CONVERGENCE THEOREMS FOR THE BIRKHOFF INTEGRAL

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Abstract. We study the validity of Vitali's convergence theorem for the Birkhoff integral of functions taking values in a Banach space X. On the one hand, we show that the theorem is true whenever X is isomorphic to a subspace of $\ell_\infty(\mathbb{N})$. On the other hand, we prove that if X is super-reflexive and has density character the continuum, then there is a uniformly bounded sequence of Birkhoff integrable X-valued functions (defined on $[0,1]$ with the Lebesgue measure) converging pointwise to a non Birkhoff integrable function.

1. Introduction

The Lebesgue integral admits different extensions to the case of functions with values in Banach spaces. In this framework the Bochner and Pettis integrals have been widely studied by many authors over the years, see [4] and [14, 15, 23], respectively. The Birkhoff integral [1] (see below for the definition) is an intermediate notion that had hardly been analyzed until the last years. In several recent papers it is shown that this integral plays a relevant role in the setting of vector integration, see [2], [6], [12], [13], [18], [19], [20] and [22].

It is well known that for the Bochner and Pettis integrals there are "good" convergence theorems (for norm and weak convergence, respectively) along the line of the classical *Vitali's theorem*, cf. [4] and [14, 15]. However, in general this is not the case for the Birkhoff integral: in [19] we present an example of a uniformly bounded sequence of Birkhoff integrable functions $f_n : [0,1] \longrightarrow c_0(\mathfrak{c})$ converging pointwise to a non Birkhoff integrable function (where $\mathfrak{c}$ stands for the cardinality of the continuum). Notice that such an example can not be constructed in a separable Banach space, since in that case Birkhoff and Pettis integrability are equivalent (and coincide with Bochner integrability for bounded functions), see [16].

The aim of this paper is to discuss the validity of Vitali's convergence theorem for the Birkhoff integral in some special classes of Banach spaces. Our main "positive" result is the following theorem that applies to subspaces of $\ell_\infty(\mathbb{N})$. It is worth to mention here that within this class of spaces Birkhoff integrability lies strictly between Bochner and Pettis integrability, see [1] and [8].

Theorem 1.1. *Let (Ω, Σ, μ) be a complete probability space and X a Banach space isomorphic to a subspace of $\ell_\infty(\mathbb{N})$. Let $f_n : \Omega \longrightarrow X$ be a sequence of Birkhoff integrable functions and $f : \Omega \longrightarrow X$ a function such that:*

(i) $\lim_n f_n(t) = f(t)$ *weakly (resp. in norm) for every* $t \in \Omega$.
(ii) *The family* $\{x^* \circ f_n : n \in \mathbb{N}, x^* \in B_{X^*}\}$ *is uniformly integrable.*

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Then f is Birkhoff integrable and for each $A \in \Sigma$ we have

$$\int_A f_n \, d\mu \longrightarrow \int_A f \, d\mu \quad \textit{weakly (resp. in norm).}$$

On the other hand, we use ideas from our previous work [19] and a paper by L. Di Piazza and D. Preiss [3] to prove the following "negative" result.

Theorem 1.2. *Let X be a super-reflexive Banach space with density character $\mathfrak{c}$. Then there is a uniformly bounded sequence $f_n : [0,1] \longrightarrow X$ of Birkhoff integrable functions converging pointwise (in norm) to a function $f : [0,1] \longrightarrow X$ that is not Birkhoff integrable.*

Recall that the density character of a topological space S, denoted by $\mathrm{dens}(S)$, is the minimal cardinality of a dense subset of S.

Throughout this paper we consider the unit interval $[0,1]$ endowed with the Lebesgue measure λ on the σ-algebra of all Lebesgue measurable subsets.

