ON BIRKHOFF INTEGRABILITY FOR SCALAR FUNCTIONS AND VECTOR MEASURES

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Abstract. The Bartle-Dunford-Schwartz integral for scalar functions with respect to vector measures is characterized by means of Riemann-type sums based on partitions of the domain into countably many measurable sets. In this setting, two natural notions of integrability (Birkhoff integrability and Kolmogoroff integrability) turn out to be equivalent to Bartle-Dunford-Schwartz integrability.

1. Introduction

Bartle, Dunford and Schwartz [1] developed a theory of integration of scalar functions with respect to vector measures in order to provide an analogue of Riesz's representation theorem for weakly compact operators defined on a Banach space of continuous functions on a compact

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Hausdorff space. Further studies on this notion of integrability were carried out by Lewis [14, 15] and Kluvanek and Knowles [12]. More recently, the spaces of scalar functions integrable with respect to vector measures have shown to play an important role within the theory of Banach lattices, see e.g. [4], [5] and [9].

The original definition of the Bartle-Dunford-Schwartz integral is based on approximation by simple functions, whereas Lewis' approach uses scalar measures through a barycentric formula. In this paper we provide two characterizations of Bartle-Dunford-Schwartz integrability involving Riemann-type sums based on partitions of the domain into countably many measurable sets, with a *Birkhoff integral* flavour. The notion of Birkhoff integrability for vector functions with respect to non-negative finite measures was introduced in [2] and has attracted the attention of several authors pretty recently, being an important notion in vector integration, see e.g. [3, 11] and [17, 19]. The difficulties of adapting Birkhoff's approach to the case of scalar functions and vector measures were already pointed out by Lewis in [15, p. 307].

We relate the Bartle-Dunford-Schwartz integral and the S^**-integral*. This notion of integral has its origins in Kolmogoroff's approach to integration theory [13, 21] and was intensively studied by Dobrakov [7] in the more general case of vector functions and operator-valued vector measures, nowadays known as the *Dobrakov integral.* Further research on the S^*-integral has been done recently in [18, 20]. We stress that no measurability assumption is needed to define the notion of S^*-integrability.

Our main results state that a scalar function is Bartle-Dunford-Schwartz integrable (with respect to a vector measure m) if and only if it is S^*-integrable (see Theorem 13), which in turn is also equivalent to saying that f is measurable and there is a partition of the domain into countably many measurable sets such that, for every finer (countable) partition $\{A_n\}_n$ and any choice of points $\omega_n \in A_n$, the series $\sum_n f(\omega_n)m(A_n)$ is unconditionally convergent (see Theorem 9).

Notation and terminology. Our standard reference on vector measures and integration is [6]. Throughout this paper $m : \Sigma \longrightarrow X$ will be a countably additive vector measure defined on a σ-algebra Σ on a non-empty set Ω with values in a real Banach space X. We denote by $\|\cdot\|$ the norm of X if it is needed explicitly. We denote by X' the dual of X and by $B(X')$ the closed unit ball of X'. The semivariation of m is the (monotone) set function defined by

$$\|m\|(A) := \sup\left\{|\langle m, x'\rangle|(A) : \ x' \in B(X')\right\}, \quad A \in \Sigma,$$

where $|\langle m, x' \rangle|$ is the total variation measure of the scalar measure $\langle m, x' \rangle$ given by $\langle m, x' \rangle (A) := \langle m(A), x' \rangle$, for all $A \in \Sigma$. It is well-known that $\|m\|(\Omega) < \infty$, since m is strongly additive, cf. [6], Proposition 11 on p. 4 and Corollary 19 on p. 9. A control measure of m is a non-negative finite measure μ on Σ such that $\|m\|(A) = 0$ if and only if $\mu(A) = 0$. It is a classical result of Bartle, Dunford and Schwartz [1] that a control measure always exists. Moreover, Rybakov proved (see [6, Theorem 2, p. 268]) that a control measure μ of m can be taken of the form $\mu = |\langle m, x' \rangle|$ for some $x' \in B(X')$. Such control measures are called *Rybakov control measures*. We will need Rybakov control measures in order to show that S^*-integrability implies measurability (see Proposition 12).

