“Banach Spaces (and topology) II (For Encyclopedia on General Topology -Elsevier)”

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This article is the continuation of “Banach spaces and topology (I)” (referred to as BT(I)). Whereas BT(I) mainly deals with properties of the norm topology and the weak topologies in Banach spaces by themselves, the present article will stress the interplay between properties of weak and norm topologies. To save space, some references are given by the author(s)'s names and Math. Reviews ID numbers. We shall also refer to articles cited in BT(I) as *e.g.* [5,BT(I)]. In this article, Banach spaces are over the reals, unless otherwise indicated.

1. Properties related to weak, weak* and norm topologies. If a Banach space X is WCG or more generally if (X, w) is K-analytic, then (X, w) is a Lindelöf space. An important class of weak-Lindelöf Banach spaces are the so called **weakly Lindelöf determined (WLD)** Banach spaces, and they coincide with Banach spaces with *Corson compact* weak* dual unit ball (Mercourakis-Negrepontis survey [HvM]). They provide a framework where Amir-Lindenstrauss constructions for WCG Banach spaces [1, BT(I)] can be formulated and some of its consequences derived. More general than the notion of weak-Lindelöf space is the following: a Banach space X is said to have the **property ($\mathcal{C}$) (of Corson)** if each family of closed convex subsets with the countable intersection property has non-empty intersection. A Banach space X has property ($\mathcal{C}$) if and only if, whenever $A \subset B_{X^*}$, each element of $\overline{A}^{w^*}$ is in the weak*-closed convex hull of a countable subset of A (Pol, **MR**82a:46022). Whereas property ($\mathcal{C}$) is stable under taking finite products, it is yet an open problem to decide if the product of a weak-Lindelöf Banach space by itself is again weak-Lindelöf.

Given a subset A of a Banach space X, a point $x \in A$ is said to be a **(weak to norm) continuity point of** A if the identity map $id : (A, w) \to (A, \| \|)$ is continuous at x. Weak-compact subsets of Banach spaces have points of continuity. A Banach space X is said to have the **point of continuity property (PCP)** if each nonempty weakly closed and bounded subset A of X has a point of continuity. Edgar and Wheeler [3] proved that a Banach space X has the PCP if and only if (B_X, w) is *hereditarily Baire.* If X is separable, then (B_X, w) is *Polish* if and only if X^* is separable and X has the PCP. A Banach space X is isomorphic to the direct sum of a reflexive subspace and a separable subspace with Polish ball if and only if (B_X, w) is *Čech-complete.* If X is a $(\mathcal{F} \cup \mathcal{G})_\delta$ subset of (X^{**}, w^*), then X has the PCP, and the converse is true if the Banach space X is assumed to be separable, see [3]. Ghoussoub and Maurey proved that if X is a separable Banach space, then X has the PCP if and only if there is a separable subspace Y of X^* such that $(B_X, \sigma(X, Y))$ is Polish, where $\sigma(X, Y)$ is the weakest topology for X that makes $x \mapsto f(x)$ continuous for each $f \in Y$ (Ghoussoub and Maurey **MR**88i:46022).

If C is a convex subset of a vector space, a point $a \in C$ is said to be an **extreme point** of C if it is not the midpoint of any proper segment in C; the set of extreme points of C will be denoted by $ext(C)$. The Krein-Milman theorem states that each compact convex subset K of a locally convex space is the closed convex hull of $ext(K)$, or equivalently, for each point $x \in K$, there exists a regular Borel measure μ on K supported by the closure of $ext(K)$ such that $f(x) = \int_K f\, d\mu$ for each continuous real-valued affine function f on K. When K is metrizable it is possible to choose μ supported by $ext(K)$ (**Choquet's theorem**). Among its consequences, we have the **Rainwater theorem**: a bounded sequence $\{x_n\}$ in a Banach X weak-converges to $a \in X$ if $\{f(x_n)\}$ converges to $f(a)$ for each $f \in ext(B_{X^*})$. More generally, it follows from Simon's inequality (Simons **MR**47#755) that Rainwater theorem remains true when $ext(B_{X^*})$ is replaced by any subset $B \subset S_{X^*}$ with the property that for every $x \in X$ there is $x^* \in B$ such that $\|x\| = x^*(x)$.

