THE BIRKHOFF INTEGRAL AND THE PROPERTY OF BOURGAIN

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ABSTRACT. In this paper we study the Birkhoff integral of functions $f : \Omega \longrightarrow X$ defined on a complete probability space (Ω, Σ, μ) with values in a Banach space X. We prove that if f is bounded then its Birkhoff integrability is equivalent to the fact that the set of compositions of f with elements of the dual unit ball $Z_f = \{\langle x^*, f\rangle : x^* \in B_{X^*}\}$ has the Bourgain property. A non necessarily bounded function f is shown to be Birkhoff integrable if, and only if, Z_f is uniformly integrable and has the Bourgain property. As a consequence it turns out that the range of the indefinite integral of a Birkhoff integrable function is relatively norm compact. We characterize the weak Radon-Nikodým property in dual Banach spaces via Birkhoff integrable Radon-Nikodým derivatives. We also point out that a recently introduced notion of unconditional Riemann-Lebesgue integrability coincides with the notion of Birkhoff integrability. Some other applications are given.

1. INTRODUCTION

Our concern here is to study the Birkhoff integral. Birkhoff integrability lies strictly between Bochner and Pettis integrability. We link Birkhoff integrability with the Bourgain property —a measurability notion— and then we succeed in replacing Pettis integrability by Birkhoff integrability in some classical results. We stress that the Birkhoff integral is: *a)* more restrictive than the Pettis integral and then closer to the Bochner integral; *b)* not as restrictive as the Bochner integral because every Riemann integrable function is Birkhoff integrable; *c)* a bit more tangible than the Pettis integral because it is defined using a limit (as the classical integrals are defined) instead of a barycentric formula.

Throughout this paper (Ω, Σ, μ) is a complete probability space and $(X, \|\cdot\|)$ is a real Banach space. The starting point of our investigation goes back to the beautiful paper by Birkhoff [1], dated in 1935, in which he studied the integration of functions $f : \Omega \longrightarrow X$. Birkhoff's idea was to extend, to the setting of Banach-valued functions, *"Fréchet's elegant interpretation of the Lebesgue integral"*. For some reason or other the approaches used by Fréchet and Birkhoff in setting up an integral were almost passed over.

Fréchet considers in [4] functions $f : \Omega \longrightarrow \mathbb{R}$ and for each partition Γ of Ω into countably many sets (A_n) of Σ assigns a relative *upper* and *lower* integral by the expressions

$$J^*(f, \Gamma) = \sum_n \sup f(A_n)\, \mu(A_n) \quad \text{and} \quad J_*(f, \Gamma) = \sum_n \inf f(A_n)\, \mu(A_n),$$

respectively, assuming both series are well defined and absolutely convergent. One has the inequality $J_*(f, \Gamma) \le J^*(f, \Gamma')$ whenever $J_*(f, \Gamma)$ and $J^*(f, \Gamma')$ are defined. Therefore the intersection of the "relative integral ranges"

$$J_*(f, \Gamma) \le x \le J^*(f, \Gamma),$$

for variable Γ is not empty. This intersection is a single point x if, and only if, f is Lebesgue integrable and $x = \int_\Omega f\, d\mu$. Fréchet presented his approach to the Lebesgue integral with the following sentence, [4, page 249]:

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> *This way of presenting the theory of integration due to M. Lebesgue has the advantage, over the way M. Lebesgue presented his theory himself, that is very much close to the views of Riemann-Darboux to which many students are familiar with.*

Fréchet's views inspired Birkhoff to give the following definition in [1, Definition 3]:

Definition 1. *Let $f : \Omega \longrightarrow X$ be a function. If Γ is a partition of Ω into countably many sets (A_n) of Σ, the function f is called summable with respect to Γ if the restriction $f|_{A_n}$ is bounded whenever $\mu(A_n) > 0$ and the set of sums*

$$J(f, \Gamma) = \Big\{ \sum_n f(t_n)\mu(A_n) : \ t_n \in A_n \Big\} \tag{1}$$

is made up of unconditionally convergent series. The function f is said integrable if for every $\varepsilon > 0$ there is a partition $\Gamma = (A_n)$ of Ω in Σ for which f is summable and $\|\cdot\|$-diam$(J(f, \Gamma)) < \varepsilon$.

When f is integrable according to the previous definition, then the Birkhoff integral $(B)\int_\Omega f\, d\mu$ of f is the only point in the intersection

$$\bigcap \{\overline{\text{co}(J(f, \Gamma))} : \ f \text{ is summable with respect to } \Gamma\},$$

[1, Theorem 12]. As said at the beginning of the introduction, it has been known for long that

$$f \text{ Bochner integrable} \quad \Rightarrow \quad f \text{ Birkhoff integrable} \quad \Rightarrow \quad f \text{ Pettis integrable},$$

and none of the reverse implications holds in general, see [1], [14] and [15]. If f is Birkhoff integrable then $(B)\int_\Omega f\, d\mu =$ (Pettis) $\int_\Omega f\, d\mu$ and both integrals are, from now onwards, simply written as $\int_\Omega f\, d\mu$. The first example of a Pettis integrable function which is not Birkhoff integrable was obtained by Phillips [15, Example 10.2]. When the range space X is separable, Birkhoff and Pettis integrability are the same [14].

The Bourgain property, Definition 3 in Section 2, is a nice tool of measurability for families of functions widely studied over the years, see amongst others [7, 12, 16] and the references therein. If $\mathcal{F} \subset \mathbb{R}^\Omega$ has the Bourgain property, then $\mathcal{F}$ is stable, see [19, 9-5-4]. While speaking about a *bounded* function $f : \Omega \longrightarrow X$, its Bochner integrability is equivalent to strong measurability; a deep result by Talagrand, [19, Theorem 6-1-2], establishes that f is Pettis integrable when $Z_f = \{\langle x^*, f\rangle : \ x^* \in B_{X^*}\}$ is stable. We prove that the Bourgain property is to Birkhoff integrability what strong measurability is to Bochner integrability: Theorem 2.4 establishes that f is Birkhoff integrable if, and only if, Z_f has the Bourgain property. We complete Section 2 with Proposition 2.6 where Birkhoff integrability is characterized via the existence of the limit *"refining partitions"* of the net of sums $J(f, \Gamma)$. This last proposition shows that the notion of unconditional Riemann-Lebesgue integrability, recently studied in [10], coincides with the notion of Birkhoff integrability.

Section 3 is mainly devoted to prove Theorem 3.5 —our characterization of Birkhoff integrable functions that are non necessarily bounded— and Theorem 3.8 —the characterization of the weak Radon-Nikodým property for dual Banach spaces via Birkhoff integrability. Technically speaking, Theorems 3.5 and 3.8 need of Lemmas 3.1, 3.2 and Corollary 3.4, which might be of independent interest. Lemma 3.2 also corrects Theorem 15 in [1] that appears to be wrong. We prove that the range of the indefinite integral of a Birkhoff integrable function is relatively norm compact in Corollary 3.6. Our approach to this subject allows us to easily obtain some other classical results as well as widen and strengthen some other results spread out in [5], [9] and [16]. We finish the paper by paying a visit to the weak Radon-Nikodým property (WRNP, for short) in dual Banach spaces. Recall that a dual Banach space X^* is said to have the WRNP (see [11] and [3, Definition 5.8]) if for every complete probability space (Ω, Σ, μ) and every μ-continuous countably additive vector measure $\nu : \Sigma \longrightarrow X^*$ of σ-finite variation there is a Pettis integrable function $f : \Omega \longrightarrow X^*$ such that

$$\nu(E) = \int_E f \, d\mu$$

for every $E \in \Sigma$. Efforts of several mathematicians (Musiał, Ryll-Nardzewski, Janicka and Bourgain) led to the well-known characterization of Banach spaces X without copies of ℓ^1 as those for which X^* has the WRNP, see [3, 11, 12, 19] and the references therein. Our Theorem 3.8 states that for dual spaces X^* the presence of WRNP entitle us to change Pettis integrable Radon-Nikodým derivatives to Birkhoff integrable Radon-Nikodým derivatives. In the very last result of the paper, Proposition 3.10, we characterize the Riemann-Lebesgue integrable functions studied in [9, 10] using the Bourgain property. As a consequence, Riemann-Lebesgue integrable functions do satisfy the law of large numbers [20].

