. It has been noted by Namioka [10] that the classes of quasi Radon-Nikodým and strongly fragmentable compacta are equal. The other superclass of continuous images of Radon-Nikodým compacta, called *countably lower fragmentable* compacta, was introduced by Fabian, Heisler and Matoušková [5]. In section 3, we recall its definition and we prove that this class is equal to the other two.

The main result in section 1 is the following:

Theorem 3. *If K is a quasi Radon-Nikodým compact space of weight less than* $\mathbf{b}$, *then K is Radon-Nikodým compact.*

The weight of a topological space is the least cardinality of a base for its topology. We also recall the definition of cardinal $\mathbf{b}$. In the set $\mathbb{N}^{\mathbb{N}}$ we consider the order relation given by $\sigma \leq \tau$ if $\sigma_n \leq \tau_n$ for all $n \in \mathbb{N}$. Cardinal $\mathbf{b}$ is the least cardinality of a subset of $\mathbb{N}^{\mathbb{N}}$ which is not *σ-bounded* for this order (a set is *σ-bounded* if it is a countable union of bounded subsets). It is consistent that $\mathbf{b} > \omega_1$. In fact, Martin's axiom and the negation of the continuum hypothesis imply that $\mathbf{c} = \mathbf{b} > \omega_1$, cf. [6, 11D and 14B]. It is also possible that $\mathbf{c} > \mathbf{b} > \omega_1$, cf. [17, section 5]. On the other hand, cardinal $\mathbf{d}$ is the least cardinality of a cofinal subset of $(\mathbb{N}^{\mathbb{N}}, \leq)$, that is, a set A such that for each $\sigma \in \mathbb{N}^{\mathbb{N}}$ there is some $\tau \in A$ such that $\sigma \leq \tau$. In a sense, the following proposition puts a rough bound on the size of the class of quasi Radon-Nikodým compacta with respect to Radon-Nikodým compacta.

Proposition 4. *Every quasi Radon-Nikodým compact space embeds into a product of Radon-Nikodým compact spaces with at most* $\mathbf{d}$ *factors.*

In section 2 we discuss the Banach space counterpart to Theorem 3. A Banach space V is Asplund generated, or *GSG*, if there is some Asplund space V' and a bounded linear operator $T : V' \longrightarrow V$ such that $T(V')$ is dense in V. Our main result for this class is the following:

Theorem 5. *Let V be a Banach space of density character less than* $\mathbf{b}$ *and such that the dual unit ball (B_{V^*}, w^*) is quasi Radon-Nikodým compact, then V is Asplund generated.*

The density character of a Banach space is the least cardinal of a norm-dense subset, and it equals the weight of its dual unit ball in the weak* topology.

Examples constructed by Rosenthal [13] and Argyros [4, section 1.6] show that there exist Banach spaces which are subspaces of Asplund generated spaces but which are not Asplund generated. However, since the dual unit ball of a subspace of an Asplund generated space is a continuous image of a Radon-Nikodým compact [4, Theorem 1.5.6], we have the following corollary to Theorem 5:

Corollary 6. *If a Banach space V is a subspace of an Asplund generated space and the density character of V is less than* $\mathbf{b}$*, then V is Asplund generated.*

Also, a Banach space is weakly compactly generated (WCG) if it is the closed linear span of a weakly compact subset. The same examples mentioned above show that neither is this property inherited by subspaces. A Banach space V is weakly compactly generated if and only if it is Asplund generated and its dual unit ball (B_{V^*}, w^*) is Corson compact [11], [14]. Having Corson dual unit ball is a hereditary property since a continuous image of a Corson compact is Corson compact [7], hence:

Corollary 7. *If a Banach space V is a subspace of a weakly compactly generated space and the density character of V is less than* $\mathbf{b}$*, then V is weakly compactly generated.*

Corollary 7 can also be obtained from the following theorem, essentially due to Mercourakis [8]:

Theorem 8. *If a Banach space V is weakly $\mathcal{K}$-analytic and the density character of V is less than* $\mathbf{b}$*, then V is weakly compactly generated.*

The class of weakly $\mathcal{K}$-analytic spaces is larger than the class of subspaces of weakly compactly generated spaces. We recall its definition in section 2. The result of Mercourakis [8, Theorem 3.13] is that, under Martin's axiom, weakly $\mathcal{K}$-analytic Banach spaces of density character less than $\mathbf{c}$ are weakly compactly generated, but his arguments prove in fact the more general Theorem 8. We give a more elementary proof of this theorem, obtaining it as a consequence of a purely topological result: Any $\mathcal{K}$-analytic topological space of density character less than $\mathbf{b}$ contains a dense σ-compact subset. We also remark that it is not possible to generalize Theorem 8 for the class of weakly countably determined Banach spaces.

