BOUNDED TIGHTNESS FOR WEAK TOPOLOGIES

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To the memory of our friend Klaus

ABSTRACT. In every space for which there exists a strictly finer topology than its weak topology but with the same bounded sets (like for instance, all infinite dimensional Banach spaces, the space of distributions $\mathfrak{D}'(\Omega)$ or the space of analytic functions $A(\Omega)$ in an open set $\Omega \subset \mathbb{R}^d$, etc.) there is a set A such that 0 is in the weak closure of A but 0 is not in the weak closure of any bounded subset B of A. A consequence of this is that a Banach space X is finite dimensional if, and only if, the following property [P] holds: for each set $A \subset X$ and each x in the weak closure of A there is a bounded set $B \subset A$ such that x belongs to the weak closure of B. More generally, a complete locally convex space X satisfies property [P] if, and only if, either X is finite dimensional or linearly topologically isomorphic to $\mathbb{R}^{\mathbb{N}}$.

1. INTRODUCTION

The following exercise (attributed to von Neumann) appears in page 83 of [14]:

Exercise A. *Let $A \subset L^2([-\pi, \pi])$ be the set of all functions*

$$f_{m,n}(t) = e^{imt} + me^{int},$$

where m, n are integers and $0 \le m < n$. Show that 0 belongs to the weak closure of A but there is no sequence contained in A that weakly converges to 0.

When solving the exercise one realizes that this is so because the sequence $(e_n)_n$,

$$e_n(t) = \frac{1}{\sqrt{2\pi}} e^{int}, n = 0, 1, 2 \ldots,$$

is an infinite orthonormal sequence in the Hilbert space $L^2([-\pi, \pi])$. Consequently, for every infinite dimensional separable Hilbert space H there is a set $A \subset H$ such that $0 \in \overline{A}^w$ but there is no sequence contained in A that is w-convergent to 0, where w denotes the weak topology in H. It is a well known fact that Hilbert spaces are reflexive, and therefore bounded sets are w-relatively compact. Thus, in separable Hilbert spaces bounded sets are w-metrizable. With all these pieces of information together we can assert by now that every separable infinite dimensional Hilbert space H has property [Q] below:

> *There exists a set $A \subset H$ such that $0 \in \overline{A}^w$ and there is no bounded set $B \subset A$ such that $0 \in \overline{B}^w$.*

Since infinite dimensional Hilbert spaces do contain infinite dimensional separable subspaces we can be sure that all infinite dimensional Hilbert spaces enjoy property [Q].

If one tries to extend the previous construction to infinite dimensional Banach spaces X one might be misled, at first glance, by the previous exercise and then try to perform some tricks to get [Q], for instance, in some class of Banach spaces with nice *bases*.

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The aim of this paper is to show that the construction in [Q] is a simple *topological matter* in infinite dimensional spaces, see Lemma 1. This construction is the tool to obtain the other two results that have been announced in the abstract, see Theorems 3.2 and 4.2. The paper is completed showing that for spaces X in a la large class $\mathfrak{G}$, X with its weak topology is a Fréchet-Urysohn space if, and only if, X is linearly topologically isomorphic to a subspace of $\mathbb{R}^{\mathbb{N}}$, see Theorem 4.3 and its Corollary 4.4. This class $\mathfrak{G}$ introduced in [5] includes, amongst other, inductive limits of metrizable spaces, the spaces of distributions $\mathfrak{D}'(\Omega)$ and the space $A(\Omega)$ of analytic functions in an open set $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$ –see, respectively, [13, Page 219], [14, Part Two] and [8] for the definitions. For this last part of the paper we use our main result in [3]; we also refer the reader to [11] for related results.

Our notation and terminology are standard. We take the books by Engelking, Kelley, Diestel and Köthe, [9, 12, 7] and [13], as our references for topology, Banach spaces and topological vector spaces. All our vector spaces X are real vector spaces —our results here are true for complex spaces too, as the reader will convince himself easily. All our topological spaces are Hausdorff topological spaces. If X is a Banach space, B_X denotes its closed unit ball, and X^* its (topological) dual space. For X locally convex space the (topological) dual is denoted, as usual, by X'. For both Banach and locally convex spaces the weak topology is denoted by w and the weak* topology is denoted by w^*.