Without compactness of C, $ext(C)$ may be empty. A closed convex subset C of a Banach space is said to have the **Krein-Milman property (KMP)** if each nonempty bounded closed convex subset of C has an extreme point. If this is the case, C is the closed convex hull of $ext(C)$. Given a non-empty bounded subset A of a Banach space X, a **slice of** A is a set of the form $\{a \in A : f(a) > \sup\{f(x) : x \in A\} - r\}$ with $f \in X^*$, $r > 0$. A point x of a closed bounded convex subset C is called a **denting point of** C if there are slices of C containing x of arbitrarily small $\| \|$-diameter. Note that each denting point of C is both a continuity and extreme point of C. The converse is true and non-trivial (Lin-Lin-Troyanski, **MR**91g:46016). A non-empty bounded subset A of a Banach space is said to be **dentable** if A has non-empty slices of arbitrarily small diameter. It is known that a Banach space X has the *Radon-Nikodým property (RNP)* if and only if each nonempty bounded closed convex subset of X is dentable (*cf.* [1]). The RNP implies both the KMP and the PCP (Lindenstrauss, *cf.* [1]). Conversely the PCP and KMP implies the RNP (Schachermayer, **MR**89c:46030). It is an open problem if the KMP implies the RNP. This is known to be true for some special cases. For instance this is true for dual Banach spaces (Huff and Morris, **MR**50#14220) and for a Banach space which is isomorphic to its square (Schachermayer, **MR**87e:46032).

Let c be a point of a bounded closed convex set C in a Banach space X and let $f \in X^*$. If $f(x) \le f(c)$ for all $x \in C$, then we say that c is a **support point** of C and f a **support functional** of C. If, for some $g \in X^*$, $g(x) < g(c)$ for all $x \in C$, $x \neq c$, then we say that c is an **exposed point** of C. The point c is a **strongly exposed point** of C if, for some $g \in X^*$, the $\| \|$-diameter of the sets $\{x \in C : g(c) - r < g(x)\}$ tends to 0 as $r \downarrow 0$, and, if this is the case, the functional g is said to **expose** c **strongly**. A strongly exposed point is both a denting point and an exposed point, and an exposed point is both a support point and an extreme point. Bishop-Phelps theorem states that for any nonempty closed convex bounded subset C of a Banach space X, the support functionals of C is norm dense in X^* (*cf.* [2]). Let C be a bounded closed convex subset of a Banach space X with the RNP. Then C is the closed convex hull of its strongly exposed points, and the set of all continuous functionals that strongly expose points of C is a dense $\mathcal{G}_\delta$ subset of $(X^*, \| \|)$. The last conclusion has a strong converse: Let X be a Banach space. If, for each bounded closed convex subset C, the set of all support functionals of C is of the second category in $(X^*, \| \|)$, then X has the RNP (Bourgain-Stegall (ca. 1975), see [1]).

If in a dual Banach space X^* we replace the weak by the weak* topology in the definition of the PCP, we arrive at the notion of weak* point of continuity property (w^*-PCP for short). The w^*-PCP, the RNP, and the KMP are equivalent for a dual Banach space X^* (see [1]). If this is the case, X^* satisfies a stronger dentability condition: every nonempty convex bounded subset of X^* has nonempty relatively w^*-open slices of arbitrarily small diameter. There are examples of dual Banach spaces having the PCP but not the RNP: therefore PCP and w^*-PCP are not equivalent in dual Banach spaces (see [3]).