Cardinal $\mathbf{b}$ is best possible for Theorem 5, Theorem 8 and their corollaries, as it is shown by slight modifications of the mentioned example of Argyros [4, section 1.6] and of the example of Talagrand [15] of a weakly $\mathcal{K}$-analytic Banach space which is not weakly compactly generated, so that we get examples of density character exactly **b**.

For information about cardinals **b** and **d** we refer to [17]. Concerning Banach spaces, our main reference is [4].

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1. Quasi Radon-Nikodým compacta of low weight

In this section, we characterize quasi Radon-Nikodým compacta in terms of embeddings into cubes $[0,1]^\Gamma$ and from this, we will derive proofs of Theorem 3 and Proposition 4. Techniques of Arvanitakis [2] will play an important role in this section, as well as the following theorem of Namioka [9]:

Theorem 9. *Let K be a compact space. The following are equivalent.*

(1) *K is Radon-Nikodým compact.*
(2) *There is an embedding $K \subset [0,1]^\Gamma$ such that K is fragmented by the uniform metric $d(x,y) = \sup_{\gamma\in\Gamma} |x_\gamma - y_\gamma|$.*

Let $\mathrm{P} \subset \mathbb{N}^\mathbb{N}$ be the set of all strictly increasing sequences of positive integers. Note that this is a cofinal subset of $\mathbb{N}^\mathbb{N}$. For each $\sigma \in \mathrm{P}$ we consider the lower semicontinuous non decreasing function $h^\sigma : [0,+\infty] \longrightarrow \mathbb{R}$ given by:

- $h^\sigma(0) = 0$,
- $h^\sigma(t) = \frac{1}{\sigma_n}$ whenever $\frac{1}{n+1} < t \leq \frac{1}{n}$.
- $h^\sigma(t) = \frac{1}{\sigma_1}$ whenever $t > 1$.

Also, if $f : X \times X \longrightarrow \mathbb{R}$ is a map and $A, B \subset X$, we will use the notation $f(A,B) = \inf\{f(x,y) : x \in A,\ y \in B\}$.

Theorem 10. *Let K be a compact subset of the cube $[0,1]^\Gamma$. The following are equivalent:*

(1) *K is quasi Radon-Nikodým compact.*
(2) *There is a map $\sigma : \Gamma \longrightarrow \mathrm{P}$ such that K is fragmented by*

$$f(x,y) = \sup_{\gamma\in\Gamma} h^{\sigma(\gamma)}(|x_\gamma - y_\gamma|)$$

which is a lower semicontinuous quasi metric.

PROOF: Observe that f in (2) is expressed as a supremum of lower semicontinuous functions, and therefore, it is lower semicontinuous. Also, $f(x,y)=0$ if and only if $h^{\sigma(\gamma)}(|x_\gamma - y_\gamma|)=0$ for all $\gamma\in\Gamma$ if and only if $|x_\gamma - y_\gamma|=0$ for all $\gamma\in\Gamma$. Hence, f is indeed a lower semicontinuous quasi metric and it is clear that (2) implies (1). Assume now that K is quasi Radon-Nikodým compact and let $g: K\times K\longrightarrow[0,1]$ be a lower semicontinuous quasi metric which fragments K. For $\gamma\in\Gamma$, we call $p_\gamma : K\longrightarrow[0,1]$ the projection on the coordinate γ, $p_\gamma(x)=x_\gamma$, and we define a quasi metric g_γ on $[0,1]$ by the rule:

$$g_\gamma(t,s)=\begin{cases} g(p_\gamma^{-1}(t),p_\gamma^{-1}(s)) & \text{if } p_\gamma^{-1}(t) \text{ and } p_\gamma^{-1}(s) \text{ are nonempty,}\\ 1 & \text{otherwise.}\end{cases}$$