Our notation is standard and our standard references are [2], [17] and [19]. We write X^* to denote the dual of our Banach spaces X. The weak (resp. weak*) topology of X (resp. X^*) is denoted by w (resp. w^*). B_X is the unit ball of X. For any set $B \subset X$ we write $\|\cdot\|$-diam$(B) = \sup_{x,y\in B} \|x-y\|$ and $\mathrm{co}(B)$ for its convex hull. For the probability space (Ω, Σ, μ) we write $\mathcal{L}^1(\mu)$ (resp. $\mathcal{L}^\infty(\mu)$) to denote the space of real μ-integrable (resp. measurable μ-essentially bounded) functions defined on Ω and $L^1(\mu)$ (resp. $L^\infty(\mu)$) for the corresponding Banach space of equivalence classes with its usual norm $\|\cdot\|_1$ (resp. $\|\cdot\|_\infty$). The topology of pointwise convergence in $\mathbb{R}^\Omega$ is denoted by τ_p.

2. The Birkhoff integral for bounded functions

For a bounded set $B \subset X$ we use the notation $\|B\| = \sup\{\|x\| : x \in B\}$. Given a sequence $B_1, B_2, \dots$ of sets of X, the symbol $\sum_{n=1}^\infty B_n$ denotes a formal series. The series $\sum_{n=1}^\infty B_n$ is said to be *unconditionally convergent* provided that for every choice $b_n \in B_n$, $n \in \mathbb{N}$, the series $\sum_{n=1}^\infty b_n$ is unconditionally convergent in X. Equivalently, $\sum_{n=1}^\infty B_n$ is unconditionally convergent if and only if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\|\sum_{i\in S} B_i\| < \varepsilon$ for every finite set $S \subset \mathbb{N} \setminus \{1, \dots, N\}$.

Given a function $f : \Omega \longrightarrow X$ and a countable partition $\Gamma = (A_n)$ of Ω in Σ, the notion "*f is summable with respect to Γ*", recalled in the introduction, is simply said now: *$f|_{A_n}$ is bounded whenever $\mu(A_n) > 0$ and the series of sets $\sum_{n=1}^\infty f(A_n)\mu(A_n)$ is unconditionally convergent*. For such an f and Γ we retain the notation $J(f, \Gamma)$ defined in equation (1) and for a given *choice* $T = (t_n)$ in Γ (i.e. $t_n \in A_n$ for every n), we write

$$S(f, \Gamma, T) := \sum_n f(t_n)\mu(A_n).$$

As usual, we say that another partition Γ' of Ω, into countably many elements of Σ, is finer than Γ when each element of Γ' is contained in some element of Γ.

The following lemma is a special case of [1, Theorem 9]. We provide a proof for the convenience of the readers.

Lemma 2.1. *Let $f : \Omega \longrightarrow X$ be a function. Suppose that there is a countable partition Γ of Ω in Σ such that f is summable with respect to Γ. If Γ' is any countable partition of Ω in Σ finer than Γ, then f is summable with respect to Γ' and*

$$\overline{\mathrm{co}(J(f, \Gamma'))} \subset \overline{\mathrm{co}(J(f, \Gamma))}. \tag{2}$$

Proof. Write $\Gamma = (A_n)$ and $\Gamma' = (A_{n,k})$, with $\bigcup_k A_{n,k} = A_n$ for every n. Now, set $B_n := f(A_n)\mu(A_n)$ and $B_{n,k} := f(A_{n,k})\mu(A_{n,k})$. We will show first that $\sum_{n,k} B_{n,k}$ is unconditionally convergent.

Fix $\varepsilon > 0$. Since $\sum_n B_n$ is unconditionally convergent, there is $N \in \mathbb{N}$ such that

$$\Big\|\sum_{n\in S} B_n\Big\| < \frac{\varepsilon}{2} \tag{3}$$

for every finite set $S \subset \mathbb{N} \setminus \{1, \dots, N\}$. Take

$$M > \max\{\|f(A_i)\| : 1 \le i \le N, \mu(A_i) > 0\}.$$

There is $K \in \mathbb{N}$ big enough such that

$$\sum_{n=1}^{N} \sum_{k>K} \mu(A_{n,k}) < \frac{\varepsilon}{2M}. \tag{4}$$

We claim that

$$\Big\| \sum_{(n,k)\in S} B_{n,k} \Big\| < \varepsilon$$

for every finite set $S \subset (\mathbb{N} \times \mathbb{N}) \setminus (\{1, \dots, N\} \times \{1, \dots, K\})$. Indeed, for such an S, let us write

$$S' := \{(n, k) \in S : \ 1 \le n \le N\} \quad \text{and} \quad S'' = \{(n, k) \in S : \ n > N\}.$$

On the one hand, inequality (4) applies to obtain

$$\Big\| \sum_{(n,k)\in S'} B_{n,k} \Big\| < \frac{\varepsilon}{2}.$$

On the other hand, if we define $N' = \max\{n > N : \text{there is } k \text{ with } (n, k) \in S\}$, then some computations and inequality (3) give us (with the convention $0/0 = 0$)

$$\begin{aligned}
\Big\| \sum_{(n,k)\in S''} B_{n,k} \Big\| &\le \Big\| \sum_{(n,k)\in S''} \frac{\mu(A_{n,k})}{\mu(A_n)} B_n \Big\| \\
&\le \Big\| \sum_{N<n\le N'} \mathrm{co}(B_n \cup \{0\}) \Big\| = \Big\| \mathrm{co}\Big(\sum_{N<n\le N'} \big(B_n \cup \{0\}\big)\Big) \Big\| \\
&= \Big\| \sum_{N<n\le N'} \big(B_n \cup \{0\}\big) \Big\| = \sup_{F\subset\{N+1,\dots,N'\}} \Big\| \sum_{k\in F} B_k \Big\| < \frac{\varepsilon}{2}.
\end{aligned} \tag{5}$$

The claim is proved and therefore f is summable with respect to Γ'.

To finish the proof we will show that $J(f, \Gamma') \subset \overline{\mathrm{co}(J(f, \Gamma))}$ by contradiction. Suppose that the previous inclusion does not hold. Then for some choice T' in Γ' we have $S(f, \Gamma', T') \notin \overline{\mathrm{co}(J(f, \Gamma))}$. The Hahn-Banach separation theorem ensures us of the existence of $x^* \in X^*$ such that

$$\langle x^*, S(f, \Gamma', T')\rangle > \sup\{\langle x^*, y\rangle : \ y \in J(f, \Gamma)\}. \tag{6}$$

At the same time we have

$$\begin{aligned}
\langle x^*, S(f, \Gamma', T')\rangle &\le \sum_{n,k} \sup_{A_n} \langle x^*, f\rangle\, \mu(A_{n,k}) = \sum_{n} \sup_{A_n} \langle x^*, f\rangle\, \mu(A_n) \\
&= \sup\{\langle x^*, S(f, \Gamma, T)\rangle : \ T \text{ choice in } \Gamma\} = \sup\{\langle x^*, y\rangle : \ y \in J(f, \Gamma)\},
\end{aligned}$$

which contradicts inequality (6) and the proof is finished. □

The Birkhoff property as defined below is a natural condition to characterize Birkhoff integrability of $f : \Omega \longrightarrow X$ in terms of

$$Z_f = \{\langle x^*, f\rangle : \ x^* \in B_{X^*}\}.$$

Definition 2. *We say that a family $\mathcal{F} \subset \mathbb{R}^\Omega$ has the Birkhoff property if for every $\varepsilon > 0$ there is a countable partition $\Gamma = (A_n)$ of Ω in Σ such that for each $t_k, t'_k \in A_k$, $k \in \mathbb{N}$, we have*

$$\Big|\sum_{k=1}^m f(t_k)\mu(A_k) - \sum_{k=1}^m f(t'_k)\mu(A_k)\Big| < \varepsilon \tag{7}$$

for every $m \in \mathbb{N}$ and every $f \in \mathcal{F}$.