2. [Q] AS A TOPOLOGICAL MATTER IN INFINITE DIMENSIONAL SPACES

If $\mathfrak{p}$ is a seminorm in a vector space X we write

$$B_{\mathfrak{p}} = \{x \in X : \mathfrak{p}(x) \leq 1\} \text{ and } S_{\mathfrak{p}} = \{x \in X : \mathfrak{p}(x) = 1\}.$$

Lemma 1. *Let X be vector space and let δ and τ be two Hausdorff locally convex topologies with τ strictly coarser than δ.*

(i) *Let $\mathfrak{p}$ be a δ-continuous seminorm on X which is not τ-continuous. Then we have $B_{\mathfrak{p}} \subset \overline{S_{\mathfrak{p}}}^{\tau}$;*
(ii) *If Y is a τ-closed linear subspace of finite co-dimension in X then, the topology τ restricted to Y, $\tau|_Y$, is strictly coarser than the topology δ restricted to Y, $\delta|_Y$;*
(iii) *There exists a set $A \subset X$ such that $0 \in \overline{A}^{\tau}$ and there is no δ-bounded set $B \subset A$ such that $0 \in \overline{B}^{\tau}$.*

Proof. To establish (i) we do the following. We get started by observing that every τ-neighbourhood of 0 in X must be $\mathfrak{p}$-unbounded: indeed, the existence of a $\mathfrak{p}$-bounded τ-neighbourhood of 0 would imply that $\mathfrak{p}$ is τ-continuous. Take U any convex τ-neighbourhood of 0, and fix $x_0 \in X$ with $\mathfrak{p}(x_0) < 1$. The set $x_0 + U$ is convex and $\mathfrak{p}$-unbounded and therefore we can take $x_1 \in x_0 + U$ with $\mathfrak{p}(x_1) > 1$. The segment $[x_0, x_1]$ is contained in $x_0 + U$ and meets both

$$\{x \in X : \mathfrak{p}(x) < 1\} \text{ and } \{x \in X : \mathfrak{p}(x) > 1\}.$$

Therefore $[x_0, x_1] \cap S_{\mathfrak{p}} \neq \emptyset$. This implies that $(x_0 + U) \cap S_{\mathfrak{p}} \neq \emptyset$ and thus we have $B_{\mathfrak{p}} \subset \overline{S_{\mathfrak{p}}}^{\tau}$ as we wanted.

Let us establish (ii). Assume that $Y \subset X$ is a τ-closed subspace of co-dimension m. The subspace Y is also δ-closed and then we have that for the direct topological sums

$$(Y, \tau|_Y) \oplus \mathbb{R}^m \cong (X, \tau)$$
$$(Y, \delta|_Y) \oplus \mathbb{R}^m \cong (X, \delta) \tag{1}$$

after [13, §15.8.(2)] and because in $\mathbb{R}^m$ there is a unique Hausdorff locally convex topology [13, §15.5.(1)]. The equalities (1) imply that $\tau|_Y$ has to be strictly coarser than $\delta|_Y$ because the same happens in X by hypothesis.

Let us finish by constructing a set A matching the requirements stated in (iii). Fix $a \in X$ and $f : X \to \mathbb{R}$ a τ-continuous linear form such that $f(a) = 1$ (in particular, a and f are both not null). Define $Y = \{x \in X : f(x) = 0\}$ and apply (ii) to obtain a δ-continuous seminorm $\mathfrak{p}$ on X such that $\mathfrak{q} := \mathfrak{p}|_Y$ is not $\tau|_Y$-continuous. For every $n \in \mathbb{N}$ we consider the set

$$S_n = \{x \in Y : \mathfrak{q}(x) = n\}.$$

We claim that $0 \in \overline{S_n}^\tau$ for every $n \in \mathbb{N}$: we simply observe that we can apply (i) to obtain $0 \in \overline{Y \cap S_\mathfrak{q}}^\tau$ and that $S_n = n(Y \cap S_\mathfrak{q})$. Now we define