Note that g_γ is lower semicontinuous because for $r<1$

$$\{(t,s): g_\gamma(t,s)\leq r\} = \bigcap_{r'>r}(p_\gamma\times p_\gamma)\{(x,y)\in K^2 : g(x,y)\leq r'\}$$

Observe also that if $x,y\in K$, then $g_\gamma(x_\gamma,y_\gamma)=g_\gamma(p_\gamma(x),p_\gamma(y))\leq g(x,y)$. Hence, K is fragmented by

$$g'(x,y)=\sup_{\gamma\in\Gamma} g_\gamma(x_\gamma,y_\gamma)\leq g(x,y)$$

The proof finishes by making use of the following lemma, where we put $g_0 := g_\gamma$:

Lemma 11. *Let $g_0 : [0,1]\times[0,1]\longrightarrow[0,1]$ be a lower semicontinuous quasi metric on $[0,1]$. Then, there exists $\tau\in\mathrm{P}$ such that $h^\tau(|t-s|)\leq g_0(t,s)$ for all $t,s\in[0,1]$.*

PROOF: We define τ recursively. Suppose that we have defined $\tau_1,\ldots,\tau_n$ in such a way that if $|t-s|>\frac{1}{n+1}$, then $h^\tau(|t-s|)\leq g_0(t,s)$. Let

$$K_m=\left\{(t,s)\in[0,1]\times[0,1] : |t-s|\geq\frac{1}{n+2} \text{ and } g_0(t,s)\leq\frac{1}{m}\right\}$$

Then, $\{K_m\}_{m=1}^\infty$ is a decreasing sequence of compact subsets of $[0,1]^2$ with empty intersection. Hence, there is m_1 such that K_m is empty for $m\geq m_1$. We define $\tau_{n+1}=\max\{m_1,\tau_n+1\}$. □

Now, we state a lemma which is just a piece of the proof of [2, Proposition 3.2]. We include its proof for the sake of completeness.

Lemma 12. *Let K,L be compact spaces, let $f: K\times K\longrightarrow\mathbb{R}$ be a symmetric map which fragments K and $p: K\longrightarrow L$ a continuous surjection. Then L is fragmented by $g(x,y)=f(p^{-1}(x),p^{-1}(y))$ and in particular, L is fragmented by any g' with $g'\leq g$.*

PROOF: Let M be a closed subset of L and $\varepsilon>0$. By Zorn's lemma a set $N\subset K$ can be found such that $p: N\longrightarrow M$ is onto and irreducible (that is, for every $N'\subset N$ closed, $p: N'\longrightarrow M$ is not onto). We find $U\subset N$ a relative open subset of N of f-diameter less than ε. By irreducibility, $p(U)$ has nonempty relative interior in M. This interior is a nonempty relative open subset of M of g-diameter less than ε. □

In the sequel, we use the following notation: If $A \subset \Gamma$ are sets, d_A states for the pseudometric in $[0,1]^\Gamma$ given by $d_A(x,y) = \sup_{\gamma\in A}|x_\gamma - y_\gamma|$.

Lemma 13. *Let K be a compact subset of the cube $[0,1]^\Gamma$ and let $\sigma : \Gamma \longrightarrow \mathrm{P}$ be a map such that the quasi metric*

$$f(x,y) = \sup_{\gamma\in\Gamma} h^{\sigma(\gamma)}(|x_\gamma - y_\gamma|)$$

fragments K and such that $\sigma(\Gamma)$ is a σ-bounded subset of $\mathbb{N}^\mathbb{N}$. Then, K is Radon-Nikodým compact. In addition, there exist sets $\Gamma_n \subset \Gamma$ such that $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$ and each d_{Γ_n} fragments K.