2. Smoothness and renorming. In a Banach space, one can change the norm to an equivalent one without affecting the norm, weak and weak* topologies. Therefore, in many instances, topological questions reduce to that of replacing the given norm with one with better geometric properties.

(A) Classical renorming results. The norm in any Hilbert space X satisfies the **Parallelogram Law**: $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ for all $x, y \in X$, and this law characterizes Hilbert spaces amongst complex Banach spaces (Von Neumann 1935). Notice that, when $x, y \in S_X$, the distance $\|x-y\|$ depends only on how close $\|2^{-1}(x+y)\|$ is to 1. In particular, the midpoint of two distinct points of S_X is never on S_X. The norm $\| \|$ of a Banach space is said to be **rotund** or **strictly convex** if the unit sphere does not contain a non-trivial segment. The norm of a Hilbert space also enjoys the **smoothness** property of being Fréchet differentiable away from 0. Properties of rotundity and smoothness are in *"duality"* through Šmulyan criterion type results, see [2, Theorem I.1.4]. For instance, the norm of X is Gâteaux differentiable away from 0 if the dual norm on X^* is rotund.

Given $\varepsilon > 0$, a **dyadic ε-tree with root** $x \in X$ **of length** $N \in \mathbb{N} \cup \{\infty\}$ is a family $\{x(s)\}$ of elements of X indexed by $s \in \{-1,1\}^{<N+1}$ such that $x = x(\emptyset)$, $x(s) = 2^{-1}(x(s,-1) + x(s,1))$ and $\|x(s,-1) - x(s,1)\| \geq \varepsilon$ for each $s \in \{-1,1\}^{<N}$. An infinite dyadic ε-tree is not dentable. Thus a Banach space with the RNP does not contain an infinite bounded dyadic ε-tree for any $\varepsilon > 0$ but the converse is not true in general (Bourgain and Rosenthal, **MR**82g:46044). A Banach space X is said to be **superreflexive** if for each $\varepsilon > 0$ there is $N(\varepsilon) \in \mathbb{N}$ such that each dyadic ε-tree contained in the unit ball B_X has length $N \leq N(\varepsilon)$. Superreflexive Banach spaces are reflexive. If X is superreflexive, then X^* is also superreflexive.

The **modulus of convexity** of a norm $\| \|$ is defined for $\varepsilon \in [0,2]$ as follows: $\delta(\varepsilon) = \inf\{1 - 2^{-1}\|x+y\| : x, y \in S_X, \|x-y\| \geq \varepsilon\}$. The norm is said to be **uniformly rotund** if $\delta(\varepsilon) > 0$ for $\varepsilon \in (0,2]$. The norm of X is uniformly rotund if and only if the dual norm on X^* is **uniformly smooth**, *i.e.* $\lim_{t\to 0} t^{-1}(\|x+th\| - \|x\|)$ exists uniformly in $(x,h) \in S_X \times S_X$. A Banach space with a uniformly rotund norm is necessarily reflexive. The spaces L^p with $1 < p < \infty$ are both uniformly rotund and uniformly smooth. A celebrated theorem of Enflo (**MR**49#1073) states that X is superreflexive if and only if X admits an equivalent uniformly rotund norm (equivalently, a uniformly smooth norm). By a probabilistic method, Pisier (**MR** 52#14940) has improved Enflo's result by showing that the modulus of convexity of an equivalent uniformly rotund norm can be made to satisfy $\delta(\varepsilon) \geq C\varepsilon^p$ with $C > 0$, $p \geq 2$.