Observe that if $\mathcal{F}$ satisfies (7), then each $f \in \mathcal{F}$ is bounded on A_n whenever $\mu(A_n) > 0$. Consequently, $\mathcal{F}$ has the Birkhoff property if, and only if, for every $\varepsilon > 0$ there is a countable partition $\Gamma = (A_n)$ such that

$$\sum_{\mu(A_n)>0} |\cdot|\text{-diam}(f(A_n))\mu(A_n) < \varepsilon$$

for every $f \in \mathcal{F}$. Let us notice that if $\mathcal{F}$ has the Birkhoff property then its absolutely convex hull $\operatorname{aco}(\mathcal{F})$ and its pointwise closure $\overline{\mathcal{F}}^{\tau_p}$ also have the Birkhoff property.

Proposition 2.2. *Let $f : \Omega \longrightarrow X$ be a function. The following conditions are equivalent:*

(i) *f is Birkhoff integrable;*
(ii) *f is summable with respect to some countable partition Γ_0 of Ω in Σ and Z_f has the Birkhoff property.*

Proof. The implication (i)⇒(ii) is obvious and thus we will only prove (ii)⇒(i). Fix $\varepsilon > 0$. Since Z_f has the Birkhoff property, there is a countable partition $\Gamma_1 = (A_n)$ of Ω in Σ such that

$$\sum_{\mu(A_n)>0} |\cdot|\text{-diam}(\langle x^*, f\rangle(A_n))\mu(A_n) < \varepsilon$$

for every $x^* \in B_{X^*}$. Take $\Gamma = (B_m)$ finer than both Γ_0 and Γ_1. Since f summable with respect to Γ_0, then f is also summable with respect to Γ after Lemma 2.1. On the other hand, since Γ is finer than Γ_1, we have

$$\begin{aligned}\sum_{\mu(B_m)>0} &|\cdot|\text{-diam}(\langle x^*,f\rangle(B_m))\mu(B_m)\\ &\le \sum_{\mu(A_n)>0}\sum_{B_m\subset A_n} |\cdot|\text{-diam}(\langle x^*, f\rangle(A_n))\mu(B_m) < \varepsilon\end{aligned} \tag{8}$$

for every $x^* \in B_{X^*}$. Therefore, if T and T' are choices in Γ inequality (8) implies that

$$\|S(f, \Gamma, T) - S(f, \Gamma, T')\| \le \varepsilon,$$

and consequently f is Birkhoff integrable and the proof is over. □

Some readers familiar with the Bourgain property might have realized by now of the relationship of the Birkhoff and Bourgain properties when the latter is viewed suitably. First we recall the definition of the Bourgain property.

Definition 3. *We say that a family $\mathcal{F} \subset \mathbb{R}^\Omega$ has the Bourgain property if for every $\varepsilon > 0$ and every $A \in \Sigma$ with $\mu(A) > 0$ there are $B_1, \dots, B_n \subset A$, $B_i \in \Sigma$ with $\mu(B_i) > 0$, such that for every $f \in \mathcal{F}$*

$$\inf_{1\le i\le n} |\cdot|\text{-diam}(f(B_i)) < \varepsilon.$$

Lemma 2.3. *Let $\mathcal{F} \subset \mathbb{R}^\Omega$ be a family of functions. The following statements hold:*

(i) *if $\mathcal{F}$ has the Birkhoff property, then $\mathcal{F}$ has the Bourgain property;*
(ii) *if $\mathcal{F}$ is uniformly bounded and has the Bourgain property, then $\mathcal{F}$ has the Birkhoff property.*

Proof. In order to prove (i) fix $\varepsilon > 0$ and $A \in \Sigma$ with $\mu(A) > 0$. By hypothesis there is a countable partition $\Gamma = (A_n)$ of Ω in Σ such that for each $t_k, t'_k \in A_k$, $k \in \mathbb{N}$, we have

$$\Big|\sum_{k=1}^{n} f(t_k)\mu(A_k) - \sum_{k=1}^{n} f(t'_k)\mu(A_k)\Big| < \frac{\varepsilon\mu(A)}{2} \tag{9}$$

for every $n \in \mathbb{N}$ and every $f \in \mathcal{F}$.

Fix $n \in \mathbb{N}$ big enough such that $\sum_{i=1}^{n} \mu(A \cap A_i) > \mu(A)/2$. We can suppose without loss of generality that $\mu(A \cap A_i) > 0$ for all $1 \leq i \leq n$. Now, we prove by contradiction that for every $f \in \mathcal{F}$

$$\inf_{1\leq i\leq n} |\cdot|\text{-diam}(f(A \cap A_i)) \leq \varepsilon. \tag{10}$$

If for some $f_0 \in \mathcal{F}$ inequality (10) does not hold, then for each $1 \leq i \leq n$ we can select points $t_i, t'_i \in A \cap A_i$ such that $f_0(t_i) - f_0(t'_i) > \varepsilon$. Thus, we conclude that

$$\frac{\varepsilon\mu(A)}{2} < \sum_{i=1}^{n} \big(f_0(t_i) - f_0(t'_i)\big)\mu(A_i),$$

which contradicts inequality (9) and finishes the proof of (i).

The proof of (ii) imitates the proof of (d)⇒(a) in [7, Proposition IV.8] but dealing now with an abstract probability space and with oscillations instead of *essential* oscillations. Fix $\varepsilon > 0$ and define the uniformly bounded set

$$K_\varepsilon = \Big\{g \in [0,1]^\Omega : \ g \text{ is measurable}, \ \int_\Omega g \, d\mu \geq \varepsilon\Big\}.$$