PROOF: There is a decomposition $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$ such that each $\sigma(\Gamma_n)$ has a bound τ_n in $(\mathbb{N}^\mathbb{N}, \leq)$. We choose $\tau_n \in \mathrm{P}$. First, we prove that each d_{Γ_n} fragments K. For every $n \in \mathbb{N}$, K is fragmented by the map

$$f_n(x,y) = \sup_{\gamma\in\Gamma_n} h^{\sigma(\gamma)}(|x_\gamma - y_\gamma|) \leq f(x,y)$$

and

$$\begin{aligned} f_n(x,y) &= \sup_{\gamma\in\Gamma_n} h^{\sigma(\gamma)}(|x_\gamma - y_\gamma|) \geq \sup_{\gamma\in\Gamma_n} h^{\tau_n}(|x_\gamma - y_\gamma|) \\ &= h^{\tau_n}\left(\sup_{\gamma\in\Gamma_n} |x_\gamma - y_\gamma|\right) = h^{\tau_n}(d_{\Gamma_n}(x,y)). \end{aligned}$$

Hence, a set of f_n-diameter less than $\frac{1}{\tau_n}$ in K is a set of d_{Γ_n}-diameter less than $\frac{1}{n}$ and therefore, since f_n fragments K, also d_{Γ_n} fragments K.

Consider now $p_n : [0,1]^\Gamma \longrightarrow [0,1]^{\Gamma_n}$ the natural projection and $K_n = p_n(K)$. By Lemma 12, since K is fragmented by f_n, K_n is fragmented by

$$g_n(x,y) = \sup_{\gamma\in\Gamma_n} h^{\sigma(\gamma)}(|x_\gamma - y_\gamma|).$$

and hence, K_n is Radon-Nikodým compact. Moreover, since $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$, K embeds into the product $\prod_{n\in\mathbb{N}} K_n$ and the class of Radon-Nikodým compacta is closed under taking countable products and under taking closed subspaces [9], so K is Radon-Nikodým compact. □

PROOF OF THEOREM 3: If the weight of K is less than $\mathbf{b}$, then K can be embedded into a cube $[0,1]^\Gamma$ with $|\Gamma| < \mathbf{b}$. Any subset of $\mathbb{N}^\mathbb{N}$ of cardinality less than $\mathbf{b}$ is σ-bounded, so Theorem 3 follows directly from Theorem 10 and Lemma 13.□

PROOF OF PROPOSITION 4: Let K be quasi Radon-Nikodým compact, suppose K is embedded into some cube $[0,1]^\Gamma$ and let $\sigma : \Gamma \longrightarrow \mathrm{P}$ be as in Theorem 10. Let $A \subset \mathrm{P}$ be a cofinal subset of P of cardinality $\mathbf{d}$. For $\alpha \in A$, let

$$\Gamma_\alpha = \{\gamma \in \Gamma : \sigma(\gamma) \leq \alpha\},$$

let $p_\alpha : [0,1]^\Gamma \longrightarrow [0,1]^{\Gamma_\alpha}$ be the natural projection, and let $K_\alpha = p_\alpha(K)$. Again, since $\Gamma = \bigcup_{\alpha\in A} \Gamma_\alpha$, K embeds into the product $\prod_{\alpha\in A} K_\alpha$. By Lemma 12, K_α is fragmented by

$$g_\alpha(x,y) = \sup_{\gamma\in\Gamma_\alpha} h^{\sigma(\gamma)}(|x_\gamma - y_\gamma|)$$

The set $\{\sigma(\gamma) : \gamma \in \Gamma_\alpha\}$ is a bounded, and hence σ-bounded, set. Hence, by Lemma 13, K_α is Radon-Nikodým compact. □

We note that from Lemma 13, we obtain something stronger than Theorem 3:

Theorem 14. *For every quasi Radon-Nikodým compact subset of a cube* $[0,1]^\Gamma$ *with* $|\Gamma| < \mathbf{b}$ *there is a countable decomposition* $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$ *such that* d_{Γ_n} *fragments* K *for all* $n \in \mathbb{N}$.