Recall that a Σ-measurable function $f : \Omega \longrightarrow \mathbb{R}$ is called *Bartle-Dunford-Schwartz integrable* (with respect to m) if:

(1) f is integrable with respect to $|\langle m, x' \rangle|$ for all $x' \in X'$; and
(2) for each $A \in \Sigma$ there exists a vector $\int_A f dm \in X$ (necessarily unique and called the *integral* of f over A) such that

$$\left\langle \int_A f dm, x' \right\rangle = \int_A f d \langle m, x' \rangle$$

for all $x' \in X'$.

We use here this equivalent definition of Bartle-Dunford-Schwartz integrability obtained by Lewis in [14, Theorem 2.4] (cf. [1, Definition 2.5]).

2. The results

We denote by $\mathcal{P}(\Omega)$ the set of all countable partitions of Ω in Σ. We say that $\mathcal{A} \in \mathcal{P}(\Omega)$ is finer than $\mathcal{B} \in \mathcal{P}(\Omega)$ (and we write $\mathcal{A} \succeq \mathcal{B}$ for short) if each $A \in \mathcal{A}$ is a subset of some $B \in \mathcal{B}$. Given a function $f : \Omega \longrightarrow \mathbb{R}$, denote also by $\mathcal{P}(\Omega, f)$ the set of all $\mathcal{A} := \{A_n\}_n \in \mathcal{P}(\Omega)$ such that the series $\sum_n f(\omega_n)m(A_n)$ converges unconditionally in X for every choice of points $(\omega_n)_n \in \prod_n A_n$.

As we mentioned in the introduction we will use quite simple techniques involving countable partitions as a tool in proving results about integrability of scalar functions with respect to vector measures. This approach goes back to the pioneering papers by Frechet [10], Kolmogoroff [13, 21] and Birkhoff [2].

Following [7, Definition 1] in the particular case of scalar functions, we have:

Definition 1. *A function $f : \Omega \longrightarrow \mathbb{R}$ is called S^*-integrable (with respect to m), with integral $s^*\int_\Omega f dm \in X$, if for every $\varepsilon > 0$ there is*

$\mathcal{A}_0 \in \mathcal{P}(\Omega)$ such that $\mathcal{A} = \{A_n\}_n \in \mathcal{P}(\Omega, f)$ for every $\mathcal{A} \succeq \mathcal{A}_0$, and

$$\left\| \sum_{n=1}^{\infty} f(\omega_n) m(A_n) - s^*\!\!\int_{\Omega} f dm \right\| \leq \varepsilon$$

for every choice of points $(\omega_n)_n \in \prod_n A_n$.

Note that the vector $s^*\!\int_{\Omega} f dm$ in the above definition is necessarily unique.

The notion of S^*-integrability becomes the analogue of Birkhoff integrability in our setting, that is, scalar functions and vector measures (cf. [3, Proposition 2.6]). Lewis [15, p. 307] already noted that the original definition of Birkhoff (for vector functions and non-negative finite measures) does not work properly in our setting: for instance, it may happen that $\mathcal{B} \in \mathcal{P}(\Omega, f)$ but $\mathcal{A} \notin \mathcal{P}(\Omega, f)$ for some $\mathcal{A} \succeq \mathcal{B}$, as the following simple example shows.

Example 2. Let λ be the Lebesgue measure defined on the σ-algebra $\mathcal{M}$ of all Lebesgue measurable subsets of the interval $[0, 1]$. Define the vector measure

$$m : \mathcal{M} \longrightarrow \mathbb{R}, \quad m(E) := \lambda\left(E \cap \left[0, \frac{1}{2}\right]\right) - \lambda\left(E \cap \left[\frac{1}{2}, 1\right]\right).$$

Any function $f : [0, 1] \longrightarrow \mathbb{R}$ satisfies $f(\omega)\, m([0, 1]) = 0$, for all $\omega \in [0, 1]$, and so the trivial partition $\{[0, 1]\}$ belongs to $\mathcal{P}([0, 1], f)$. For a particular function f it is easy to construct partitions $\mathcal{A} \in \mathcal{P}([0,1])$ such that $\mathcal{A} \notin \mathcal{P}([0,1], f)$.