Given a Banach space X, $C \subset X$ a non-empty bounded set and $\varepsilon > 0$ define

$$D_\varepsilon(C) = \{x \in C : \|\cdot\|\text{-diam}(S) > \varepsilon \text{ for each slice } S \text{ of } C \text{ containing } x\}.$$

By induction we can define a transfinite sequence of sets by letting: $B_{\varepsilon,0} = B_X, B_{\varepsilon,\alpha+1} = D_\varepsilon(B_{\varepsilon,\alpha})$ and, for a limit ordinal α, $B_{\varepsilon,\alpha} = \bigcap\{B_{\varepsilon,\beta} : \beta < \alpha\}$. Then X is superreflexive if and only if for every $\varepsilon > 0$ there is $n(\varepsilon) < \omega$ such that $B_{\varepsilon,n(\varepsilon)} = \emptyset$ (Lancien used this fact in (**MR**96e:46009) to give a non-probabilistic proof of Pisier's result.) The Banach space X has the RNP if and only if for each $\varepsilon > 0$ there is an ordinal $\alpha(\varepsilon)$ such that $B_{\varepsilon,\alpha(\varepsilon)} = \emptyset$. If $\alpha(\varepsilon)$ is a countable ordinal for each $\varepsilon > 0$, then X has an equivalent locally uniformly rotund norm (Lancien, **MR**94h:46026). The norm $\|\ \|$ of a Banach space X is said to be **locally uniformly rotund (LUR)** if for each $x \in S_X$ and each $\varepsilon > 0$ there is a $\delta > 0$ such that $\|x-y\| < \varepsilon$ whenever $y \in S_X$ and $\|2^{-1}(x+y)\| > 1-\delta$. Certain topological properties of X ensure the existence of equivalent LUR norms, for instance X being WLD. The norm of X is Fréchet differentiable away from 0 if the dual norm on X^* is LUR. This fact provides a standard technique for finding an equivalent Fréchet differentiable norm as, for instance, for Banach spaces X such that X^* is weakly countably determined. See subsection (C) for further discussion on LUR-renorming.

(B) Asplund spaces. A Banach space X is said to be an **Asplund space** (resp. **weak Asplund space**) if each continuous convex real-valued function defined on a convex open subset of X is Fréchet (resp. Gâteaux) differentiable at all points of a dense $\mathcal{G}_\delta$ subset of its domain (*cf.* Sec. 3(A), BT(I)). Each separable Banach space is weak Asplund (Mazur 1933), and if X^* is separable, then X is indeed an Asplund space (Asplund, **MR** 37#6754).

The Banach space X is Asplund if and only if X^* has the RNP, and this is the case if and only if each separable subspace of X has the separable dual (Namioka-Phelps and Stegall, see [1]). In particular, for a compact Hausdorff space K the space $C(K)$ is Asplund if and only if K is scattered. Preiss (**MR**91g:46051) established that real-valued Lipschitz functions on Asplund spaces are Fréchet differentiable on a dense subset. In the case of a separable space X, there is a very tight connection between renormability of the space and the Asplund property. Indeed, for separable X the following are all equivalent: separability of the dual space X^*; the existence on X of an equivalent norm with LUR dual norm; the existence on X of an equivalent norm which is Fréchet differentiable in $X \setminus \{0\}$. Day asked if the Fréchet renormability is necessary or sufficient condition for a non-separable Banach space to be Asplund. In one direction we have that X is an Asplund space (resp. weak Asplund) whenever it has an equivalent Fréchet (resp. Gâteaux) differentiable norm, Ekeland and Lebourg (**MR**55#4254) (resp. Preiss-Phelps-Namioka, **MR**92h:46021). The converse is not true (Haydon, **MR**91h:46045). The result by Ekeland and Lebourg cited above needs only the existence of a non-trivial Fréchet differentiable function of bounded support. This type of functions are called **bump functions**. The question whether the Asplund property is characterized by the existence of a bump function remains open.