A similar result holds also for generalized Cantor cubes (cf. [5, Theorem 3], [2, Theorem 3.6]): If K is a quasi Radon-Nikodým compact subset of $\{0,1\}^\Gamma$, then there is a decomposition $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$ such that d_{Γ_n} fragments K for all $n \in \mathbb{N}$. We give now an example which shows that this phenomenon does not happen for general cubes, even if the compact K has weight less than $\mathbf{b}$ or it is zero-dimensional:

Proposition 15. *There exist a set* Γ *of cardinality* $\mathbf{b}$ *and a compact subset* K *of* $[0,1]^\Gamma$ *homeomorphic to the metrizable Cantor cube* $\{0,1\}^\mathbb{N}$ *such that for any decomposition* $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$ *there exists* $n \in \mathbb{N}$ *such that* d_{Γ_n} *does not fragment* K.

PROOF: First, we take Γ a subset of $\mathbb{N}^\mathbb{N}$ of cardinality $\mathbf{b}$ which is not σ-bounded. We call $A = \{\gamma_n : \gamma \in \Gamma, n \in \mathbb{N}\}$ the set of all terms of elements of Γ. We define

$$K' = \{x \in \{0,1\}^{\Gamma\times\mathbb{N}} : x_{\gamma,n} = x_{\gamma',n'} \text{ whenever } \gamma_n = \gamma'_{n'}\}.$$

Observe that K' is homeomorphic to $\{0,1\}^\mathbb{N}$: namely, for each $a \in A$ choose some $\gamma^a, n^a \in \Gamma\times\mathbb{N}$ such that $\gamma^a_{n^a} = a$; in this case we have a homeomorphism $K' \longrightarrow \{0,1\}^A$ given by $x \mapsto (x_{\gamma^a,n^a})_{a\in A}$.

Now, we consider the embedding $\phi : \{0,1\}^{\Gamma\times\mathbb{N}} \longrightarrow [0,1]^\Gamma$ given by

$$\phi(x) = \left(\sum_{n\in\mathbb{N}} \left(\frac{2}{3}\right)^n x_{\gamma,n}\right)_{\gamma\in\Gamma}$$

We claim that the space $K = \phi(K') \subset [0,1]^\Gamma$ verifies the statement. Let $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$ any countable decomposition of Γ. Since Γ is not σ-bounded, there is some $n \in \mathbb{N}$ such that Γ_n is not bounded. For this fixed n, since Γ_n is not bounded, there is some $m \in \mathbb{N}$ such that the set $S = \{\gamma_m : \gamma \in \Gamma_n\} \subset A$ is infinite. We consider

$$K_0 = \{x \in K' : x_{\gamma,k} = 0 \text{ whenever } \gamma_k \notin S\} \subset K.$$

By the same arguments as for K', K_0 is homeomorphic to the Cantor cube $\{0,1\}^\mathbb{N}$ by a map $K_0 \longrightarrow \{0,1\}^S$ given by $x \mapsto (x_{\gamma^a,n^a})_{a\in S}$. Now, we take two different elements $x, y \in K_0$. Then, there must exist some $\gamma \in \Gamma_n$ such that $x_{\gamma,m} \neq y_{\gamma,m}$, and this implies that $|\phi(x)_\gamma - \phi(y)_\gamma| \geq 3^{-m}$ and therefore $d_{\Gamma_n}(\phi(x),\phi(y)) \geq 3^{-m}$. This means that any nonempty subset of $\phi(K_0)$ of d_{Γ_n}-diameter less than 3^{-m} must be a singleton. If d_{Γ_n} fragmented K, this would imply that $\phi(K_0)$ has an isolated point, which contradicts the fact that it is homeomorphic to $\{0,1\}^\mathbb{N}$. □

2. Banach spaces of low density character

In this section we find that cardinal $\mathbf{b}$ is the least possible density character of Banach spaces which are counterexamples to several questions. First, we introduce some notation: If A is a subset of a Banach space V, we call d_A to the pseudometric $d_A(x^*, y^*) = \sup_{x\in A} |x^*(x) - y^*(x)|$ on B_{V^*}. Also, we recall the following definition [4, Definition 1.4.1]:

Definition 16. A nonempty bounded subset M of a Banach space V is called an *Asplund set* if for each countable set $A \subset M$ the pseudometric space (B_{V^*}, d_A) is separable.

By [3, Theorem 2.1], M is an Asplund subset of V if and only if d_M fragments (B_{V^*}, w^*). Also, by [4, Theorem 1.4.4], a Banach space V is Asplund generated if and only if it is the closed linear span of an Asplund subset.