Its canonical image $\hat{K}_\varepsilon$ in $L^1(\mu)$ is uniformly integrable, $\|\cdot\|_1$-bounded and weakly closed, hence weakly compact, [2, Theorem 15, p. 76]. The set $A^g := \{w \in \Omega : g(w) > 0\}$ is of positive μ-measure for every $g \in K_\varepsilon$. Since $\mathcal{F}$ has the Bourgain property, there are sets $A_1^g, \dots, A_{n(g)}^g \subset A^g$ with $A_i^g \in \Sigma$ and $\mu(A_i^g) > 0$ such that

$$\inf_{1\leq i\leq n(g)} |\cdot|\text{-diam}(f(A_i^g)) \leq \varepsilon \tag{11}$$

for every $f \in \mathcal{F}$. Note that the canonical image in $L^1(\mu)$ of the set

$$V(g) = \bigcap_{i=1}^{n(g)} \Big\{h \in \mathcal{L}^1(\mu) : \ \int_{A_i^g} h \, d\mu > 0\Big\}$$

is a weak open neighborhood of the class $\hat{g}$ of g in $L^1(\mu)$. The weak compactness of $\hat{K}_\varepsilon$ implies that there are finitely many $g_1, \dots, g_k \in K_\varepsilon$ such that

$$K_\varepsilon \subset \bigcup_{j=1}^{k} V(g_j).$$

Let $\{U_1, \dots, U_N\}$ be a partition of Ω in Σ with the property that if $U_l \cap A_i^{g_j} \neq \emptyset$ for some $1 \leq j \leq k$, $1 \leq i \leq n(g_j)$, then $U_l \subset A_i^{g_j}$. Given $f \in \mathcal{F}$ consider

$$C = \bigcup\big\{U_l : \ |\cdot|\text{-diam}(f(U_l)) > \varepsilon\big\}.$$

We now prove that the characteristic function χ_C of C does not belong to K_ε. If $\chi_C \in K_\varepsilon$, then $\chi_C \in V(g_j)$ for some $1 \leq j \leq k$. This means that $\mu(C \cap A_i^{g_j}) > 0$ for every $1 \leq i \leq n(g_j)$. So, for every $1 \leq i \leq n(g_j)$ there is U_{l_i} with $U_{l_i} \cap A_i^{g_j} \neq \emptyset$ and $|\cdot|$-diam$(f(U_{l_i})) > \varepsilon$. Since U_{l_i} is contained in $A_i^{g_j}$ we infer that $|\cdot|$-diam$(f(A_i^{g_j})) > \varepsilon$

for every $1 \le i \le n(g_j)$, which contradicts (11) and henceforth we conclude that the function $\chi_C \notin K_\varepsilon$, i.e. $\mu(C) < \varepsilon$. Now, we finish the proof of (ii) by noticing that

$$\begin{aligned}\sum_{l=1}^{N} |\cdot|\text{-diam}(f(U_l))\mu(U_l) = \\ &= \sum_{U_l \subset C} |\cdot|\text{-diam}(f(U_l))\mu(U_l) + \sum_{U_l \not\subset C} |\cdot|\text{-diam}(f(U_l))\mu(U_l) \\ &\le 2M\varepsilon + \varepsilon\mu(\Omega) = (2M+1)\varepsilon,\end{aligned}$$

where $M := \sup\{|f(w)| : w \in \Omega, f \in \mathcal{F}\}$. □

In general, when $\mathcal{F} \subset \mathbb{R}^\Omega$ is not necessarily uniformly bounded and has the Bourgain property we do not know if $\mathcal{F}$ has the Birkhoff property: we can prove though that this is the case for families $\mathcal{F} = Z_f$, where $f : \Omega \longrightarrow X$, see Corollary 3.4.

We can now characterize Birkhoff integrability for bounded functions. Recall that a subset $B \subset B_{X^*}$ is said to be *norming* if

$$\|x\| = \sup_{x^* \in B} |\langle x^*, x\rangle|$$

for every $x \in X$.

Theorem 2.4. *Let $f : \Omega \longrightarrow X$ be a bounded function. The following statements are equivalent:*

(i) *f is Birkhoff integrable;*
(ii) *Z_f has the Bourgain property;*
(iii) *there is a norming set $B \subset B_{X^*}$ such that*

$$Z_{f,B} = \{\langle x^*, f\rangle : x^* \in B\} \subset \mathbb{R}^\Omega$$

has the Bourgain property.

Proof. If f is bounded, then $Z_f = \{\langle x^*, f\rangle : x^* \in B_{X^*}\} \subset \mathbb{R}^\Omega$ is a uniformly bounded family of functions and f is summable with respect to any countable partition of Ω in Σ. Hence, the equivalence (i)⇔(ii) is exactly what has been proved in Proposition 2.2 if we bear in mind Lemma 2.3. The implication (ii)⇒(iii) is trivial and the proof for the implication (iii)⇒(ii) goes as follows. If (iii) holds, then $Z_{f,B} \subset \mathbb{R}^\Omega$ has the Birkhoff property because it is uniformly bounded and has the Bourgain property —apply Lemma 2.3. Since B is norming, the Hahn-Banach separation theorem guarantees that $\overline{\operatorname{aco}(Z_{f,B})}^{\tau_p} = Z_f$. As the Birkhoff property is preserved by taking absolutely convex hulls and pointwise closures we conclude that Z_f has Birkhoff (Bourgain) property and the proof is over. □

Riddle and Saab proved in [16, Theorem 13] that any bounded function $f : \Omega \longrightarrow X^*$ is Pettis integrable whenever $Z_{f,B_X} = \{\langle f, x\rangle : x \in B_X\}$ has the Bourgain property. The particular case of Theorem 2.4 that is isolated below improves their result.

Corollary 2.5. *Let $f : \Omega \longrightarrow X^*$ be a bounded function. Then f is Birkhoff integrable if, and only if, $\{\langle f, x\rangle : x \in B_X\}$ has the Bourgain property.*

We end up the section by highlighting that for a function $f : \Omega \longrightarrow X$ its Birkhoff integral (upon its existence) can be realized as the limit of the net $\{S(f, \Gamma, T)\}_\Gamma$, where we order partitions by refinement. Note that for any f, Lemma 2.1 implies that the set $\mathcal{S}_f$ of all pairs (Γ, T), where Γ is a countable partition of Ω in Σ for which f is summable and T is a choice in Γ, is a directed set with the binary relation

$$(\Gamma, T) \preceq (\Gamma', T') \;\Leftrightarrow\; \Gamma' \text{ is finer than } \Gamma.$$

Proposition 2.6. *Let $f : \Omega \longrightarrow X$ be a function. The following conditions are equivalent:*

(i) *f is Birkhoff integrable;*

(ii) *there is $x \in X$ with the following property: for every $\varepsilon > 0$ there is a countable partition Γ of Ω in Σ such that f is summable with respect to Γ and*

$$\|S(f,\Gamma,T) - x\| < \varepsilon$$

for every choice T in Γ;

(iii) *there is $y \in X$ with the following property: for every $\varepsilon > 0$ there is a countable partition Γ of Ω in Σ such that f is summable with respect to each countable partition Γ' finer than Γ and*

$$\|S(f,\Gamma',T') - y\| < \varepsilon$$

for every choice T' in Γ'.

In this case, $x = y = \int_\Omega f\, d\mu$.

Proof. The implications (iii)⇒(ii)⇒(i) are obvious. To see that (i)⇒(iii), we simply notice that Birkhoff integrability of f and Lemma 2.1 imply that the net

$$\mathcal{S}_f \longrightarrow X, \quad (\Gamma,T) \mapsto S(f,\Gamma,T),$$

is a Cauchy net and, therefore, it converges to some $y \in X$.

The last statement in this proposition straightforwardly follows from the very definition of the Birkhoff integral. □

We mention that functions $f : \Omega \longrightarrow X$ satisfying (iii) in the previous proposition are called *unconditionally Riemann-Lebesgue integrable* functions in [10], where some results about this type of integrable functions are proved. The previous proposition makes clear that the notion of unconditional Riemann-Lebesgue integrability coincides with Birkhoff's one.

3. Birkhoff integrability for arbitrary functions

We start this section by establishing the following criterion for the unconditional convergence of double series in Banach spaces.

Lemma 3.1. *Let $(x_{n,k})_{n,k\in\mathbb{N}}$ be a double sequence in X such that:*

(i) *the series $\sum_k x_{n,k}$ is unconditionally convergent for every $n \in \mathbb{N}$;*

(ii) *there are an unconditionally convergent series $\sum_{n,k} y_{n,k}$ in X and a sequence of non-negative real numbers (a_n) with $\sum_{n=1}^\infty a_n < \infty$ such that*

$$\Big\|\sum_{k\in Q}(x_{n,k} - y_{n,k})\Big\| \le a_n$$

for every finite subset $Q \subset \mathbb{N}$ and every $n \in \mathbb{N}$.