$$A_n := n^{-1}a + S_n \quad \text{for each } n \in \mathbb{N}$$

and

$$A := \bigcup_{n=1}^{\infty} A_n.$$

A translation argument implies that $n^{-1}a \in \overline{A}_n^\tau$, for each $n \in \mathbb{N}$, that finally leads to $0 \in \overline{A}^\tau$. For no δ-bounded set $B \subset A$ we have $0 \in \overline{B}^\tau$. Indeed, for every $n \in \mathbb{N}$ one easily estimates

$$\mathfrak{p}(x) \geq n - \frac{\mathfrak{p}(a)}{n} \quad \text{for each } x \in A_n.$$

This implies that if $B \subset A$ is δ-bounded then it meets just finitely many A_n's. Consequently there is $N \in \mathbb{N}$ such that $B \subset \bigcup_{n=1}^N A_n$. A direct computation shows now that

$$f(x) \geq \frac{1}{N} \quad \text{for each } x \in \bigcup_{n=1}^{N} A_n.$$

The inequality above remains true for each $x \in \overline{\bigcup_{n=1}^N A_n}^\tau$ by the τ-continuity of f. Therefore we deduce that $0 \notin \overline{B}^\tau \subset \overline{\bigcup_{n=1}^N A_n}^\tau$ because $f(0) = 0 < 1/N$. □

3. Characterization of finite dimensional Banach spaces

As a straightforward consequence of Lemma 1 we have the following:

Corollary 3.1. *Let X be a normed space and let τ be a Hausdorff locally convex topology which is coarser than the norm topology. The following statements are equivalent:*

(i) *The topology τ coincides with the norm topology;*
(ii) *For each set $A \subset X$ and each point $x \in \overline{A}^\tau$ there is a norm bounded set $B \subset A$ such that $x \in \overline{B}^\tau$.*

As we already know that (i) and (ii) are equivalent in Corollary 3.1, if moreover τ-bounded sets are norm bounded, it is readily seen that both conditions are also equivalent to the following ones below:

(iii) *For each set $A \subset X$ and each point $x \in \overline{A}^\tau$ there is a sequence $(x_n)_n$ in A such that $x = \tau - lim_n x_n$;*
(iv) *For each set $A \subset X$ and each point $x \in \overline{A}^\tau$ there is a bounded and countable set $B \subset A$ such that $x \in \overline{B}^\tau$.*

Condition (iii) above is usually referred in topology by saying that (X, τ) is a Fréchet-Urysohn space, see [1, page 7].

It is a well known fact that a Banach space X is finite dimensional if, and only if, the weak topology coincides with the norm topology. The same characterization holds replacing weak topology by the weak* topology and norm topology by dual norm topology.

Theorem 3.2. *For a Banach space X the following statements are equivalent:*

(i) *X is finite dimensional;*

(ii) *For each set $A \subset X$ and each point $x \in \overline{A}^w$ there is a bounded set $B \subset A$ such that $x \in \overline{B}^w$;*

(iii) *For each set $A \subset X^*$ and each point $x \in \overline{A}^{w^*}$ there is a bounded set $B \subset A$ such that $x \in \overline{B}^{w^*}$.*

Proof. To establish the equivalence (i)⇔(ii) apply Corollary 3.1 to $\tau = w$ in X. To establish the equivalence (i)⇔(iii) apply Corollary 3.1 to X^* and to $\tau = w^*$ in X^*. □

Let us reflect a bit upon Theorem 3.2. If $A \subset X$ is assumed to be convex then, Hahn-Banach theorem says that for every $x \in \overline{A}^w = \overline{A}^{norm}$ there is a sequence $(x_n)_n$ in A such that $x = \lim_n x_n$ in the norm topology. Our statement in (ii) is, of course and luckily, for *each* set $A \subset X$. However statement (iii) in Theorem 3.2 does not hold for convex sets (even subspaces) in an arbitrary dual Banach space.