A Banach space X admits C^k**-smooth partitions of unity** ($k \in \mathbb{N}$ or $k = +\infty$) when each norm-open cover of X has a *partition of unity* subordinate to it consisting of Fréchet C^k-smooth functions. Toruńczyk (**MR**49#4016) showed that a Banach space admits C^k-smooth partitions of unity if and only if there is a set Γ and a homeomorphic embedding ϕ from X into $c_0(\Gamma)$ such that the γ-th coordinate function $x \mapsto \phi(x)(\gamma)$ is C^k-smooth on X, for each $\gamma \in \Gamma$. As a consequence he obtained that any Hilbert space admits C^∞-smooth partitions of unity, proved the existence of high order smooth partitions of unity on L^p-spaces and showed that any reflexive space admits C^1 -smooth partitions of unity. When X is WCG and X admits a C^k-bump function then X admits a C^k-smooth partition of unity (Godefroy-Troyanski-Whitfield-Zizler, **MR**85d:46020). All spaces $C_0(\Upsilon)$ for Υ a tree considered in [8] also admits C^∞-smooth partitions of unity. It is an open problem if the existence of a C^k-bump function in X implies that the space admits C^k-smooth partitions of unity. Hajek and Haydon have shown recently that this is the case when X is a space $C(K)$, to be published.

(C) Fragmentability conditions. The notion of fragmentability provides a tool to measure how far apart the weak and the norm topologies of a Banach space are. This term was introduced by Jayne and Rogers (**MR**87a:28011) in their work on Borel selectors for functions taking values among subsets of Banach spaces. Let (T, τ) be a topological space, d a metric on T and $\varepsilon > 0$; the space T is said to be ε**-fragmented by** d if for each nonempty subset C of T there exists a τ-open subset V of T with $C \cap V \neq \emptyset$ and d-diam$(C \cap V) < \varepsilon$. When T is ε-fragmented by d for each $\varepsilon > 0$, we say that T is **fragmented by** d. Each bounded subset of a Banach space with the RNP is fragmented by the norm, but the converse is not true. However this is the case in dual Banach spaces: the dual ball (B_{X^*}, w^*) is fragmented by the norm if and only if X is an Asplund space (Namioka-Phelps, *cf.* [1]), *i.e.* X^* has the RNP. For Asplund spaces X, Jayne-Rogers' selection theorem provides a *first Baire class* map $f : X \to S_{X^*}$ with $\langle f(x), x\rangle = \|x\|$ for every $x \in X$; this selector for the duality map is a main tool to deal with the Amir-Lindenstrauss type construction in X^*. For further discussion, see [2] and [5].

There is not much known about the permanence of the weak Asplund property under the standard operations of Banach spaces. It is not even known if $X \times \mathbb{R}$ is weak Asplund when X is. However there is a much better subclass of weak Asplund spaces due to Stegall defined below (*cf.* [5] and references therein). Let T and S be topological spaces. A map $F : S \to 2^T$ is said to be **usco** if F is *upper semicontinuous* (*i.e.* whenever $U \subset T$ is open $\{s \in S : F(s) \subset U\}$ is open in S) and for each $s \in S$, $F(s)$ is compact and non-empty. A *Tychonoff space* T is said to belong to **class** $\mathcal{S}$ if, whenever B is a Baire space and $F : B \to 2^T$ is usco, there is a selector f of F (*i.e.* a map $f : B \to T$ such that $f(b) \in F(b)$ for each $b \in B$) which is continuous at each point of a dense $\mathcal{G}_\delta$ subset of B. It is shown that, if T is fragmented by a metric, then $T \in \mathcal{S}$. A Banach space X is said to belong to class $\tilde{\mathcal{S}}$ **(Stegall's class)** if $(B_{X^*}, w^*) \in \mathcal{S}$. Each Banach space in $\tilde{\mathcal{S}}$ is weak Asplund and this class has very good permanence properties. When a compact space K is fragmented, the Banach space $C(K)$ belongs to $\tilde{\mathcal{S}}$ (Ribarska, **MR**89e:54063). Compact spaces K such that $C(K)$ is weak Asplund are sequentially compact and contain dense $\mathcal{G}_\delta$ completely metrizable subset (Čoban-Kenderov, **MR**91c:90119). A Banach space with a Gâteaux differentiable norm belongs to $\tilde{\mathcal{S}}$ (Preiss-Phelps-Namioka, *op. cit.*).