Our standard reference about Banach spaces is [5]. All the Banach spaces X considered in this work are real. If the norm of X is needed explicitly, we denote it by $\|\cdot\|$. We write X^* to denote the dual of X and $B_{X^*} = \{x^* \in X^* : \|x^*\| \le 1\}$. For our purposes, *super-reflexivity* of X may be defined as existence of an equivalent uniformly convex norm on X. Recall that a norm $\|\cdot\|$ on X is called *uniformly convex* if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\|x - y\| \le \varepsilon$ for every $x, y \in X$, $\|x\| \le 1$, $\|y\| \le 1$, with $\|\frac{x+y}{2}\| \ge 1 - \delta$. Standard examples of super-reflexive spaces are the Hilbert spaces and, more generally, the L^p spaces of non negative measures, where $1 < p < \infty$. For more information on super-reflexive Banach spaces we refer the reader to [5, Chapter 9] and the references therein.

Our standard references about vector integration and vector measures are [4] and [23]. Let (Ω, Σ, μ) be an arbitrary complete probability space. A function $f : \Omega \longrightarrow X$ is *Birkhoff integrable*, with integral $\int_\Omega f \, d\mu \in X$, if for every $\varepsilon > 0$ there is a partition of Ω into countably many measurable sets $A_1, A_2, \dots$ such that $\| \sum_n \mu(A_n) f(t_n) - \int_\Omega f \, d\mu \| \le \varepsilon$ for every choice of points $t_n \in A_n$, the series involved being unconditionally convergent. Notice that the last requirement holds true automatically if f is bounded. The notion of Birkhoff integrability does not change if we consider another equivalent norm on X.

Recall also that a family $\mathcal{H}$ of real-valued integrable functions defined on Ω is *uniformly integrable* if $\sup_{h \in \mathcal{H}} \int_\Omega |h| \, d\mu < \infty$ and, for each $\varepsilon > 0$, there is $\delta > 0$ such that $\sup_{h \in \mathcal{H}} \int_A |h| \, d\mu \le \varepsilon$ for every $A \in \Sigma$ satisfying $\mu(A) \le \delta$.

2. Proof of Theorem 1.1

Throughout this section (Ω, Σ, μ) is a complete probability space and X is a Banach space. In order to prove Theorem 1.1 we will use the relationship between Birkhoff integrability and the so-called Bourgain property of a family of real-valued functions (see [2] and [18]). Recall that a family $\mathcal{H} \subset \mathbb{R}^\Omega$ has the *Bourgain property* [17] if for each $\varepsilon > 0$ and each $A \in \Sigma$ with $\mu(A) > 0$, there exist $A_1, \dots, A_n \subset A$, $A_i \in \Sigma$ with $\mu(A_i) > 0$, such that $\min_{1 \le i \le n} \mathrm{osc}(h|_{A_i}) \le \varepsilon$ for every $h \in \mathcal{H}$, where we write $\mathrm{osc}(h|_B) = \sup\{|h(t) - h(t')| : t, t' \in B\}$. In [2], together with B. Cascales, we have proved the following characterization.

Fact 2.1. *A function $f : \Omega \longrightarrow X$ is Birkhoff integrable if and only if the family*

$$Z_f = \{x^* \circ f : \; x^* \in B_{X^*}\} \subset \mathbb{R}^\Omega$$

is uniformly integrable and has the Bourgain property.

Therefore, for our purposes it is sufficient to study the stability of uniform integrability and Bourgain property under limit operations.

Lemma 2.2. *Let $f_n : \Omega \longrightarrow X$ be a sequence of functions and $f : \Omega \longrightarrow X$ a function such that:*

(i) $\lim_n f_n(t) = f(t)$ *weakly for every* $t \in \Omega$.
(ii) *The family* $\bigcup_{n\in\mathbb{N}} Z_{f_n}$ *is uniformly integrable.*

Then Z_f is uniformly integrable.

Proof. Fix $\varepsilon > 0$. By the uniform integrability of $\bigcup_{n\in\mathbb{N}} Z_{f_n}$, we have

(a) $M = \sup_{x^*\in B_{X^*}} \sup_{n\in\mathbb{N}} \int_\Omega |x^* \circ f_n| \, d\mu < \infty$.
(b) There exists $\delta > 0$ such that $\sup_{x^*\in B_{X^*}} \sup_{n\in\mathbb{N}} \int_A |x^* \circ f_n| \, d\mu \le \varepsilon$ for every $A \in \Sigma$ satisfying $\mu(A) \le \delta$.