In order to compare the Bartle-Dunford-Schwartz integral with the S^*-integral it is convenient to use the following concept, which is somehow an adaptation of the notion of Birkhoff integrability to our context of scalar functions and vector measures (cf. [3, Lemma 3.2.(iii)]).

Definition 3. *We say that a Σ-measurable function $f : \Omega \longrightarrow \mathbb{R}$ is B-integrable (with respect to m) if there exists $\mathcal{A}_0 \in \mathcal{P}(\Omega)$ such that $\mathcal{A} \in \mathcal{P}(\Omega, f)$ for every $\mathcal{A} \succeq \mathcal{A}_0$.*

The following lemma about scalar absolutely convergent series is well-known and will be used in the proof of Lemma 5.

Lemma 4. *Let $(\alpha_n)_n$ be a sequence of real numbers such that the series $\sum_n \alpha_n$ is absolutely convergent. Let $\{J_k\}_k \in \mathcal{P}(\mathbb{N})$ and denote by β_k the sum $\beta_k := \sum_{n\in J_k} \alpha_n$. Then the series $\sum_k \beta_k$ is absolutely convergent and $\sum_{n=1}^{\infty} \alpha_n = \sum_{k=1}^{\infty} \beta_k$.*

Given a function $f : \Omega \longrightarrow \mathbb{R}$ and $A \subset \Omega$, we write

$$\operatorname{osc}(f, A) := \sup \{|f(\omega) - f(\varsigma)| : \omega, \varsigma \in A\}.$$

Lemma 5. *Let* $f : \Omega \longrightarrow \mathbb{R}$ *be a function and let* $\mathcal{A} = \{A_n\}_n$ *and* $\mathcal{B} = \{B_k\}_k$ *be two partitions in* $\mathcal{P}(\Omega, f)$ *such that* $\mathcal{A} \succeq \mathcal{B}$. *Suppose* $\operatorname{osc}(f, B_k) \leq K$ *for every* $k = 1, 2, ...$ *Then*

$$\left\| \sum_{n=1}^{\infty} f(\omega_n) m(A_n) - \sum_{k=1}^{\infty} f(\varsigma_k) m(B_k) \right\| \leq K \|m\|(\Omega)$$

for all choices $(\omega_n)_n \in \prod_n A_n$ *and* $(\varsigma_k)_k \in \prod_k B_k$.

Proof. Since $\mathcal{A} \succeq \mathcal{B}$, each set A_n is a subset of a unique set B_k and we can write $B_k = \bigcup_{n \in J_k} A_n$, where $J_k := \{n \in \mathbb{N} : A_n \subseteq B_k\}$ for all $k = 1, 2, \ldots$ For each $n \in \mathbb{N}$ we denote by $k(n)$ the unique $k \in \mathbb{N}$ such that $A_n \subseteq B_k$. Take two sequences $(\omega_n)_n \in \prod_n A_n$ and $(\varsigma_k)_k \in \prod_k B_k$. Note that $\omega_n \in A_n \subseteq B_{k(n)}$ for all $n = 1, 2, \ldots$ Note also that $k(n) = k$ for all $n \in J_k$. Since $m(B_k) = \sum_{n \in J_k} m(A_n)$, we have

$$\begin{aligned} \sum_{k=1}^{\infty} f(\varsigma_k) m(B_k) &= \sum_{k=1}^{\infty} f(\varsigma_k) \left(\sum_{n \in J_k} m(A_n) \right) \\ &= \sum_{k=1}^{\infty} \left(\sum_{n \in J_k} f(\varsigma_{k(n)}) m(A_n) \right). \end{aligned} \tag{1}$$

Let us show that the series $\sum_n f(\varsigma_{k(n)}) m(A_n)$ is unconditionally convergent. Each of its terms $f(\varsigma_{k(n)}) m(A_n)$ can be written as

$$\begin{aligned} f(\varsigma_{k(n)}) m(A_n) &= \big(f(\varsigma_{k(n)}) - f(\omega_n) + f(\omega_n) \big) m(A_n) \\ &= \big(f(\varsigma_{k(n)}) - f(\omega_n) \big) m(A_n) + f(\omega_n) m(A_n). \end{aligned}$$