A topological space (X, τ) is said to be **σ-fragmentable by a metric** d if for each $\varepsilon > 0$, $X = \bigcup_{n=1}^{\infty} X_{n,\varepsilon}$ where each $X_{n,\varepsilon}$ is ε-fragmented by d. The class of Banach spaces such that (X, w) is σ-fragmentable by the norm metric has been extensively studied following the work of Jayne, Namioka and Rogers (**MR**93i:46027, 94c:46028 and 94c:4601). Such a Banach space is said to be **σ-fragmentable**. They have established the connection of this notion with descriptive topology, renormings and *property* $\mathcal{N}^*$. A norm in a Banach space X is called a **Kadec norm** if the norm and the weak topologies coincide on the unit sphere S_X. Each LUR norm is a Kadec norm because each point of the unit sphere of a LUR norm is a strongly exposed point of the unit ball. If a Banach space admits an equivalent Kadec norm, X is a Borel subset of (X^{**}, w^*) and the Borel sets for the weak and the norm topology of X coincide (Edgar, **MR**81d:28016). A Banach space X which is obtained through the *Souslin operation* applied to Borel subsets of $(X^{**}, weak^*)$ is called **weak-Čech-analytic**. Each weak-Čech-analytic Banach space is σ-fragmentable (Jayne-Namioka-Rogers, *op. cit.*). Therefore for a Banach space we have

LUR renormable $\Rightarrow$ Kadec renormable $\Rightarrow$ weak-Čech-analytic $\Rightarrow$ σ-fragmentable

No example of σ-fragmentable Banach space without an equivalent Kadec norm is known. The first example of a Banach space with Kadec norm and without equivalent LUR norm is given by Haydon [8]. Kenderov and Moors have proved in (**MR**2001f:46026) that a Banach space X is σ-fragmentable if and only if (X, w) is fragmented by a metric whose topology is stronger than the weak topology. Their argument is game theoretic.

A modification of σ-fragmentability characterizes LUR renormability of Banach spaces: X is LUR renormable if and only if for each $\varepsilon > 0$, $X = \bigcup_{n=1}^{\infty} X_{n,\varepsilon}$ where each $X_{n,\varepsilon}$ is the union of its slices of diameter less than ε (Molto-Orihuela-Troyanski, **MR**98e:46011 and Raja [Mathematika **46** (1999),343–358]). In particular, if every point of the unit sphere is a denting point then the space is LUR renormable (Troyanski, **MR**86g:46030). When the previous sets $X_{n,\varepsilon}$ are simply the union of relatively weak open sets of diameter less than ε then X is said to have the **JNR property**. If a Banach space admits an equivalent Kadec norm , then it also has the JNR property. Conversely, the JNR property in X implies the existence of a symmetric homogeneous and weakly lower semicontinuous real-valued function F defined on X with $\| \, \| \leq F(\cdot) \leq 3 \| \, \|$ and such that the norm and the weak topologies coincide on $\{x \in X : F(x) = 1\}$, (Raja **MR**2000i:46003). A Banach space X has the JNR property if and only if there is a sequence $\{A_n\}$ of subsets of X such that the family $\{A_n \cap W : W \text{weak-open}, n \in \mathbb{N}\}$ is a *network* for the norm topology. A Banach space admits an equivalent Kadec norm if and only if the sequence $\{A_n\}$ above can be chosen to be convex (Raja **MR**2000i:46003). The JNR property is equivalent to the fact that the weak topology has a *σ-relatively discrete* network (*cf.* Hansell [Serdica Math. J. **27** (2001), 1–66] and Molto-Orihuela-Troyanski-Valdivia, **MR**2000b:46031).