For each $x^* \in B_{X^*}$, the classical Vitali's convergence theorem (see e.g. [11, p. 203]) applied to the uniformly integrable sequence $(x^* \circ f_n)$ ensures that its pointwise limit $x^* \circ f$ is integrable and that

$$\lim_n \int_\Omega |x^* \circ f_n - x^* \circ f| \, d\mu = 0. \tag{1}$$

In view of (a), (b) and (1) we conclude that $\sup_{x^*\in B_{X^*}} \int_\Omega |x^* \circ f| \, d\mu \le M < \infty$ and that $\sup_{x^*\in B_{X^*}} \int_A |x^* \circ f| \, d\mu \le \varepsilon$ whenever $\mu(A) \le \delta$. It follows that the family Z_f is uniformly integrable. □

Suppose that B_{X^*} has a countable weak*-dense subset D. Given a function $f : \Omega \longrightarrow X$, we can write $\|f(t)\| = \sup\{|(x^* \circ f)(t)| : x^* \in D\}$ for every $t \in \Omega$. It follows that if f is *scalarly measurable* (i.e. $x^* \circ f$ is measurable for every $x^* \in X^*$), then the real-valued mapping $t \mapsto \|f(t)\|$ is measurable. This easy observation will be used in the proofs of the next lemma and Theorem 1.1.

Lemma 2.3. *Suppose that X is isomorphic to a subspace of $\ell_\infty(\mathbb{N})$. Let $f_n : \Omega \longrightarrow X$ be a sequence of functions and $f : \Omega \longrightarrow X$ a function such that:*

(i) $\lim_n f_n(t) = f(t)$ *weakly for every* $t \in \Omega$.
(ii) *For each $n \in \mathbb{N}$ the family Z_{f_n} has the Bourgain property.*

Then Z_f has the Bourgain property.

Proof. Clearly, we can assume without loss of generality that X is isometric to a subspace of $\ell_\infty(\mathbb{N})$, that is, B_{X^*} is weak*-separable.

Fix $A \in \Sigma$ with $\mu(A) > 0$ and $\varepsilon > 0$. Let us consider the countable set

$$\mathcal{I} = \{(q_1, \dots, q_p) \in ([0,1] \cap \mathbb{Q})^p : \sum_{i=1}^p q_i = 1, \ p \in \mathbb{N}\}.$$

Given $(q_1, \dots, q_p) \in \mathcal{I}$, we define

$$E(q_1, \dots, q_p) = \Big\{t \in \Omega : \Big\|\sum_{i=1}^p q_i f_i(t) - f(t)\Big\| \le \varepsilon\Big\}.$$

Notice that the X-valued function $t \mapsto \sum_{i=1}^p q_i f_i(t) - f(t)$ is scalarly measurable (bear in mind that every family with the Bourgain property is made up of measurable functions, see e.g. [17, Theorem 11]). Since B_{X^*} is weak*-separable, the comments preceding this lemma say that $E(q_1, \dots, q_p) \in \Sigma$.

By (i), for every $t \in \Omega$ we have

$$f(t) \in \overline{\{f_n(t) : \ n \in \mathbb{N}\}}^{\text{weak}} \subset \overline{\text{co}(\{f_n(t) : \ n \in \mathbb{N}\})}^{\text{norm}}$$

and therefore

$$\Omega = \bigcup\{E(q_1, \dots, q_p) : \ (q_1, \dots, q_p) \in \mathcal{I}\}.$$

This equality and the fact that $\mu(A) > 0$ ensure us that there is $(q_1, \dots, q_p) \in \mathcal{I}$ such that $\mu(A \cap E(q_1, \dots, q_p)) > 0$. Define $g : \Omega \longrightarrow X$ by $g(t) = \sum_{i=1}^p q_i f_i(t)$.