Example 3.3. *There is a Banach space X and a w^*-dense subspace $Y \subset X^*$ such that for some $x^* \in X^* \setminus Y$ there is no bounded set $B \subset Y$ such that $x^* \in \overline{B}^{w^*}$.*

Proof. Take X a Banach space such that X^{**}/X is infinite dimensional. Then there is a w^*-dense subspace Y of X^* such that $B_{X^*} \cap Y$ is not norming, that is, such that

(2) $$|x| := \sup\{x^*(x) : x^* \in Y \cap B_{X^*}\}, \text{ for each } x \in X,$$

is not an equivalent norm in X, [6]. We claim that there is a point $x^* \in X^* \setminus Y$ with the property that for no bounded set $B \subset X^*$ we have $x^* \in \overline{B}^{w^*}$. Otherwise, we would have that $X^* = \bigcup_{n=1}^{\infty} \overline{Y \cap nB_{X^*}}^{w^*}$. But in this case Baire's Category theorem would imply that for some $\varepsilon > 0$ we have to have

$$\varepsilon B_{X^*} \subset \overline{Y \cap B_{X^*}}^{w^*},$$

that contradicts the fact that $|\cdot|$ given by the formula (2) is not an equivalent norm in the space X. □

Using the comments that follows Corollary 3.1 we know that statements (i), (ii) and (iii) in Theorem 3.2 are equivalent to:

(iv) *For each set $A \subset X$ and each point $x \in \overline{A}^w$ there is a bounded and countable set $B \subset A$ such that $x \in \overline{B}^w$.*

(v) *For each set $A \subset X^*$ and each point $x \in \overline{A}^{w^*}$ there is a bounded and countable set $B \subset A$ such that $x \in \overline{B}^{w^*}$.*

With the terms *bounded and countable* in statements (iv) and (v) we characterize then finite dimensional Banach spaces. If we simply write *countable* (and forget about *bounded*) in (iv) then the new statement (iv') holds for every Banach space: this property is the so-called Kaplansky property for the weak topology of a Banach space, see [13, §24.1.(2)] and [10, Corollary in page 38]. This property is named differently in topology. Topologists refer to (iv') by saying: for any Banach space X, the space (X, w) has *countable tightness*.

Recall that a topological space Z is said to have *countable tightness* if for each set $A \subset Z$ and each point $x \in \overline{A}$ there is countable set $B \subset A$ such that $x \in \overline{B}$, see [1, page 5]. The following theorem of Arkhangel'skii, see [1, Theorem II.1.1], is a nice tool both in topology and analysis. We quote a special case below.

Theorem B. *Let T be a topological space such that T^n is Lindelöf for each $n \in \mathbb{N}$. Then, the space $(C(T), \tau_p(T))$ of continuous functions on T endowed with the topology of pointwise convergence has countable tightness.*

It is easy to prove, using Theorem B, that for any Banach space X the space (X, w) has countable tightness. Moreover, using again Theorem B it can be shown that the dual space (X^*, w^*) has countable tightness when $(X, w)^n$ is Lindelöf for every $n \in \mathbb{N}$: indeed, simply bear in mind that (X^*, w^*) embeds as a subspace of $(C(X, w), \tau_p(X))$. The class of Banach spaces for which $(X, w)^n$ is Lindelöf is a wide class that contains, for instance, the Banach spaces which are K-analytic for their weak topologies: we refer the interested reader to [4] where several connections between the Lindelöf property and some other classical properties in Banach spaces are established. On the other hand, not for every Banach space X the dual (X^*, w^*) has countable tightness. Indeed, $\ell^1(\mathbb{R})$ provides us with such an example: $(\ell^\infty(\mathbb{R}), w^*)$ has not countable tightness as the reader can easily check.

Our Theorem 3.2 imposes a pretty serious limitation to the widely enjoyed countable tightness of the weak and weak* topologies in Banach spaces when countable tightness is combined with boundedness. For $C(K)$ spaces Theorem 3.2 reads as follows.

Corollary 3.4. *Let K be a compact Hausdorff space. The following statement are equivalent.*

(i) *K is finite;*

(ii) *For each set $A \subset C(K)$ and each point $x \in \overline{A}^{\tau_p(K)}$ there is a uniformly bounded set $B \subset A$ such that $x \in \overline{B}^{\tau_p(K)}$.*

4. Characterization of linear topological subspaces of $\mathbb{R}^{\mathbb{N}}$

Also as a straightforward consequence of Lemma 1 we have the following *locally convex* counterpart to Corollary 3.1.