PROOF OF THEOREM 5: Let Γ be a dense subset of the unit ball B_V of V of cardinality less than $\mathbf{b}$. Then, we have a natural embedding $(B_{V^*}, w^*) \subset [-1, 1]^\Gamma$. Since (B_{V^*}, w^*) is quasi Radon-Nikodým compact, we apply Theorem 14 and we have $\Gamma = \bigcup \Gamma_n$ and each d_{Γ_n} fragments (B_{V^*}, w^*). This means that for each n, Γ_n is an Asplund set, and by [4, Lemma 1.4.3], $M = \bigcup_{n\in\mathbb{N}} \frac{1}{n}\Gamma_n$ is an Asplund set too. Finally, since the closed linear span of M is V, by [4, Theorem 1.4.4], V is Asplund generated. □

We recall now the concepts that we need for the proof of Theorem 8. We follow the terminology and notation of [4, sections 3.1, 4.1]. Let X and Y be topological spaces. A map $\phi : X \to 2^Y$ from X to the subsets of Y is said to be an usco if the following conditions hold:

(1) $\phi(x)$ is a compact subset of Y for all $x \in X$.
(2) $\{x : \phi(x) \subset U\}$ is open in X, for every open set U of Y.

In this situation, for $A \subset X$ we denote $\phi(A) = \bigcup_{x\in A} \phi(x)$.

A completely regular topological space X is said to be $\mathcal{K}$-analytic if there exists an usco $\phi : \mathbb{N}^\mathbb{N} \to 2^X$ such that $\phi(\mathbb{N}^\mathbb{N}) = X$. A Banach space is weakly $\mathcal{K}$-analytic if it is a $\mathcal{K}$-analytic space in its weak topology.

We note that if a Banach space V contains a weakly σ-compact subset M which is dense in the weak topology, then V is WCG. This is because if $M = \bigcup_{n=1}^\infty K_n$ being K_n a weakly compact set bounded by $c_n > 0$, then $\{0\} \cup \bigcup \frac{1}{nc_n} K_n$ is a weakly compact subset of V whose linear span is (weakly) dense in V. Hence, Theorem 8 is deduced from the following:

Proposition 17. *If X is a $\mathcal{K}$-analytic topological space which contains a dense subset of cardinality less than $\mathbf{b}$, then X contains a dense σ-compact subset.*

PROOF: We have an usco $\phi : \mathbb{N}^\mathbb{N} \longrightarrow 2^X$ with $\phi(\mathbb{N}^\mathbb{N}) = X$ and also a set $\Sigma \subset \mathbb{N}^\mathbb{N}$ such that $|\Sigma| < \mathbf{b}$ and $\phi(\Sigma)$ is dense in X. Any subset of $\mathbb{N}^\mathbb{N}$ of cardinal less than

b is contained in a σ-compact subset of $\mathbb{N}^{\mathbb{N}}$ [17, Theorem 9.1]. Uscos send compact sets onto compact sets, so if $\Sigma' \supset \Sigma$ is σ-compact, then $\phi(\Sigma')$ is a dense σ-compact subset of X. □

We recall that a completely regular topological space X is $\mathcal{K}$-countably determined if there exists a subset Σ of $\mathbb{N}^{\mathbb{N}}$ and an usco $\phi : \Sigma \longrightarrow 2^X$ such that $\phi(\Sigma) = X$ and that a Banach space is weakly countably determined if it is $\mathcal{K}$-countably determined in its weak topology. Talagrand [16] has constructed a Banach space which is weakly countably determined but which is not weakly $\mathcal{K}$-analytic. A slight modification of this example gives a similar one with density character ω_1. This shows that no analogue of Theorem 8 is possible for weakly countably determined Banach spaces. The change in the example consists in substituting the set T considered in [16, p. 78] by any subset $T' \subset T$ of cardinal ω_1 such that $\{o(X) : X \in T'\}$ is uncountable and $\mathcal{A}$ by $\mathcal{A}' = \{A \subset T' : A \in \mathcal{A}_1\}$ (the notations are explained in [16]).