It is clear that Z_g has the Bourgain property. Hence we can find finitely many measurable sets with positive measure $A_1, \dots, A_k \subset A \cap E(q_1, \dots, q_p)$ such that $\min_{1\leq i\leq k} \operatorname{osc}(x^* \circ g|_{A_i}) \leq \varepsilon$ for every $x^* \in B_{X^*}$. Since $\|g(t) - f(t)\| \leq \varepsilon$ for every $t \in A_i$ and every $1 \leq i \leq k$, we infer that $\min_{1\leq i\leq k} \operatorname{osc}(x^* \circ f|_{A_i}) \leq 3\varepsilon$ for every $x^* \in B_{X^*}$. It follows that Z_f has the Bourgain property and the proof is over. □

Proof of Theorem 1.1. The Birkhoff integrability of f follows directly from Fact 2.1 and Lemmas 2.2 and 2.3. The proof of the convergence of the integrals is standard and we include it for the sake of completeness. We distinguish two cases.

Case 1.- Assume that $\lim_n f_n(t) = f(t)$ *weakly for every* $t \in \Omega$. Then, for each $A \in \Sigma$, equation (1) yields

$$\lim_n x^*\Big(\int_A f_n \, d\mu\Big) = \lim_n \int_A x^* \circ f_n \, d\mu = \int_A x^* \circ f \, d\mu = x^*\Big(\int_A f \, d\mu\Big)$$

for every $x^* \in X^*$, that is, $\lim_n \int_A f_n \, d\mu = \int_A f \, d\mu$ for the weak topology.

Case 2.- Assume that $\lim_n f_n(t) = f(t)$ *in norm for every* $t \in \Omega$. Again, we can suppose without loss of generality that B_{X^*} is weak*-separable. Given $n \in \mathbb{N}$, the weak*-separability of B_{X^*} and the fact that $f_n - f$ is scalarly measurable ensure that the real-valued function $t \mapsto \|f_n(t) - f(t)\|$ is measurable. Fix $\varepsilon > 0$. Since the family $(\bigcup_{n\in\mathbb{N}} Z_{f_n}) \cup Z_f$ is uniformly integrable, there is $\delta > 0$ such that

$$(2) \quad \Big\|\int_E f_n \, d\mu - \int_E f \, d\mu\Big\| = \sup_{x^*\in B_{X^*}} x^*\Big(\int_E (f_n - f) \, d\mu\Big) \leq \sup_{x^*\in B_{X^*}} \int_E |x^* \circ f_n - x^* \circ f| \, d\mu \leq \varepsilon$$

for every $E \in \Sigma$ satisfying $\mu(E) \leq \delta$. On the other hand, by the assumption there is $n_0 \in \mathbb{N}$ such that $\mu(\{t \in \Omega : \|f_n(t) - f(t)\| > \varepsilon\}) \leq \delta$ for every $n \geq n_0$. Take any $A \in \Sigma$ and $n \geq n_0$. Write $A' = \{t \in A : \|f_n(t) - f(t)\| \leq \varepsilon\}$ and $A'' = \{t \in A : \|f_n(t) - f(t)\| > \varepsilon\}$. Then (2) yields

$$\Big\|\int_A f_n \, d\mu - \int_A f \, d\mu\Big\| \leq \Big\|\int_{A'} (f_n - f) \, d\mu\Big\| + \Big\|\int_{A''} f_n \, d\mu - \int_{A''} f \, d\mu\Big\| \leq 2\varepsilon.$$

This proves that $\lim_n \int_A f_n \, d\mu = \int_A f \, d\mu$ for the norm topology, uniformly in $A \in \Sigma$. The proof of the theorem is complete. □

Remark 2.4. When μ is a quasi-Radon probability (e.g. a Radon probability) on a topological space, McShane's approach to integration can be extended to the case of functions $f : \Omega \longrightarrow X$, see [7, 8, 9]. D. H. Fremlin [7] showed that for the McShane integral the analogue of Theorem 1.1 holds true for arbitrary X. In general, the McShane integral lies strictly between the Birkhoff and Pettis integrals, but it coincides with Birkhoff's one when X is isomorphic to a subspace of $\ell_\infty(\mathbb{N})$, see [6]. This provides another proof of our theorem in the particular case of functions defined on a quasi-Radon topological probability space.