On the one hand, the series $\sum_n \big(f(\varsigma_{k(n)}) - f(\omega_n)\big)\, m(A_n)$ converges unconditionally in X, since $\sum_n m(A_n)$ does and

$$\big|f(\varsigma_{k(n)}) - f(\omega_n)\big| \leq \operatorname{osc}(f, B_{k(n)}) \leq K, \quad n = 1, 2, \ldots$$

because both $\varsigma_{k(n)}, \omega_n \in B_{k(n)}$ for all $n = 1, 2, \ldots$ On the other hand, $\sum_n f(\omega_n) m(A_n)$ converges unconditionally too, because we have chosen $\mathcal{A} \in \mathcal{P}(\Omega, f)$. Therefore, the series $\sum_n f(\varsigma_{k(n)}) m(A_n)$ is unconditionally convergent as well. Moreover, we have

$$\sum_{n=1}^{\infty} f(\varsigma_{k(n)}) m(A_n) = \sum_{k=1}^{\infty} f(\varsigma_k) m(B_k) \tag{2}$$

because, for every $x' \in X'$, Lemma 4 together with (1) yield

$$\begin{aligned}
\left\langle \sum_{n=1}^{\infty} f(\varsigma_{k(n)}) m(A_n), x' \right\rangle &= \sum_{n=1}^{\infty} f(\varsigma_{k(n)})\, \langle m(A_n), x' \rangle \\
&= \sum_{k=1}^{\infty} \left(\sum_{n \in J_k} f(\varsigma_{k(n)}) \langle m(A_n), x' \rangle \right) \\
&= \sum_{k=1}^{\infty} f(\varsigma_k)\, \langle m(B_k), x' \rangle \\
&= \left\langle \sum_{k=1}^{\infty} f(\varsigma_k) m(B_k), x' \right\rangle .
\end{aligned}$$

Now, for every $x' \in B(X')$ we obtain from (2)

$$\begin{aligned}
&\left|\left\langle \sum_{n=1}^{\infty} f(\omega_n)m(A_n) - \sum_{k=1}^{\infty} f(\varsigma_k)m(B_k), x' \right\rangle\right| \\
&\qquad = \left|\left\langle \sum_{n=1}^{\infty} f(\omega_n)m(A_n) - \sum_{n=1}^{\infty} f(\varsigma_{k(n)})m(A_n), x' \right\rangle\right| \\
&\qquad \leq \sum_{n=1}^{\infty} \left|f(\omega_n) - f(\varsigma_{k(n)})\right| \left|\langle m, x'\rangle\right| (A_n) \\
&\qquad \leq K \left\|m\right\| (\Omega),
\end{aligned}$$

and we conclude that

$$\left\| \sum_{n=1}^{\infty} f(\omega_n)m(A_n) - \sum_{k=1}^{\infty} f(\varsigma_k)m(B_k) \right\| \leq K\|m\|(\Omega).$$

□

Theorem 6. *Let $f : \Omega \longrightarrow \mathbb{R}$ be a B-integrable function. Then f is S^*-integrable. Moreover, if for some $\mathcal{A} = \{A_n\}_n \in \mathcal{P}(\Omega, f)$ we have* $\operatorname{osc}(f, A_n) \leq K$ *for every $n = 1, 2, \ldots,$ then*

$$\left\| \sum_{n=1}^{\infty} f(\omega_n)m(A_n) - s^*\!\!\int_{\Omega} f dm \right\| \leq K\|m\|(\Omega) \tag{3}$$

for every choice $(\omega_n)_n \in \prod_n A_n$.

Proof. Let us consider a sequence $(\varepsilon_n)_n$ of positive real numbers decreasing to zero. For $\varepsilon_1 > 0$ we take a partition $\mathcal{I}_1 := \{I_k^1\}_k$ of $\mathbb{R}$ into countably many intervals with $\operatorname{length}(I_k^1) \leq \varepsilon_1$. Since $0 < \varepsilon_2 < \varepsilon_1$ we can refine the former partition $\mathcal{I}_1$ in order to obtain a new partition