Then $\sum_{n,k} x_{n,k}$ is unconditionally convergent in X.

Proof. Fix $\varepsilon > 0$ and take $N \in \mathbb{N}$ such that

$$\sum_{n>N} a_n < \varepsilon \quad \text{and} \quad \Big\|\sum_{(n,k)\in P} y_{n,k}\Big\| < \varepsilon \tag{12}$$

for every finite subset $P \subset \mathbb{N}\times\mathbb{N}$ for which $P \cap (\{1,2,\dots,N\}\times\{1,2,\dots,N\}) = \emptyset$. Take now $M \in \mathbb{N}$, $M \ge N$, such that

$$\Big\|\sum_{k\in F} x_{n,k}\Big\| < \frac{\varepsilon}{N}, \quad n = 1,2,\dots,N, \tag{13}$$

for every finite set $F \subset \mathbb{N}$ for which $F \cap \{1,2,\dots,M\} = \emptyset$.

Given a finite set $H \subset \mathbb{N}\times\mathbb{N}$ we write $H' := \{(n,k)\in H : 1\le n\le N\}$. If $H\cap(\{1,2,\dots,N\}\times\{1,2,\dots,M\})=\emptyset$, then we have

$$\begin{aligned}\Big\|\sum_{(n,k)\in H} x_{n,k}\Big\| &= \Big\|\sum_{(n,k)\in H'} x_{n,k}+\sum_{(n,k)\in H\setminus H'} x_{n,k}\Big\|\\ &\le \sum_{n=1}^{N}\Big\|\sum_{\substack{k\\(n,k)\in H'}} x_{n,k}\Big\| + \Big\|\sum_{(n,k)\in H\setminus H'}(x_{n,k}-y_{n,k})\Big\| + \Big\|\sum_{(n,k)\in H\setminus H'} y_{n,k}\Big\|\\ &< \sum_{n=1}^{N}\frac{\varepsilon}{N}+\sum_{n>N} a_n+\varepsilon<3\varepsilon,\end{aligned}$$

after inequalities (12) and (13). This proves that the series $\sum_{n,k} x_{n,k}$ is unconditionally convergent in X and we are finished. □

If $f:\Omega\longrightarrow X$ is Birkhoff integrable, then for every $A\in\Sigma$ the restriction $f|_A$ is Birkhoff integrable with respect to (A,Σ_A,μ_A) —$\Sigma_A=\{E\cap A : E\in\Sigma\}$ and μ_A stands for the restriction of μ to Σ_A— and its Birkhoff and Pettis integrals over A coincide, see [14, 2.21].

Lemma 3.2. *Let $f:\Omega\longrightarrow X$ be a function. The following conditions are equivalent:*

(i) *f is Birkhoff integrable;*

(ii) *f is Pettis integrable and there is a countable partition $\Gamma=(A_n)$ of Ω in Σ such that $f|_{A_n}$ is Birkhoff integrable for every n;*

(iii) *there is a countable partition $\Gamma=(A_n)$ of Ω in Σ such that $f|_{A_n}$ is Birkhoff integrable for every n and for every countable partition $\Gamma'=(B_n)$ of Ω in Σ finer than Γ, the series $\sum_n\int_{B_n} f\,d\mu$ is unconditionally convergent in X.*

Proof. The implications (i)⇒(ii)⇒(iii) are clear — use that the indefinite integral of any Pettis integrable function is a countably additive vector measure, [2, Theorem 5, p. 53]. We prove now (iii)⇒(ii). Assume for the moment that f takes real values and (iii) holds. Set $\Omega^+ := \{w\in\Omega : f(w)>0\}$ and $\Omega^- := \{w\in\Omega : f(w)\le 0\}$. With the partitions Γ and $\{\Omega^+,\Omega^-\}$ we induce the partition Γ' whose members are $\Omega^+\cap A_n$ and $\Omega^-\cap A_n$. By hypothesis, the series $\sum_n\int_{\Omega^+\cap A_n} f\,d\mu+\sum_n\int_{\Omega^-\cap A_n} f\,d\mu$ is absolutely convergent, meaning

$$\begin{aligned}+\infty &> \sum_n\Big|\int_{\Omega^+\cap A_n} f\,d\mu\Big| + \sum_n\Big|\int_{\Omega^-\cap A_n} f\,d\mu\Big|\\ &= \sum_n\int_{\Omega^+\cap A_n}|f|\,d\mu+\sum_n\int_{\Omega^-\cap A_n}|f|\,d\mu=\sum_n\int_{A_n}|f|\,d\mu.\end{aligned}$$

An appeal to Lebesgue's Monotone Convergence Theorem [17, Theorem 1.26] gives us that f is an integrable function. Now we handle the general case. If $f:\Omega\longrightarrow X$ satisfies (iii), then for every $x^*\in B_{X^*}$ the scalar function $\langle x^*,f\rangle$ also satisfies (iii). Hence $\langle x^*,f\rangle\in\mathcal{L}^1(\mu)$. Moreover, if we define $x_\Omega := \sum_n\int_{A_n} f\,d\mu$, then we have

$$x^*(x_\Omega)=x^*\Big(\sum_n\int_{A_n} f\,d\mu\Big)=\sum_n\int_{A_n}\langle x^*,f\rangle\,d\mu=\int_\Omega\langle x^*,f\rangle\,d\mu.$$

If A is a subset of Σ then (iii) is satisfied for $f|_A$ and (A,Σ_A,μ_A). The previous arguments applied to any $f|_A$ allow us to conclude that f is Pettis integrable and therefore (ii) holds.

To finish we prove that (ii)⇒(i). We are going to show that f is Birkhoff integrable using the very Definition 1. Fix $\varepsilon>0$. Birkhoff integrability of $f|_{A_n}$, $n\in\mathbb{N}$, implies that there is a partition $\Gamma_n=(A_{n,k})_k$ of A_n in Σ such that $f|_{A_n}$ is summable with respect to Γ_n and

$$\Big\|S(f|_{A_n},\Gamma_n,T_n)-S(f|_{A_n},\Gamma_n,T_n')\Big\|<\frac{\varepsilon}{2^n} \tag{14}$$

for arbitrary choices T_n and T'_n in Γ_n. Take $Q \subset \mathbb{N}$ a finite set, fix $n \in \mathbb{N}$ and set $B_{n,Q} = \bigcup_{k\in Q} A_{n,k}$. Inequality (14) implies that

$$\Big\| \sum_{k\in Q} f(t_{n,k})\mu(A_{n,k}) - \sum_{k\in Q} f(t'_{n,k})\mu(A_{n,k}) \Big\| < \frac{\varepsilon}{2^n}$$

for any choices $(t_{n,k})_{k\in Q}$ and $(t'_{n,k})_{k\in Q}$ in the partition $\Gamma_{n,Q} = (A_{n,k})_{k\in Q}$. With the notation in Definition 1, this means that $\|\cdot\|\text{-diam}(J(f|_{B_{n,Q}}, \Gamma_{n,Q})) \le \varepsilon/2^n$. Since $f|_{B_{n,Q}}$ is Birkhoff integrable and $\int_{B_{n,Q}} f\, d\mu \in \overline{\text{co}(J(f|_{B_{n,Q}}, \Gamma_{n,Q}))}$, we conclude that

$$\Big\| \sum_{k\in Q} \big(f(t_{n,k})\mu(A_{n,k}) - \int_{A_{n,k}} f\, d\mu \big) \Big\| = \Big\| \sum_{k\in Q} f(t_{n,k})\mu(A_{n,k}) - \int_{B_{n,Q}} f\, d\mu \Big\| \le \frac{\varepsilon}{2^n}. \tag{15}$$