3. Heritage of S. Banach and the structural theory of Banach spaces. A sequence of vectors $\{e_n : n \in \mathbb{N}\}$ is called **a basis** of a Banach space X if every $x \in X$ has a unique representation as $x = \sum a_i x_i$ with scalars a_i. If the convergence of the series is unconditional the basis is called an **unconditional basis**. In that case every infinite subset M of integers gives a continuous linear projection $P_M(\sum a_i x_i) = \sum_{i \in M} a_i x_i$. Each infinite dimensional Banach space contains an infinite dimensional subspace with a basis and Banach asked if each separable Banach space has a basis. A famous counterexample of Enflo [4] solved even a stronger version of the problem dealing with the approximation property of Grothendieck. After Enflo's counterexample, and for a long time, it was conjectured that each infinite dimensional Banach space contains copies of c_0 or ℓ^p or, at least, an infinite dimensional subspace with an unconditional basis. This is the case for Banach spaces with a C^∞-smooth bump function, (Deville, **MR**90m:46023) and for the class of Orlicz spaces, Lindenstrauss and Tzafriri [9]. Nevertheless Tsirelson constructed a reflexive Banach space not containing ℓ^p for $1 < p < +\infty$. A Banach space X is called stable if for any two bounded sequences $\{x_n\}$ and $\{y_n\}$ in X and for any two free ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$

$$\lim_{n,\mathcal{U}} \lim_{m,\mathcal{V}} \|x_n + y_m\| = \lim_{m,\mathcal{V}} \lim_{n,\mathcal{U}} \|x_n + y_m\|.$$

Krivine and Maurey (**MR**83a:46030) have proved that stable Banach spaces contain, for every $\varepsilon > 0$, a subspace which is $(1+\varepsilon)$-isometric to ℓ_p for some p. Tsirelson's construction has been modified by Schlumprecht (**MR**93h:46023) opening the door for the construction by Gowers and Maurey [7] of a separable reflexive Banach space X that does not have any infinite-dimensional subspace with an unconditional basis (see also Gowers, **MR**94j:46024). Gowers-Maurey's example X has the property that, for each subspace Z, any continuous linear projection P of X onto Z is trivial, *i.e.*, either $\dim PZ < \infty$ or $\dim Z/PZ < \infty$. A Banach space with this property is said to be **hereditarily indecomposable**. A hereditarily indecomposable Banach space is not isomorphic to any of its proper subspaces and it provides an answer to Banach's *"hyperplane problem"* asking whether each infinite-dimensional Banach space is isomorphic to its hyperplanes, [6]. Recently Argyros has been able to construct non separable hereditarily indecomposable Banach spaces.

A dichotomy result by Gowers (**MR**97m:46017) makes clear that hereditarily indecomposable spaces are not just pathological counterexamples but they are essential in the structural theory of general Banach spaces: Each infinite dimensional Banach space has a hereditarily indecomposable subspace or a subspace with an unconditional basis. The proof is combinatorial and it uses infinite Ramsey theory which turns out to be an important tool in the infinite-dimensional setting. As a consequence Gowers solved the classical *"homogeneous space prob-*

lem" by showing that ℓ^2 is the only Banach space which is isomorphic to each infinite dimensional subspace. Another well-known open problem was the following: Assume that X and Y are Banach spaces each of them isomorphic to a complemented subspace (*i.e.* the image of a continuous linear projection) of the other. Must X be isomorphic to Y? Again Gowers gave a counterexample: a Banach space X isomorphic to $X \oplus X \oplus X$ and not to $X \oplus X$ (**MR** 97d:46009).

4. Final comment. We refer the interested reader to the references here and in BT(I) with special mention to the book by Benyamini and Lindenstrauss [3,BT(I)] where many of the topics we have commented are expanded and the non-linear functional analysis theory is presented. We also refer to the *Handbook of the Geometry of Banach spaces (Two volumes) W. B. Johnson and J. Lindenstrauss eds. Elsevier, Amsterdam (2001)*– to appear.

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