$\mathcal{I}_2 := \{I_k^2\}_k$ of $\mathbb{R}$ into countably many intervals with length$(I_k^2) \leq \varepsilon_2$. In this way we can obtain a sequence $(\mathcal{I}_n)_n$, with $\mathcal{I}_n := \{I_k^n\}_k$, of partitions of $\mathbb{R}$ into countably many intervals with length$(I_k^n) \leq \varepsilon_n$ in such a way that $\mathcal{I}_1 \preceq \cdots \preceq \mathcal{I}_n \preceq \cdots$ Now, since f is Σ-measurable, we have $B_k^n := f^{-1}(I_k^n) \in \Sigma$ for all $k, n = 1, 2, \ldots$ and we obtain a sequence $(\mathcal{B}_n)_n$ in $\mathcal{P}(\Omega)$, just taking $\mathcal{B}_n := \{B_k^n\}_k$, such that $\mathcal{B}_1 \preceq \cdots \preceq \mathcal{B}_n \preceq \cdots$ and $\operatorname{osc}(f, B_k^n) \leq \varepsilon_n$ for all $k, n = 1, 2, \ldots$ Moreover, since f is B-integrable, we can assume that $\mathcal{A} \in \mathcal{P}(\Omega, f)$ for all $\mathcal{A} \succeq \mathcal{B}_1$.

For each $n = 1, 2 \ldots$, we consider a fixed choice of points $(\omega_k^n)_k \in \prod_k B_k^n$, and the sum $x_n := \sum_{k=1}^{\infty} f(\omega_k^n) m(B_k^n)$ of the associated unconditionally convergent series.

Step 1. The sequence $(x_n)_n$ is a Cauchy sequence (converges) in X. In fact, if we apply Lemma 5 to two arbitrary partitions $\mathcal{B}_p$ and $\mathcal{B}_q$, with $p, q \geq 1$, we obtain that $\|x_p - x_q\| \leq \varepsilon_{\min\{p,q\}} \|m\|(\Omega)$. Set $x := \lim_{n\to\infty} x_n$ and note that

$$\|x_p - x\| \leq \varepsilon_p \|m\|(\Omega), \quad p = 1, 2, \ldots \tag{4}$$

Step 2. f is S^*-integrable with integral x. For a given $\varepsilon > 0$, fix $n_0 \in \mathbb{N}$ such that $\varepsilon_{n_0}\|m\|(\Omega) < \dfrac{\varepsilon}{2}$ and consider the partition $\mathcal{B}_{n_0}$. Then, if $\mathcal{A} = \{A_n\}_n \succeq \mathcal{B}_{n_0}$ and $(\omega_n)_n \in \prod_n A_n$, then the series $\sum_n f(\omega_n) m(A_n)$ is unconditionally convergent since $\mathcal{A} \succeq \mathcal{B}_{n_0} \succeq \mathcal{B}_1$. Moreover, its sum satisfies

$$\begin{aligned}\left\|\sum_{n=1}^{\infty} f(\omega_n)\, m(A_n) - x\right\| &\leq \left\|\sum_{n=1}^{\infty} f(\omega_n)\, m(A_n) - x_{n_0}\right\| + \|x_{n_0} - x\| \\ &\leq 2\varepsilon_{n_0} \|m\|(\Omega) < \varepsilon,\end{aligned}$$

which follows from inequality (4) and Lemma 5 applied to the partitions $\mathcal{A}$ and $\mathcal{B}_{n_0}$.

The *moreover* of the statement of the theorem follows from Lemma 5 in a similar way. □

It is known that any bounded Σ-measurable function $g : \Omega \longrightarrow \mathbb{R}$ is Bartle-Dunford-Schwartz integrable, see [14, p. 161]. The following lemma establishes that such a function is also S^*-integrable.

Lemma 7. *Let $g : \Omega \longrightarrow \mathbb{R}$ be a bounded Σ-measurable function. Then g is B-integrable and its S^*-integral coincides with its Bartle-Dunford-Schwartz integral.*

Proof. For every $\{C_n\}_n \in \mathcal{P}(\Omega)$, since the series $\sum_n m(C_n)$ is unconditionally convergent, the boundedness of g ensures that the series $\sum_n g(\omega_n) m(C_n)$ is unconditionally convergent for every choice of points $(\omega_n)_n \in \prod_n C_n$. This shows that g is B-integrable.