If we define $\Gamma := \bigcup \Gamma_n = (A_{n,k})_{n,k}$, then the series $S(f,\Gamma,T) = \sum_{n,k} f(t_{n,k})\mu(A_{n,k})$ converges unconditionally for every choice $T = (t_{n,k})$ in Γ. Indeed, this follows from Lemma 3.1 bearing in mind that $\sum_{n,k} \int_{A_{n,k}} f\, d\mu$ is unconditionally convergent (f is Pettis integrable) and that inequality (15) holds. Now, inequality (14) is used again to deduce that

$$\Big\| S(f,\Gamma,T) - S(f,\Gamma,T') \Big\| \le \varepsilon,$$

for any choices T and T' in Γ. This shows that f is Birkhoff integrable. □

The equivalence between (i) and (ii) in the Lemma above was first stated in the unpublished note [5]: our approach here isolates and clarifies the difficulties behind the proof via Lemma 3.1, that could be of interest by itself. We have felt somehow obliged to include (iii) in the lemma because our implication (iii)⇒(i) fixes a minor mistake in Birkhoff's paper [1]. Theorem 15 in [1] states that a function $f : \Omega \longrightarrow X$ is Birkhoff integrable as long as the following weaker form of (iii) holds:

(iii') *there is a countable partition $\Gamma = (A_n)$ of Ω in Σ such that $f|_{A_n}$ is Birkhoff integrable for every n and the series $\sum_n \int_{A_n} f\, d\mu$ is unconditionally convergent in X.*

Unfortunately this is not true even for real functions. Indeed, take any infinite countable partition $\Gamma = (A_n)$ of Borel sets of $[0,1]$ with Lebesgue measure $\lambda(A_n) > 0$. Split $A_n = C_n \cup D_n$ as the union of two disjoint measurable sets such that $\lambda(C_n) = \lambda(D_n) = \lambda(A_n)/2$, for every $n \in \mathbb{N}$. Define the function $f : [0,1] \longrightarrow \mathbb{R}$ by the sum

$$f := \sum_{n=1}^{\infty} \Big(\frac{\chi_{C_n}}{\lambda(C_n)} - \frac{\chi_{D_n}}{\lambda(D_n)} \Big).$$

Clearly, f is not integrable over $[0,1]$ meanwhile $\int_{A_n} f\, d\lambda = 0$ over each A_n. This shows that Theorem 15 in [1] is not correct and that it is certainly needed our assumption $\sum_n \int_{B_n} f\, d\lambda$ unconditionally converges for every partition $\Gamma' = (B_n)$ finer than Γ, as presented in (iii) in Lemma 3.2.

Our next goal is to show that the Bourgain and Birkhoff properties are equivalent for families of the form $Z_f = \{\langle x^*, f\rangle : x^* \in B_{X^*}\}$, where $f : \Omega \longrightarrow X$ is any function. We reduce our proof to the case of a bounded function f by using the lemma below.

Lemma 3.3. *Let $B_1, \dots, B_n \subset X$ be sets for which there is a constant $k > 0$ such that for every $x^* \in B_{X^*}$*

$$\inf_{1\le i\le n} |\cdot|\text{-diam}(x^*(B_i)) \le k.$$

Then there is $1 \le j \le n$ such that B_j is bounded.

Proof. For each $1 \le i \le n$, define $C_i := \{x^* \in B_{X^*} : |\cdot|\text{-diam}(x^*(B_i)) \le k\}$. Notice that each C_i is norm closed. $\{B_{X^*} \setminus C_i : 1 \le i \le n\}$ is a family of relatively open subsets of B_{X^*} with empty intersection, hence there is $1 \le j \le n$ such that $B_{X^*} \setminus C_j$ is not dense in B_{X^*}. Therefore $G := \{x^* \in X^* : \|x^*\| < 1\} \not\subset \overline{B_{X^*} \setminus C_j}^{\|\cdot\|}$. It follows that there exist $x_0^* \in G$ and $\delta > 0$ such that $\{x^* \in X^* : \|x_0^* - x^*\| \leq \delta\} \subset G \cap C_j$. Fix $x_0 \in B_j$. Given $x^* \in B_{X^*}$, we have $x_0^* + \delta x^* \in C_j$ and therefore for every $x \in B_j$ we obtain

$$\begin{aligned}|\delta x^*(x)| &\leq |(x_0^* + \delta x^*)(x) - (x_0^* + \delta x^*)(x_0)| + |x_0^*(x) - x_0^*(x_0)| + |\delta x^*(x_0)| \\ &\leq |\cdot|\text{-diam}((x_0^* + \delta x^*)(B_j)) + |\cdot|\text{-diam}(x_0^*(B_j)) + \delta\|x_0\| \leq 2k + \delta\|x_0\|.\end{aligned}$$

Hence $\|x\| \leq (2k)/\delta + \|x_0\|$ for every $x \in B_j$. Consequently, B_j is bounded and the proof is complete. □

It is not difficult to see that a family $\mathcal{F} \subset \mathbb{R}^\Omega$ has the Birkhoff (resp. Bourgain) property if and only if there is a countable partition (A_n) of Ω in Σ such that for each $n \in \mathbb{N}$ the family of restrictions $\{f|_{A_n} : f \in \mathcal{F}\} \subset \mathbb{R}^{A_n}$ has the Birkhoff (resp. Bourgain) property with respect to $(A_n, \Sigma_{A_n}, \mu_{A_n})$.

Corollary 3.4. *Let $f : \Omega \longrightarrow X$ be a function. The following conditions are equivalent:*

(i) *Z_f the has Birkhoff property;*
(ii) *Z_f has the Bourgain property.*

In this case, there is a countable partition (A_n) of Ω in Σ such that $f(A_n)$ is bounded whenever $\mu(A_n) > 0$.

Proof. The implication (i)⇒(ii) has already been proved in Lemma 2.3 (i). Our comments prior to this corollary and Lemma 2.3 (ii) imply that to prove (ii)⇒(i) it suffices to show that there is a countable partition (A_n) of Ω in Σ such that $f(A_n)$ is bounded whenever $\mu(A_n) > 0$. A standard exhaustion argument reduces the proof of the last condition to check that *for each $E \in \Sigma$ with $\mu(E) > 0$ there is $A \subset E$, $A \in \Sigma$ with $\mu(A) > 0$, such that $f(A)$ is bounded*. We prove this: since Z_f has the Bourgain property, there are $E_1, \dots, E_n \subset E$, $E_i \in \Sigma$ with $\mu(E_i) > 0$, such that for every $x^* \in B_{X^*}$

$$\inf_{1 \leq i \leq n} |\cdot|\text{-diam}(\langle x^*, f\rangle(E_i)) \leq 1. \tag{16}$$

If we write $B_i := f(E_i)$, inequality (16) is read as $\inf_{1 \leq i \leq n} |\cdot|\text{-diam}(x^*(B_i)) \leq 1$ for every $x^* \in B_{X^*}$. An appeal to Lemma 3.3 ensures us that there is $1 \leq j \leq n$ such that $B_j = f(E_j)$ is bounded. The proof is finished. □

Theorem 3.5. *Let $f : \Omega \longrightarrow X$ be a function. The following conditions are equivalent:*

(i) *f is Birkhoff integrable;*
(ii) *Z_f is uniformly integrable and has the Bourgain property.*

Proof. The implication (i)⇒(ii) follows from Proposition 2.2 and Lemma 2.3 (i) together with the fact that since f is Pettis integrable (recall that Birkhoff integrability implies Pettis integrability) the set Z_f is uniformly integrable, [19, Theorem 4-2-2].