In the proofs of Lemma 2.3 and Theorem 1.1 the weak*-separability of B_{X^*} is used only to ensure *the measurability of* $\|f(\cdot)\|$ *for any scalarly measurable function* $f : \Omega \longrightarrow X$. In [18, Corollary 4.6] we prove that the latter property is also valid in a more general situation, namely, when $\operatorname{dens}(B_{X^*}, \text{weak}^*)$ is smaller than the following uncountable cardinal number:

$$\kappa(\mu) = \min\{\text{cardinality of } \mathcal{E} : \ \mathcal{E} \subset \Sigma, \ \mu(E) = 0 \text{ for every } E \in \mathcal{E}, \ \mu^*(\cup\mathcal{E}) > 0\},$$

defined if there exist such families $\mathcal{E}$ (this happens, for instance, if μ is atomless). Here μ^* denotes the outer measure induced by μ. It is well known (see e.g. [21]) that Martin's Axiom implies the statement "$\kappa(\lambda) = \mathfrak{c}$", usually called Axiom M.

As a consequence, we conclude that *Theorem 1.1 is true whenever* X *admits an equivalent norm for which* $\mathrm{dens}(B_{X^*}, \mathrm{weak}^*) < \kappa(\mu)$. We stress that the inequality $\mathrm{dens}(B_{X^*}, \mathrm{weak}^*) \leq \mathrm{dens}(X)$ is valid for arbitrary X. For more information on the role played by $\kappa(\mu)$ in the Birkhoff integral theory, we refer the reader to [18].

3. Proof of Theorem 1.2

Throughout this section X is a non separable super-reflexive Banach space with $\mathrm{dens}(X) = \kappa$ and we fix an equivalent uniformly convex norm $\|\cdot\|$ on X.

Since X is reflexive (see e.g. [5, Theorem 9.12]), there exists a *projectional resolution of the identity* on X (see e.g. [5, Theorem 11.6]), that is, a family $\{P_\alpha\}_{\omega\leq\alpha\leq\kappa}$ of linear continuous projections $P_\alpha : X \longrightarrow X$ such that $P_\omega = 0$, P_κ is the identity operator and, for each $\omega < \alpha \leq \kappa$, we have

- $\|P_\alpha\| = 1$.
- $\mathrm{dens}(P_\alpha(X)) \leq \alpha$.
- $P_\alpha \circ P_\beta = P_\beta \circ P_\alpha = P_\beta$ whenever $\omega \leq \beta \leq \alpha$.
- $\bigcup_{\omega\leq\beta<\alpha} P_{\beta+1}(X)$ is norm dense in $P_\alpha(X)$.

A key ingredient for the proof of Theorem 1.2 is the following fact, due to L. Di Piazza and D. Preiss (see Lemma 5 and the proof of Theorem 1 in [3]). They used this geometric property to prove that Pettis and McShane integrability coincide for functions with values in super-reflexive spaces.

Fact 3.1. *Given* $n \in \mathbb{N}$*, let* d_n *be the supremum of the norms of those vectors* $x \in X$ *for which there are* $\omega \leq \beta_0 \leq \beta_1 \leq \cdots \leq \beta_n \leq \kappa$ *such that* $x \in (P_{\beta_n} - P_{\beta_0})(X)$ *and* $\|(P_{\beta_i} - P_{\beta_{i-1}})(x)\| \leq 1$ *for every* $1 \leq i \leq n$*. Then*

$$\lim_n \frac{d_n}{n} = 0.$$

Lemma 3.2. *Let* $f : [0,1] \longrightarrow X$ *be a bounded function satisfying the following properties:*

(i) *For each* $t \in [0,1]$ *there is some* $\omega \leq \alpha < \kappa$ *such that*
$$f(t) \in (P_{\alpha+1} - P_\alpha)(X).$$
(ii) *For each* $\omega \leq \alpha < \kappa$ *there is at most one* $t \in [0,1]$ *such that*
$$(P_{\alpha+1} - P_\alpha)(f(t)) \neq 0.$$

Then f *is Birkhoff integrable and* $\int_0^1 f \, d\lambda = 0$.