Now, in order to check the equality of the integrals it suffices to prove that for each $x' \in X'$ the equality $\left\langle s^*\!\!\int_\Omega g dm, x'\right\rangle = \int_\Omega g d\langle m, x'\rangle$ holds true. For a given $\varepsilon > 0$ we consider a partition $\{C_n\}_n \in \mathcal{P}(\Omega)$ such that $\operatorname{osc}(g, C_n) < \dfrac{\varepsilon}{2\|m\|(\Omega)}$ for every $n = 1, 2, \ldots$, and a choice of points $(\omega_n)_n \in \prod_n C_n$. Then for every $x' \in B(X')$ we have

$$\begin{aligned}
&\left|\left\langle s^*\!\!\int_\Omega g dm, x'\right\rangle - \int_\Omega g d\langle m, x'\rangle\right| \\
&\leq \left|\left\langle s^*\!\!\int_\Omega g dm - \sum_{n=1}^{\infty} g(\omega_n) m(C_n), x'\right\rangle\right| \\
&\qquad\qquad + \left|\sum_{n=1}^{\infty}\left(g(\omega_n)\langle m(C_n), x'\rangle - \int_{C_n} g d\langle m, x'\rangle\right)\right| \\
&\leq \left\|s^*\!\!\int_\Omega g dm - \sum_{n=1}^{\infty} g(\omega_n) m(C_n)\right\| + \sum_{n=1}^{\infty}\int_{C_n} |g(\omega_n) - g|\, d|\langle m, x'\rangle| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

To estimate $\left\|s^*\!\!\int_\Omega g dm - \sum_{n=1}^{\infty} g(\omega_n) m(C_n)\right\|$ we have used the inequality (3) of Theorem 6. As $\varepsilon > 0$ is arbitrary, we get

$$\left\langle s^*\!\!\int_\Omega g dm, x'\right\rangle = \int_\Omega g d\langle m, x'\rangle .$$

□

Lemma 8. *Let $g : \Omega \longrightarrow \mathbb{R}$ be a function of the form $g := \sum_{n=1}^{\infty} \alpha_n \chi_{A_n}$, where $(\alpha_n)_n$ is a sequence of real numbers and $\{A_n\}_n \in \mathcal{P}(\Omega)$. The following statements are equivalent:*

(a) *g is Bartle-Dunford-Schwartz integrable;*

(b) *g is B-integrable;*

(c) *g is S^*-integrable;*

(d) *the series $\sum_n \alpha_n m(B_n)$ is unconditionally convergent for each sequence of measurable sets $(B_n)_n$ such that $B_n \subset A_n$ for every $n = 1, 2, \ldots$*

In this case, the S^*-integral of g coincides with its Bartle-Dunford-Schwartz integral.

Proof. (a) $\Longrightarrow$ (b) Let us consider $\{C_k\}_k \in \mathcal{P}(\Omega)$ finer than $\{A_n\}_n$. Thus g is constant on each set C_k, and then, for each choice of points $(\omega_k)_k \in \prod_k C_k$, the series

$$\sum_k g(\omega_k) m(C_k) = \sum_k \int_{C_k} g dm$$

is unconditionally convergent in X, see [14, Theorem 2.2].

(b) $\Longrightarrow$ (c) Follows from Theorem 6.

(c) $\Longrightarrow$ (d) We can find $\{C_k\}_k \in \mathcal{P}(\Omega, g)$ finer than $\{B_n, A_n \setminus B_n\}_n$. Fix $(\omega_k)_k \in \prod_k C_k$. Since $\sum_k g(\omega_k) m(C_k)$ is unconditionally convergent, the same holds for its subseries $\sum_{\omega_k \in B} g(\omega_k) m(C_k)$, where $B := \bigcup_{n=1}^{\infty} B_n$. It is clear that $\sum_n \alpha_n m(B_n)$ is obtained from $\sum_{\omega_k \in B} g(\omega_k) m(C_k)$ by *introducing parentheses*, and so it is unconditionally convergent too.