By Corollary 3.4 condition (ii) is actually equivalent to:

(ii') *Z_f is uniformly integrable, Z_f has the Bourgain property and there is a countable partition $\Gamma = (A_n)$ of Ω in Σ such that $f(A_n)$ is bounded whenever $\mu(A_n) > 0$.*

We now prove (ii')⇒(i). This implication can be established in several different ways: we present the simplest one we came across with. To get started we prove that f is Pettis integrable. By [19, Theorem 4-2-3] it suffices to show that the canonical map $i : (Z_f, \tau_p) \longrightarrow (L^1(\mu), w)$, that sends every function to its equivalence class, is continuous. We are going to show that i is in fact τ_p-to-$\|\cdot\|_1$ continuous by proving that $i(\overline{A}^{\tau_p}) \subset \overline{i(A)}^{\|\cdot\|_1}$ for every $A \subset Z_f$. Fix $A \subset Z_f$ and pick $g \in \overline{A}^{\tau_p}$. Since Z_f has the Bourgain property, A has the Bourgain property too. Therefore there is a sequence (g_n) in A converging to g μ-almost everywhere, [16, Theorem 11]. The sequence (g_n) is uniformly integrable and therefore Vitali's theorem, [8, p. 203], ensures that $\lim_n \|g_n - g\|_1 = 0$. Hence $i(g) \in \overline{i(A)}^{\|\cdot\|_1}$, the inclusion $i(\overline{A}^{\tau_p}) \subset \overline{i(A)}^{\|\cdot\|_1}$ holds and the proof of Pettis integrability of f is over. To prove that f is Birkhoff integrable observe that, on the one hand, $f|_{A_n}$ is Birkhoff integrable whenever $\mu(A_n) = 0$. On the other hand, whenever $\mu(A_n) > 0$, $f|_{A_n}$ is bounded and

$$Z_{f|_{A_n}} = \{\langle x^*, f|_{A_n}\rangle : x^* \in B_{X^*}\}$$

has the Bourgain property; hence Theorem 2.4 implies that $f|_{A_n}$ is also Birkhoff integrable. Therefore, f and Γ fulfill the requirements in (ii) of Lemma 3.2 and we conclude that f is Birkhoff integrable. □

Fremlin proved in [5] that for every Birkhoff integrable function $f : \Omega \longrightarrow X$ the set Z_f is stable. Since the Bourgain property is more restrictive than stability, [19, 9-5-4], the aforementioned Fremlin's result is a weaker form of Theorem 3.5.

In the proof of (ii)⇒(i) Pettis integrability of f can be established in a different way, namely, by using that a function $f : \Omega \longrightarrow X$ is Pettis integrable if Z_f is stable and uniformly integrable, [19, Theorem 6-1-2]. Nonetheless, we think that the arguments given using the Bourgain property are easier than those using stability.

A thorough study about the continuity of the map $i : (\mathcal{F}, \tau_p) \longrightarrow (L^1(\mu), \|\cdot\|_1)$, for certain families $\mathcal{F} \subset L^1(\mu)$, can be found in [21]. Another consequence of the continuity of i proved in the implication (ii)⇒(i) in Theorem 3.5 is:

Corollary 3.6. *If $f : \Omega \longrightarrow X$ is Birkhoff integrable, then the range of the indefinite integral $\{\int_A f \, d\mu : A \in \Sigma\}$ is relatively norm compact.*

Proof. It suffices to show that $i(Z_f)$ is a compact subset of $(L^1(\mu), \|\cdot\|_1)$, [19, Proposition 4-1-5]. This follows from the compactness of (Z_f, τ_p) —Alaouglu's theorem— and the continuity of the canonical map $i : (Z_f, \tau_p) \longrightarrow (L^1(\mu), \|\cdot\|_1)$. □

Corollary 3.6, that strengthens a result in [9] regarding the separability of the range of the indefinite integral of Riemann-Lebesgue integrable functions, can be alternatively proved combining Theorem 18 in [1] with Remark 9.1 in [12]. We mention that the range of the indefinite integral of a Pettis integrable function is not relatively norm compact in general (Fremlin and Talagrand, see e.g. [19, Theorem 4-2-5]).

Observe that if $f : \Omega \longrightarrow X$ is μ-strongly measurable, then Z_f has the Bourgain property. This easily follows from the fact that μ-strong measurability for f is equivalent to the condition:

(S) *for every $\varepsilon > 0$ there is a countable partition $\Gamma_0 = (A_0, A_1, \dots)$ of Ω in Σ such that $\mu(A_0) = 0$ and $\|\cdot\|$-diam$(f(A_n)) < \varepsilon$ for every $n \geq 1$,*

see [2, Corollary 3, p. 42]. Hence, Theorem 3.5 particularly says that for μ-strongly measurable functions, Birkhoff integrability and Pettis integrability coincide, [14, Corollary 5.11], because Z_f is uniformly integrable whenever f is Pettis integrable, see [19, Theorem 4-2-2]. More particularly, every Bochner integrable function is Birkhoff integrable, [1, p. 377].

Another application of Theorem 3.5 is Corollary 3.7 below where Birkhoff integrability of a non necessarily bounded function $f : \Omega \longrightarrow X^*$ is characterized in terms of the family

$$Z_{f,B_X} = \{\langle f, x\rangle : x \in B_X\} \subset \mathbb{R}^\Omega.$$

Corollary 3.7. *Let $f : \Omega \longrightarrow X^*$ be a function. Then f is Birkhoff integrable if, and only if, $\{\langle f, x\rangle : x \in B_X\}$ is uniformly integrable and has the Bourgain property.*

Proof. In view of Theorem 3.5 we only need to check that Z_f is uniformly integrable and has the Bourgain property whenever the same holds true for Z_{f,B_X}. Observe that Goldstine's theorem applies to deduce that $\overline{Z_{f,B_X}}^{\tau_p} = Z_f$. On the one hand, since the Bourgain property is preserved by taking pointwise closures [16, Theorem 11], we conclude that Z_f has the Bourgain property. On the other hand, Z_f is uniformly integrable. Indeed, since Z_{f,B_X} has the Bourgain property and $\overline{Z_{f,B_X}}^{\tau_p} = Z_f$, every element of Z_f is the μ-almost everywhere limit of a sequence in Z_{f,B_X}, see [16, Theorem 11]. The uniform integrability of Z_{f,B_X} and Vitali's theorem ensure us that $Z_f \subset \mathcal{L}^1(\mu)$ and that every element of Z_f is the $\|\cdot\|_1$-limit of a sequence in Z_{f,B_X}. Finally, the fact that Z_{f,B_X} is uniformly integrable is used again to infer that the same holds for Z_f. The proof is over. □

We characterize now WRNP in dual Banach spaces in terms of Birkhoff integrable Radon-Nikodým derivatives.

Theorem 3.8. *Let X be a Banach space. The following statements are equivalent:*

(i) *X^* has the weak Radon-Nikodým property;*
(ii) *X does not contain a copy of ℓ^1;*
(iii) *for every complete probability space (Ω, Σ, μ) and every μ-continuous countably additive vector measure $\nu : \Sigma \longrightarrow X^*$ of σ-finite variation there is a Birkhoff integrable function $f : \Omega \longrightarrow X^*$ such that*

$$\nu(E) = \int_E f \, d\mu$$

for every $E \in \Sigma$.

Proof. The equivalence (i)⇔(ii) is well-known, see for instance [3, Theorem 6.8]. The implication (iii)⇒(i) uses the very definitions and the fact that Birkhoff integrability implies Pettis integrability. For the proof of (ii)⇒(iii) we distinguish two cases. We write $|\nu|$ to denote the variation of ν.