Proof. We can assume without loss of generality that $\|f(t)\| \leq 1$ for every $t \in [0,1]$. Fix $\varepsilon > 0$. By Fact 3.1, we can find $n \in \mathbb{N}$ large enough such that $d_{2n}/(2n) \leq \varepsilon$. Let $\{I_1, \dots, I_n\}$ be a finite partition of $[0,1]$ into Lebesgue measurable sets with $\lambda(I_i) = 1/n$ for every $1 \leq i \leq n$, and take $t_i \in I_i$ for every $1 \leq i \leq n$. Properties (i) and (ii) allow us to find $\omega \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n < \kappa$ such that

$$x := \sum_{i=1}^n \lambda(I_i) f(t_i) = \frac{1}{n} \sum_{j=1}^n y_j$$

for some $y_j \in (P_{\alpha_j+1} - P_{\alpha_j})(X)$ with $\|y_j\| \leq 1$. Set $\beta_{2j} = \alpha_j + 1$ and $\beta_{2j-1} = \alpha_j$ for every $1 \leq j \leq n$, with $\beta_0 = \beta_1$. Then $\omega \leq \beta_0 \leq \beta_1 \leq \cdots \leq \beta_{2n} < \kappa$, $nx \in (P_{\beta_{2n}} - P_{\beta_0})(X)$ and, for each $1 \leq i \leq 2n$, we have

$$(P_{\beta_i} - P_{\beta_{i-1}})(nx) = \begin{cases} y_j & \text{if } i = 2j \text{ for some } 1 \leq j \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\|nx\| \leq d_{2n}$ and therefore

$$\Big\|\sum_{i=1}^{n} \lambda(I_i)f(t_i)\Big\| \leq \frac{d_{2n}}{n} \leq 2\varepsilon.$$

As $\varepsilon > 0$ is arbitrary, f is Birkhoff integrable, with integral 0. □

Observe that a function as in Lemma 3.2 is even *Riemann integrable*, i.e. its Birkhoff integrability can be checked by means of finite partitions into intervals. We refer the reader to [10] for a detailed survey on this topic.

Notice also that Lemma 3.2 still holds true if $[0,1]$ is replaced by an arbitrary complete probability space without atoms.

In order to prove Theorem 1.2 we will use some ideas from our counterexample [19] to Vitali's convergence theorem for $c_0(\mathfrak{c})$-valued Birkhoff integrable functions. We first need to recall the construction isolated in Fact 3.3 below, see [19, Lemma 2.2] for a proof.

Fact 3.3. *Let $\{\Gamma_\beta : \beta < \mathfrak{c}\}$ be an enumeration of the collection of all countable partitions of $[0,1]$ by Borel sets. Then there exist collections $\{A_\beta\}_{\beta<\mathfrak{c}}$ and $\{A'_\beta\}_{\beta<\mathfrak{c}}$ of countable subsets of $[0,1]$ such that:*

(i) *$A_\beta \cap A_\gamma = \emptyset$ and $A'_\beta \cap A'_\gamma = \emptyset$ for every $\beta, \gamma < \mathfrak{c}$ with $\beta \neq \gamma$.*
(ii) *$A_\beta \cap A'_\gamma = \emptyset$ for every $\beta, \gamma < \mathfrak{c}$.*
(iii) *For each $\beta < \mathfrak{c}$ and each $E \in \Gamma_\beta$ with $\lambda(E) > 0$, we have $A_\beta \cap E \neq \emptyset$ and $A'_\beta \cap E \neq \emptyset$.*