(d) $\Longrightarrow$ (a) The sequence $(g_n)_n$ of simple functions $g_n := \sum_{k=1}^{n} \alpha_k \chi_{A_k}$ converges to g pointwise. In order to prove that g is Bartle-Dunford-Schwartz integrable we only need to check that, for each $E \in \Sigma$, the sequence

$$\left(\int_E g_n dm\right)_n = \left(\sum_{k=1}^{n} \alpha_k m(A_k \cap E)\right)_n$$

converges in X, see [14, Theorem 2.4]. For this, it is enough to take $B_n := A_n \cap E$ in (d). In fact, [14, Theorem 2.4] also yields the equality

$$\int_\Omega g dm = \sum_{n=1}^{\infty} \alpha_n m(A_n).$$

Finally, it is not difficult to check that the S^*-integral of g is necessarily $\sum_{n=1}^{\infty} \alpha_n m(A_n)$. Indeed, from the implication (a) $\Longrightarrow$ (b) we know that $\mathcal{C} = \{C_k\}_k \in \mathcal{P}(\Omega, g)$ for every $\mathcal{C} \succeq \mathcal{A} = \{A_n\}_n$. Moreover, since $\operatorname{osc}(g, A_n) = 0$ for every $n = 1, 2, \dots$, Lemma 5 tells us that, for every choice $(\omega_k)_k \in \prod_k C_k$, we have

$$\sum_{k=1}^{\infty} g(\omega_k) m(C_k) = \sum_{n=1}^{\infty} \alpha_n m(A_n).$$

Then it is clear from the definition that $s^*\!\!\int_\Omega g dm = \sum_{n=1}^{\infty} \alpha_n m(A_n)$, and the proof is over. $\square$

We arrive at our first *Birkhoff-type* characterization of Bartle-Dunford-Schwartz integrability. Both notions of integrability deal with Σ-measurable functions.

Theorem 9. *Let $f : \Omega \longrightarrow \mathbb{R}$ be a function. Then f is B-integrable if and only if it is Bartle-Dunford-Schwartz integrable. In this case*

$$s^*\!\!\int_A f\,dm = \int_A f\,dm, \quad A \in \Sigma.$$

Proof. The Σ-measurability of f allows us to write $f = g + h$, where $g := \sum_{n=1}^{\infty} \alpha_n \chi_{A_n}$ for some sequence of real numbers $(\alpha_n)_n$ and some $\{A_n\}_n \in \mathcal{P}(\Omega)$, and h is Σ-measurable and bounded. The result now follows from Lemmas 7 and 8. □

The following folk characterization of measurability will be used in the proof of Lemma 11. Recall that a function $f : \Omega \longrightarrow \mathbb{R}$ is said to be μ-measurable, μ being a non-negative measure on (Ω, Σ), if there exist a sequence of simple functions $(\varphi_n)_n$ and a μ-null set N such that $\lim_{n\to\infty} \varphi_n(\omega) = f(\omega)$ for all $\omega \notin N$.

Lemma 10. *Suppose μ is a non-negative finite measure on (Ω, Σ). Let $f : \Omega \longrightarrow \mathbb{R}$ be a function. The following conditions are equivalent:*

(i) *f is μ-measurable.*

(ii) *For each $\varepsilon > 0$ and each $E \in \Sigma$ with $\mu(E) > 0$ there is $B \subset E$, $B \in \Sigma$ with $\mu(B) > 0$, such that $\operatorname{osc}(f, B) \leq \varepsilon$.*

It is a well-known fact that every Σ-measurable function is μ-measurable, and for every μ-measurable function f there exists a Σ-measurable function g such that $f = g$ μ-a.e. If the measure space (Ω, Σ, μ) is complete, both notions of measurability coincide.

The following lemma goes back to Fréchet [10] and, naturally, it is part of the proof that S^*-integrability is equivalent to Lebesgue integrability when m is a non-negative finite measure. We include the proof here for the sake of completeness.