Now, we turn to the fact that cardinal **b** is best possible in Theorem 5, Theorem 8 and their corollaries. We fix a subset S of $\mathbb{N}^{\mathbb{N}}$ of cardinality **b** which is not σ-bounded.

Following the exposition of the example of Argyros in [4, section 1.6] we just substitute the space $Y = \overline{span}\{\pi_\sigma : \sigma \in \mathbb{N}^{\mathbb{N}}\}$ in [4, Theorem 1.6.3] by $Y' = \overline{span}\{\pi_\sigma : \sigma \in S\}$ and we obtain a Banach space of density character **b** which is a subspace of a WCG space $C(K)$ but which is not Asplund generated. The same arguments in [4, section 1.6] hold just changing $\mathbb{N}^{\mathbb{N}}$ by S where necessary. Only the proof of [4, Lemma 1.6.1] is not good for this case. It must be substituted by the following:

Lemma 18. *Let Γ_n, $n \in \mathbb{N}$, be any subsets of S such that $\bigcup_{n\in\mathbb{N}} \Gamma_n = S$. Then there exist $n, m \in \mathbb{N}$ and an infinite set $A \in \mathcal{A}_m$ such that $A \subset \Gamma_n$.*

Here, as in [4, section 1.6], $\mathcal{A}_m$ is the family of all subsets $A \subset \mathbb{N}^{\mathbb{N}}$ such that if $\sigma, \tau \in A$ and $\sigma \neq \tau$, then $\sigma_i = \tau_i$ if $i \leq m$ and $\sigma_{m+1} \neq \tau_{m+1}$. Also, $\mathcal{A} = \bigcup_{m=1}^{\infty} \mathcal{A}_m$.

PROOF OF LEMMA 18: We consider $\Gamma_{i,j} = \{\sigma \in \Gamma_i : \sigma_1 = j\}$, $i, j \in \mathbb{N}$. Note that $S = \bigcup_{i,j} \Gamma_{i,j}$. Since S is not σ-bounded, there exist n, l with $\Gamma_{n,l}$ unbounded. This implies that for some m, the set $\{\sigma_m : \sigma \in \Gamma_{n,l}\}$ is infinite. We take m the least integer with this property ($m > 1$). Let $B \subset \Gamma_{n,l}$ be an infinite set such that $\sigma_m \neq \sigma'_m$ for $\sigma, \sigma' \in B$, $\sigma \neq \sigma'$. Since all σ_k with $\sigma \in B$, $k < m$, lie in a finite set, an infinite set $A \subset B$ can be chose such that $A \in \mathcal{A}_{m-1}$. □

On the other hand, if we follow the proof in [4, section 4.3] that the Banach space $C(K)$ of Talagrand is weakly $\mathcal{K}$-analytic but not WCG, and we change K in [4, p. 76] by $K' = \{\chi_A : A \in \mathcal{A},\ A \subset S\} \subset \{0,1\}^S$ then $C(K')$ still verifies this conditions and has density character **b**. Observe that $C(K')$ is weakly $\mathcal{K}$-analytic because K' is a retract of the original K. The fact that $C(K')$ is not WCG (not even a subspace of a WCG space) follows from [4, Theorem 4.3.2] and Lemma 18 above by the same arguments as in [4, p. 78].

3. Countably lower fragmentable compacta

In this section we prove that the concept of quasi Radon Nikodým compact [2] is equivalent to that of countably lower fragmentable compact [5]. The main result for this class in [5] is that if K is countably lower fragmentable, then so is $(B_{C(K)^*}, w^*)$. We note that, with these two facts at hand, together with the fact that if $C(K)$ is Asplund generated, then K is Radon-Nikodým [4, Theorem 1.5.4], Theorem 3 is deduced from Theorem 5.

We need some notation: if K is a compact space and $A \subset C(K)$ is a bounded set of continuous functions over K, we define the pseudometric d_A on K as $d_A(x,y) = \sup_{f\in A}|f(x)-f(y)|$. If X is a topological space, $d : X\times X \longrightarrow \mathbb{R}$ is a map, and Δ is a positive real number, it is said that d Δ-*fragments* X if for each subset L of X there is a relative open subset U of L of d-diameter less than or equal to Δ.