Corollary 4.1. *Let X be vector space and let δ and τ be Hausdorff locally convex topologies with τ coarser than δ. Assume that for each set $A \subset X$ and each point $x \in \overline{A}^{\tau}$ there is δ-bounded set $B \subset A$ such that $x \in \overline{B}^{\tau}$. Then $\tau = \delta$.*

Of course we could write and prove the counterpart to Corollary 3.1 for metrizable locally convex spaces but we are rather interested in classifying via Corollary 4.1, that is via Lemma 1, the complete locally convex spaces as was announced in the abstract. To do that we will need an extra topological tool. We shall use the concept of k-space. A topological space Z is said to be a k-space when the following property holds: if a subset A of Z intersects each compact subset of Z in a closed set, then A is closed, see [12, page 230] and [9, Theorem 3.3.18].

We began our paper referring to an exercise in Rudin's book and now we have to solve the following exercise in Kelley's book [12, page 240] to finally prove the last result announced in our abstract.

Exercise C. *The product of uncountable many copies $\mathbb{R}^I$ of the real line is not a k-space.*

Solution after Kelley's hint. Consider the subset $A \subset \mathbb{R}^I$ made up of the members x such that for some $n \in \mathbb{N}$ each coordinate of x is equal to n except for a set of at most n indices, and on this set x is zero. Denote by 0 the element of $\mathbb{R}^I$ with each coordinate equal to zero. Then the following holds:

(i) $\overline{A} = A \cup \{0\}$, thus A is not closed.
(ii) $0 \notin \overline{A \cap K}$ for any compact subset $K \subset \mathbb{R}^I$, thus $A \cap K$ is closed.

Proof. Statement (i) is easy to prove. Let us prove (ii). If $K \subset \mathbb{R}^I$ is compact, then its projection on each coordinate is bounded, so we may assume without loss of generality that $K = \prod_{i \in I}[-N_i, N_i]$, where $N_i \in \mathbb{N}$. As I is uncountable, there exists $N \in \mathbb{N}$ such that $N_i = N$ for infinitely many indices $i \in I$. Pick any $x \in A \cap K$. Since $x \in A$, the coordinates of x take some value $n \in \mathbb{N}$ except for finitely many indices. Using that $x \in K$, we deduce that $n \leq N$ and therefore 0 cannot be a cluster point of $A \cap K$. Statement (ii) has been proved. □

A good account of properties for $\mathbb{R}^I$ as topological vector space can be found in [2, 2.6].

Theorem 4.2. *Let (X, δ) be a complete locally convex space. The following statements are equivalent:*

(i) *Either X is finite dimensional or X is linearly and topologically isomorphic to $\mathbb{R}^{\mathbb{N}}$ with its product topology;*
(ii) *For each set $A \subset X$ and each point $x \in \overline{A}^w$ there is a bounded set $B \subset A$ such that $x \in \overline{B}^w$.*

Proof. The implication (i)⇒(ii) is easy. When X is finite dimensional then (ii) holds. In the worst case when $X = \mathbb{R}^{\mathbb{N}}$ the topology δ is metrizable and $w = \delta$. Therefore the statament (ii) also holds, because the w-closure of any set is attained trough limits of sequences from the set, and convergent sequences are bounded. The other way around, let us prove (ii)⇒(i). If (ii) holds then Corollary 4.1 applies to tell us that $w = \delta$. Take $\{f_i : i \in I\}$ an algebraic basis of X' and consider $(\mathbb{R}^I, \tau_p)$ with its product topology τ_p. The map $\varphi : X \to \mathbb{R}^I$ given by,

$$\varphi(x) = (f_i(x))_{i \in I}, x \in X,$$

is a linear topological isomorphism from (X, w) onto its image. Moreover $\varphi(X) \subset \mathbb{R}^I$ is dense. Therefore, as (X, w) is complete, we get that $\varphi(X) = \mathbb{R}^I$ and so φ is a linear topological isomorphism from (X, w) onto $(\mathbb{R}^I, \tau_p)$. Tychonoff's theorem allows us to re-read (ii) in the space $(\mathbb{R}^I, \tau_p)$ as follows:

> *For each set $A \subset \mathbb{R}^I$ and each point $x \in \overline{A}^{\tau_p}$ there is a τ_p-relatively compact set $B \subset A$ such that $x \in \overline{B}^{\tau_p}$.*

A moment of reflection is now enough to realize that if $A \subset \mathbb{R}^I$ intersects each τ_p-compact subset of $\mathbb{R}^I$ in a τ_p-closed set, then A is τ_p-closed. We have proved then that $\mathbb{R}^I$ is a k-space and Exercise C applies to say that I is at most countable and we are done. □

Observe that we can re-phrase Theorem 4.2 to obtain that for every complete non metrizable locally convex space (X, δ) –for $\mathfrak{D}'(\Omega)$, $A(\Omega)$, etc.– there is a set $A \subset X$ such that $0 \in \overline{A}^w$ and there is no bounded set $B \subset A$ such that $0 \in \overline{B}^w$.

If we replace the completeness assumption about δ by metrizability in the theorem above, then we can characterize all subspaces of $\mathbb{R}^{\mathbb{N}}$.

Theorem 4.3. *Let (X, δ) be a metrizable locally convex space. The following statements are equivalent:*

(i) *(X, δ) is linearly and topologically isomorphic to a subspace of $\mathbb{R}^{\mathbb{N}}$;*

(ii) *For each set $A \subset X$ and each point $x \in \overline{A}^w$ there is a bounded set $B \subset A$ such that $x \in \overline{B}^w$.*

Proof. The implication (i)⇒(ii) goes as the proof of (i)⇒(ii) in Theorem 4.2 using that weak topologies induce weak topologies in subspaces, after Hahn-Banach theorem. Let us prove (ii)⇒(i). If (ii) is satisfied then Corollary 4.1 applies to obtain that $w = \delta$ is metrizable and, from here, the reader familiar with these kind of arguments might know already that (X, δ) is linearly and topologically isomorphic to a subspace of $\mathbb{R}^{\mathbb{N}}$. To keep selfcontained the paper we provide a proof of this simple fact. If (X, w) is metrizable then the topological dual X' is at most of countable algebraic dimension. Indeed, for given finitely many vector $y_1^*, y_2^*, \dots, y_m^* \in X'$ and $\varepsilon > 0$ let us write a generic w-neighbourhood of the origin in X as

$$V(0, y_1^*, ..., y_m^*, \varepsilon) := \{x \in X : |y_i^*(x)| \le \varepsilon, j = 1, 2, \dots, m\}.$$

If (X, w) is metrizable then there are sequences $(x_n^*)_n$ in X' and $(\varepsilon_n)_n$ of positive real numbers such that $\{V(0, x_1^*, ..., x_n^*, \varepsilon_n) : n \in \mathbb{N}\}$ a basis of w-neighborhoods of the origin in X. Given any $x^* \in X'$ there is some $N \in \mathbb{N}$ such that

$$V(0, x_1^*, ..., x_N^*, \varepsilon_N) \subset V(0, x^*, 1). \tag{3}$$

We have $\cap_{i=1}^N \ker\ x_i^* \subset V(0, x_1^*, ..., x_N^*, \varepsilon_N)$. This inclusion together with the one in (3) imply that $\cap_{i=1}^N \ker\ x_i^* \subset \ker x^*$. The last implies that x^* is a linear combination of the x_i^*'s, see [7, Lemma in page 10], and consequently X' is at most of countable algebraic dimension. Take $\{f_i : i \in I\}$, I at most countable, an algebraic basis of X'. The map $\varphi : X \to \mathbb{R}^I$ given by,

$$\varphi(x) = (f_i(x))_{i \in I}, x \in X,$$

is a linear topological isomorphism from (X, w) onto its image. The proof is over. □

Quite briefly let us comment that again (i) and (ii) in Theorem 4.3 are also equivalent to the condition:

(iii) *For each set $A \subset X$ and each point $x \in \overline{A}^w$ there is a bounded and countable set $B \subset A$ such that $x \in \overline{B}^w$.*

It also happens to be true that for any metrizable locally convex space (X, δ) the space (X, w) has countable tightness –play a bit with Theorem B and prove it or see [10, Corollary in page 38] or [13, §24.1.6]–. In the metrizable case the combination *bounded* and *countable tighness* for the weak topology is comparative as strong as it was for normed spaces.