Lemma 11. *Suppose μ is a non-negative finite measure on (Ω, Σ). Let $f : \Omega \longrightarrow \mathbb{R}$ be an S^*-integrable function (with respect to μ). Then f is μ-measurable.*

Proof. We will apply the criterion of Lemma 10. Fix $\varepsilon > 0$ and $E \in \Sigma$ with $\mu(E) > 0$. Since f is S^*-integrable with respect to μ, there is $\{A_n\}_n \in \mathcal{P}(\Omega, f)$ such that

$$\left| \sum_{n=1}^{\infty} f(\omega_n)\mu(A_n) - \sum_{n=1}^{\infty} f(\varsigma_n)\mu(A_n) \right| \leq \frac{\varepsilon\mu(E)}{2}$$

whenever $(\omega_n)_n$ and $(\varsigma_n)_n$ are in $\prod_n A_n$, or equivalently

$$\sum_{n=1}^{\infty} |f(\omega_n) - f(\varsigma_n)| \, \mu(A_n) \leq \frac{\varepsilon\mu(E)}{2} \tag{5}$$

whenever $(\omega_n)_n$ and $(\varsigma_n)_n$ are in $\prod_n A_n$. Take $N \in \mathbb{N}$ such that

$$\sum_{n=1}^{N} \mu(A_n \cap E) > \frac{\mu(E)}{2}$$

and define $\mathcal{I} := \{1 \leq n \leq N : \mu(A_n \cap E) > 0\}$. We claim that there is $n \in \mathcal{I}$ for which $\operatorname{osc}(f, A_n \cap E) \leq \varepsilon$. Indeed, suppose not. Then for each $n \in \mathcal{I}$ we can take $\omega_n, \varsigma_n \in A_n \cap E$ such that $f(\omega_n) - f(\varsigma_n) > \varepsilon$, hence

$$\frac{\varepsilon\mu(E)}{2} < \sum_{n \in \mathcal{I}} (f(\omega_n) - f(\varsigma_n))\, \mu(A_n \cap E) \leq \sum_{n \in \mathcal{I}} (f(\omega_n) - f(\varsigma_n))\, \mu(A_n),$$

which contradicts inequality (5) and finishes the proof. □

Proposition 12. *Let $f : \Omega \longrightarrow \mathbb{R}$ be an S^*-integrable function (with respect to m). Then f is μ-measurable with respect to any control measure μ of m.*

Proof. Obviously, it is enough to prove that f is μ-measurable, where μ is any Rybakov control measure of m, that is, a control measure of the form $\mu := |\langle m, x_0' \rangle|$ for some $x_0' \in B(X')$. By Hahn's decomposition theorem, there is $H \in \Sigma$ such that

$$\begin{aligned} \langle m, x_0' \rangle (A) &\geq 0 \ \text{ for all } A \in \Sigma,\ A \subset H, \text{ and} \\ \langle m, x_0' \rangle (B) &\leq 0 \ \text{ for all } B \in \Sigma,\ B \subset \Omega \setminus H. \end{aligned}$$

The function $f|_H$ is S^*-integrable with respect to the restriction of m to the σ-algebra $\Sigma_H := \{C \cap H : C \in \Sigma\}$ on H and, therefore, it is also S^*-integrable with respect to the non-negative finite measure

μ^+, defined as the restriction of $\langle m, x_0' \rangle$ to Σ_H. Therefore, $f|_H$ is μ^+-measurable (Lemma 11). A similar argument shows that $f|_{\Omega\setminus H}$ is μ^--measurable, where μ^- is the non-negative finite measure defined as the restriction of $-\langle m, x_0' \rangle$ to the σ-algebra $\Sigma_{\Omega\setminus H} := \{C \setminus H : C \in \Sigma\}$. It follows that f is μ-measurable and the proof is over. □

We are now able to prove the *equivalence* of S^*-integrability and Bartle-Dunford-Schwartz integrability. The *if part* of the following theorem can bee seen as a particular case of a result of Dobrakov [7] saying that a *measurable* vector function is S^*–integrable with respect to an operator-valued vector measure if and only if it is Dobrakov integrable. Recall that Dobrakov and Bartle-Dunford-Schwartz integrability coincide for scalar functions and vector measures, see the papers by Dobrakov and Panchapagesan [8] and Panchapagesan [16].

Theorem 13. *Let $f : \Omega \longrightarrow \mathbb{R}$ be a function. Then f is S^*-integrable if and only if it coincides $\|m\|$-a.e. with a Bartle-Dunford-Schwartz integrable function. If, in addition, m is complete (i.e. any / every control measure of m is complete), then f is S^*-integrable if and only if it is Bartle-Dunford-Schwartz integrable.*

Proof. The *if part* follows from Theorems 6 and 9. The *only if part* is a consequence of Proposition 12 and Theorem 9. □

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