Proof of Theorem 1.2. By the assumption, $\kappa = \text{dens}(X) = \mathfrak{c}$. For each $\omega \leq \alpha < \mathfrak{c}$, fix $x_\alpha \in (P_{\alpha+1} - P_\alpha)(X)$ with $\|x_\alpha\| = 1$. Fix two injective maps $\phi, \psi : \mathfrak{c} \longrightarrow [\omega, \mathfrak{c})$ with disjoint ranges. Define $f : [0,1] \longrightarrow X$ by

$$f(t) = \begin{cases} x_{\phi(\beta)} & \text{if } t \in A_\beta,\ \beta < \mathfrak{c}, \\ x_{\psi(\beta)} & \text{if } t \in A'_\beta,\ \beta < \mathfrak{c}, \\ 0 & \text{if } t \notin \bigcup_{\beta<\mathfrak{c}}(A_\beta \cup A'_\beta). \end{cases}$$

Enumerate $A_\beta = \{a_{\beta,1}, a_{\beta,2}, \dots\}$ and $A'_\beta = \{a'_{\beta,1}, a'_{\beta,2}, \dots\}$ for every $\beta < \mathfrak{c}$ and define $D_k = \{a_{\beta,k} : \beta < \mathfrak{c}\} \cup \{a'_{\beta,k} : \beta < \mathfrak{c}\}$ for every $k \in \mathbb{N}$. It is easy to check that each $f\chi_{D_k} : [0,1] \longrightarrow X$ satisfies properties (i) and (ii) in Lemma 3.2 (as usual, χ_{D_k} denotes the characteristic function of D_k), hence it is Birkhoff integrable. Therefore, $f_n := \sum_{k=1}^{n} f\chi_{D_k}$ is Birkhoff integrable for every $n \in \mathbb{N}$. It is clear that (f_n) is a uniformly bounded sequence that converges pointwise (in norm) to f.

To finish the proof we will show that f is not Birkhoff integrable. Suppose, if possible, otherwise. Then there exists a countable partition of $[0,1]$ into Lebesgue measurable sets $B_1, B_2, \dots$ such that

$$\Big\|\sum_n \lambda(B_n)f(t_n) - \sum_n \lambda(B_n)f(t'_n)\Big\| < \frac{1}{2} \tag{3}$$

for arbitrary choices $t_n, t'_n \in B_n$. By the inner regularity of λ with respect to the Borel σ-algebra of $[0,1]$, we can assume further that each B_n is a Borel set, that is, $\{B_1, B_2, \dots\} = \Gamma_\beta$ for some $\beta < \mathfrak{c}$. By property (iii) in Fact 3.3, for each $B_m \in \Gamma_\beta$ we can choose two points $t_m, t'_m \in B_m$ such that $\sum_n \lambda(B_n)f(t_n) = x_{\phi(\beta)}$ and $\sum_n \lambda(B_n)f(t'_n) = x_{\psi(\beta)}$. It follows that

$$\begin{aligned} 2 \cdot \Big\|\sum_n \lambda(B_n)f(t_n) - \sum_n \lambda(B_n)f(t'_n)\Big\| &= 2 \cdot \|x_{\phi(\beta)} - x_{\psi(\beta)}\| \\ &\geq \|(P_{\phi(\beta)+1} - P_{\phi(\beta)})(x_{\phi(\beta)} - x_{\psi(\beta)})\| = \|x_{\phi(\beta)}\| = 1, \end{aligned}$$

which contradicts inequality (3). This shows that f is not Birkhoff integrable. The proof of the theorem is complete. □

Remark 3.4. Observe that the super-reflexivity of X has only been used to handle the property isolated in Fact 3.1. We stress that this property is also valid for $c_0(\Gamma)$ (where Γ is any uncountable set) equipped with its canonical projectional resolution of the identity. Thus, as regards the non validity of Vitali's convergence theorem for the Birkhoff integral, the approach presented here unifies the already known case of $c_0(\mathfrak{c})$ and the case of super-reflexive spaces with density character $\mathfrak{c}$.

The comments at the end of Section 2 show that, at least under Axiom M, the conclusion of Theorem 1.2 (and its analogue for spaces of the form $c_0(\Gamma)$) does not remain true if $\mathrm{dens}(X) < \mathfrak{c}$; in fact, in this case every Birkhoff integrable function $f : [0, 1] \longrightarrow X$ is strongly measurable, see e.g. [18, Corollary 4.12].

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