Definition 19. A compact space K is said to be countably lower fragmentable if there are bounded subsets $\{A_{n,p} : n,p \in \mathbb{N}\}$ of $C(K)$ such that $C(K) = \bigcup_{n\in\mathbb{N}} A_{n,p}$ for every $p \in \mathbb{N}$, and the pseudometric $d_{A_{n,p}}$ $\frac{1}{p}$-fragments K.

This is the definition as it appears in [5]. However, variable p is superfluous in it. If the sets $A_{n,1}$ exist, it is sufficient to define $A_{n,p} = \{\frac{1}{p}f : f \in A_{n,1}\}$.

On the other hand, we recall a concept introduced by Namioka [9]: For a topological space K, a set $L \subset K\times K$ is said to be an *almost neighborhood of the diagonal* if it contains the diagonal $\Delta_K = \{(x,x) : x \in K\}$ and satisfies that for every nonempty subset X of K there is a nonempty relative open subset U of X such that $U\times U \subset L$. The use of this was suggested to us by I. Namioka and simplifies our original proof.

Theorem 20. *For a compact subset K of $[0,1]^\Gamma$ the following are equivalent:*

1. *K is quasi Radon-Nikodým compact*
2. *K is countably lower fragmentable.*
3. *There are subsets $\Gamma_{n,p}$, $n,p \in \mathbb{N}$, of Γ such that $d_{\Gamma_{n,p}}$ $\frac{1}{p}$-fragments K for every $n,p \in \mathbb{N}$.*

PROOF: Suppose K is quasi Radon-Nikodým compact and let ϕ be a lower semicontinuous quasi metric which fragments K. Then, we just define

$$A_{n,p} = \left\{ f \in C(K) : |f(x)-f(y)| < \frac{1}{p} \text{ whenever } \phi(x,y) \leq \frac{1}{n} \right\} \cap \{f : \|f\|_\infty \leq n\}$$

Clearly, $d_{A_{n,p}}$ $\frac{1}{p}$-fragments K because any subset of K of ϕ-diameter less than $\frac{1}{n}$ has $d_{A_{n,p}}$-diameter less than $\frac{1}{p}$, and we know that ϕ fragments K. On the other hand, for a fixed $p \in \mathbb{N}$, in order to prove that $C(K) = \bigcup_{n\in\mathbb{N}} A_{n,p}$, observe that, if $f \in C(K)$, then

$$C_n = \left\{ (x,y) \in K\times K : |f(x)-f(y)| \geq \frac{1}{p} \text{ and } \phi(x,y) \leq \frac{1}{n} \right\}$$

is a decreasing sequence of compact subsets of $K \times K$ with empty intersection so there is some $n > \|f\|_\infty$ such that C_n is empty, and then, $f \in A_{n,p}$.

That (2) implies (3) is evident, just to take $\Gamma_{n,p} = A_{n,p} \cap \Gamma$ whenever $A_{n,p}$, $n, p \in \mathbb{N}$ are the sets in the definition of countably lower fragmentability.

Now, suppose (3). For every $n, p \in \mathbb{N}$, since $d_{A_{n,p}}$ $\frac{1}{p}$-fragments K, this means that the set $C_{n,p} = \{(x, y) \in K \times K : d_{\Gamma_{n,p}}(x, y) \le \frac{1}{p}\}$ is an almost neighborhood of the diagonal which, in addition, is closed. On the other hand, observe that, for each $n, p \in \mathbb{N}$, $(x, y) \in C_{n,p}$ if and only if $|x_\gamma - y_\gamma| \le \frac{1}{p}$ for all $\gamma \in \Gamma_{n,p}$ so that

$$\bigcap_{n,p\in\mathbb{N}} C_{n,p} = \bigcap_{p\in\mathbb{N}} \left\{(x, y) : |x_\gamma - y_\gamma| \le \frac{1}{p}\ \forall\gamma \in \bigcup_{n\in\mathbb{N}} \Gamma_{n,p} = \Gamma\right\} = \Delta_K$$

Now, K is quasi Radon-Nikodým by virtue of [10, Theorem 1], which states that K is quasi Radon-Nikodým compact if and only if there is a countable family of closed almost neighborhoods of the diagonal whose intersection is the diagonal Δ_K. □

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