Particular Case.- Suppose that there is $M > 0$ such that $|\nu|(E) \leq M\mu(E)$ for every $E \in \Sigma$. Fix a lifting ρ on $L^\infty(\mu)$, [3, Theorem G.1, p. 145]. By [3, Proposition 6.2] there is a bounded w^*-scalarly measurable function $f : \Omega \longrightarrow X^*$ such that:

(a) $\rho(\langle f, x\rangle) = \langle f, x\rangle$ for every $x \in X$;
(b) $\langle \nu(E), x\rangle = \int_E \langle f, x\rangle \, d\mu$ for every $E \in \Sigma$ and every $x \in X$.

Notice that $Z_{f,B_X} \subset \ell^\infty(\Omega)$ is a uniformly bounded subset that cannot contain an ℓ^1-sequence because otherwise there are $\delta > 0$ and a sequence (x_n) in B_X such that for every $n \in \mathbb{N}$ and every $a_1, \dots, a_n \in \mathbb{R}$

$$\delta \sum_{i=1}^n |a_i| \leq \Big\| \sum_{i=1}^n a_i \langle f, x_i\rangle \Big\|_\infty = \Big\| \langle f, \sum_{i=1}^n a_i x_i\rangle \Big\|_\infty \leq M \Big\| \sum_{i=1}^n a_i x_i \Big\|;$$

this means that ℓ^1 embeds in X contradicting our hypothesis. Hence Z_{f,B_X} does not contain ℓ^1-sequences that together with the equality $\rho(Z_{f,B_X}) = Z_{f,B_X}$ allow us to use Corollary 12.1 in [12] to obtain that Z_{f,B_X} has the Bourgain property. Therefore f is Birkhoff integrable after Corollary 2.5. Hence for every $E \in \Sigma$, both $\nu(E)$ and $\int_E f \, d\mu$ belong to X^* and according to (b) above the equality $\nu(E) = \int_E f \, d\mu$ holds.

General Case.- Since $|\nu|$ is a σ-finite measure and $|\nu|(E) = 0$ whenever $\mu(E) = 0$, standard arguments, see the proof of [3, Lemma 5.9], provide us with a countable partition $\Gamma = (A_n)$ of Ω in Σ such that $|\nu|(E) \leq n\mu(E)$ for every $E \in \Sigma_{A_n}$ and every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. The *Particular Case*, already proved, implies the existence of a Birkhoff integrable function $f_n : A_n \longrightarrow X^*$ such that

$$\nu(E) = \int_E f_n \, d\mu, \quad E \in \Sigma_{A_n}.$$

Define $f : \Omega \longrightarrow X^*$ by $f(\omega) = f_n(\omega)$ if $\omega \in A_n$. The proof of [3, Lemma 5.9] reveals that f is Pettis integrable with indefinite Pettis integral ν. So, Lemma 3.2 says that f is Birkhoff integrable and the proof is finished. □

It is well-known that there is a one-to-one correspondence between bounded linear operators $T : L^1(\mu) \longrightarrow X^*$ and measures $\nu : \Sigma \longrightarrow X^*$ satisfying $|\nu|(E) \leq M\mu(E)$,

$E \in \Sigma$, for some $M < \infty$: simply put $\nu(E) := T(\chi_E)$, $E \in \Sigma$, see [3, Lemma 5.9]. Theorem 3.8 can be completed as follows:

Corollary 3.9. *Let X be a Banach space. The following statements are equivalent:*

(i) *X^* has the weak Radon-Nikodým property;*

(iv) *for every complete probability space (Ω, Σ, μ) and for every bounded operator $T : L^1(\mu) \longrightarrow X^*$ there is a bounded Birkhoff integrable function $f : \Omega \longrightarrow X^*$ such that*

$$\langle x^{**}, T(g)\rangle = \int_\Omega g\langle x^{**}, f\rangle \, d\mu, \quad x^{**} \in X^{**},\ g \in L^1(\mu). \tag{17}$$

We mention that Saab proved in [18, Proposition 9] —using martingale techniques— that X^* has the WRNP if, and only if, for every bounded operator $T : L^1[0,1] \longrightarrow X^*$ there is a bounded function $f : [0,1] \longrightarrow X^{***}$ such that Z_f has the Bourgain property (with respect to the Lebesgue measure) and equation (17) holds for every $g \in L^1[0,1]$.

Another consequence of Theorem 3.8 is that if X^* has the WRNP, then every Pettis integrable function $f : \Omega \longrightarrow X^*$ is scalarly equivalent to a Birkhoff integrable function —bear in mind that the indefinite integral associated to a Pettis integrable function is countably additive and has σ-finite variation, [3, Proposition 5.6].

Functions $f : \Omega \to X$ for which Z_f is stable and such that $\|f\|$ has a μ-integrable majorant have gotten the attention of several authors over the years, see [6, 13, 19, 20] amongst others. These functions are called *Talagrand integrable functions* by Fremlin and Mendoza, see [6], and they were characterized by Talagrand as those functions satisfying the *law of large numbers*, see [20]. As the last application of our techniques here we characterize those functions f for which Z_f has the Bourgain property and $\|f\|$ has a μ-integrable majorant.

Recall that a function $f : \Omega \longrightarrow X$ is said to be Riemann-Lebesgue integrable, [9, 10], if there is $x \in X$ with the following property: for every $\varepsilon > 0$ there is a countable partition Γ of Ω in Σ such that for every countable partition Γ' finer than Γ and every choice T' in Γ', the series $S(f, \Gamma', T')$ is absolutely convergent and $\|S(f, \Gamma', T') - x\| < \varepsilon$. Every Riemann-Lebesgue integrable function is Birkhoff integrable after Proposition 2.6.

Proposition 3.10. *Let $f : \Omega \longrightarrow X$ be a function. The following conditions are equivalent:*

(i) *f is Riemann-Lebesgue integrable;*

(ii) *Z_f has the Bourgain property and there is $g \in \mathcal{L}^1(\mu)$ such that $\|f\| \leq g$ μ-almost everywhere.*

Proof. (i)⇒(ii) Theorem 3.5 ensures that Z_f has the Bourgain property because f is Birkhoff integrable. Now, take $\Gamma = (A_n)$ a countable partition of Ω in Σ such that $\|S(f, \Gamma, T) - S(f, \Gamma, T')\| < 1$, for any choices T and T' in Γ, being the series involved absolutely convergent. Notice that $f(A_n)$ is bounded whenever $\mu(A_n) > 0$. The series $\sum_{\mu(A_n)>0} \|f(A_n)\|\mu(A_n)$ is convergent and therefore the function defined by $g = \sum_{\mu(A_n)>0} \|f(A_n)\|\chi_{A_n}$ is μ-integrable and satisfies $\|f\| \leq g$ μ-almost everywhere.

Conversely, (ii)⇒(i). Since Z_f has the Bourgain property, Z_f is made up of measurable functions, [16, Theorem 11]. The inequality $\|f\| \leq g$ μ-almost everywhere implies: *a)* Z_f is uniformly integrable; *b)* there is a countable partition (A_n) of Ω in Σ such that $f(A_n)$ is bounded whenever $\mu(A_n) > 0$ and $\sum_{\mu(A_n)>0} \|f(A_n)\|\mu(A_n)$ is convergent. An appeal to Theorem 3.5 establishes that f is Birkhoff integrable. Clearly $S(f, \Gamma', T')$ is absolutely convergent for every countable partition Γ' of Ω in Σ finer than Γ and every choice T' in Γ'. Proposition 2.6 finishes the proof. □

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