It is not possible however, for metrizable (even complete) non normable locally convex spaces, to add to Theorem 4.2 a third condition analogous to (iii) in Theorem 3.2. Indeed, for $\omega = \mathbb{R}^{\mathbb{N}}$ we have that $\omega' = \bigoplus_{n=1}^{\infty} \mathbb{R} = \varphi$ and it is not true that:

> *For each set $A \subset \varphi$ and each point $x \in \overline{A}^{w^*}$ there is a bounded set $B \subset A$ such that $x \in \overline{B}^{w^*}$.*

If the last statement were true then Corollary 4.1 would imply that w^* has to coincide with the topology $\beta(\varphi, \omega)$ on φ of uniform convergence on bounded sets of ω. But this is not the case because ω has bounded sets which are not finite dimensional.

Let us finish the paper with another consequence of our results here, to be more specific, a consequence of Theorem 4.3. In the paper [5] one of us, with J. Orihuela, introduced a large class $\mathfrak{G}$ of locally convex spaces that is stable by the usual operations of countable type and that contains many important spaces –it contains, as said in the introduction, the inductive limits of metrizable spaces, the dual of metrizable spaces, the spaces of distributions $\mathfrak{D}'(\Omega)$ and spaces of analytic functions $A(\Omega)$ in an open Ω of $\mathbb{R}^d$, etc–. A locally convex space (X, δ) is in the class $\mathfrak{G}$ if there is a family $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of subsets in the topological dual X' such that:

$$(a)\ \ X' = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\};$$
$$(b)\ \ A_\alpha \subset A_\beta \text{ when } \alpha \leq \beta \text{ in } \mathbb{N}^\mathbb{N}\ (\alpha \leq \beta \text{ coordinatewise});$$
$$(c)\ \ \text{in each } A_\alpha,\ \text{sequences are } \delta-\text{equicontinuous.}$$

This class $\mathfrak{G}$ is made up of spaces whose compact sets are metrizable and whose w-compact sets behave likewise to the w-compact sets of Banach spaces. The class $\mathfrak{G}$ has gotten the attention of several different authors and pretty recently in the paper [3] it has been proved that for a space (X, δ) in $\mathfrak{G}$, (X, δ) is Fréchet-Urysohn if, and only if, (X, δ) is metrizable. Fréchet-Uryshon property for the weak topology of spaces in $\mathfrak{G}$ is quite restrictive.

Corollary 4.4. *Let (X, δ) be a space in $\mathfrak{G}$. The following statements are equivalent:*

(i) *(X, δ) is linearly and topologically isomorphic to a subspace of $\mathbb{R}^\mathbb{N}$;*
(ii) *(X, w) is a Fréchet-Urysohn space.*

Proof. The implication (i)⇒(ii) is exactly what we wrote for the same implication in Theorems 4.2 and 4.3. The proof for implication (ii)⇒(i) almost matches word by word the proof the correspondent implication in Theorem 4.3, but we need some extra input first. Assume that statement (ii) here is true. Then Corollary 4.1 applies to tell us $(X, w) = (X, \delta)$. But, now read what this information means in this way:

$$\mathfrak{G} \ni (X, \delta) = (X, w) \text{ is Fréchet Urysohn.}$$

We use that in class $\mathfrak{G}$ the Fréchet-Urysohn property is equivalent to metrizability, see [3, Theorem 2.2], and so $(X, \tau) = (X, w)$ is metrizable. Theorem 4.3 says that (X, δ) is linearly and topologically isomorphic to a subspace of $\mathbb{R}^\mathbb{N}$, and the proof finishes. □

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