INTEGRATION IN HILBERT GENERATED BANACH SPACES

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Abstract. We prove that McShane and Pettis integrability are equivalent for functions taking values in a subspace of a Hilbert generated Banach space. This generalizes simultaneously all previous results on such equivalence. On the other hand, for any super-reflexive generated Banach space having density character greater than or equal to the continuum, we show that Birkhoff integrability lies strictly between Bochner and McShane integrability. Finally, we give a ZFC example of a scalarly null Banach space-valued function (defined on a Radon probability space) which is not McShane integrable.

1. Introduction

The comparison between different generalizations of the Lebesgue integral is a milestone of the modern theory of integration of Banach space-valued functions. For functions taking values in a *separable* Banach space the situation is well understood thanks to Pettis' measurability theorem, which reduces the problem to the analysis of the convergence character of certain series. On the contrary, for *non-separable* Banach spaces several difficulties appear enriching the field of vector integration.

The *McShane integral* of vector-valued functions (see Section 2 for precise definitions) was first studied by Gordon [19], Fremlin and Mendoza [18] in the case of functions defined on $[0, 1]$ (with the Lebesgue measure). Then Fremlin [16] developed a generalized McShane integral theory for functions defined in a wide class of topological measure spaces (including Radon probability spaces). These authors proved that, in general, McShane integrability lies strictly between Bochner and Pettis integrability, while McShane and Pettis integrability are equivalent for functions taking values in separable Banach spaces.

More recently, Di Piazza and Preiss [6] studied the equivalence of McShane and Pettis integrability in certain classes of non-separable Banach spaces. They showed that such equivalence holds true for functions taking values in $c_0(I)$ (for any set I) or in arbitrary super-reflexive Banach spaces. From the technical point of view, the methods of Di Piazza and Preiss gathered the particular geometrical properties of these spaces together with some powerful tools of linear geometry and topology of Banach spaces (projectional resolutions of the identity and weak measure-compactness) which are actually available in more general classes of spaces, like the weakly compactly generated (WCG for short) ones. In this way, they asked [6, p. 1178] whether McShane and Pettis integrability are equivalent for functions taking values in arbitrary WCG Banach spaces. The second named author [27] gave

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another partial answer to this question by showing that this is always the case for $L^1(\nu)$ (where ν is any probability measure).

On the other hand, the Pettis integrable ℓ^∞-valued function which is not McShane integrable constructed by Fremlin and Mendoza [18] gave no information regarding the following question attributed to Musial in [6, p. 1177]: *Is every scalarly null Banach space-valued function McShane integrable?* Under the Continuum Hypothesis, Di Piazza and Preiss [6] and the second named author [27] provided examples of scalarly null functions defined on $[0,1]$ which are not McShane integrable.

In this paper we discuss the equivalence of McShane and Pettis integrability in certain classes of non-separable Banach spaces. Also, further comparison results involving the Bochner and Birkhoff integrals are presented. We next summarize the contents of our work.

Section 2 introduces the terminology and preliminaries that are needed throughout the rest of the paper.

Section 3 deals with the McShane and Pettis integrals in subspaces of Hilbert generated Banach spaces. This well known class of Banach spaces includes all separable spaces, $c_0(I)$ (for any set I), all super-reflexive spaces and $L^1(\nu)$ (for any probability measure ν). Our main result states that *McShane and Pettis integrability are equivalent for functions taking values in subspaces of Hilbert generated Banach spaces* (Theorem 3.7). This generalizes simultaneously all previous results on such equivalence. Our approach relies heavily on the special properties of the Markushevich bases of those Banach spaces.

Section 4 is devoted to show that, whenever X is a super-reflexive generated Banach space with density character greater than or equal to the continuum, one can find X-valued functions defined on $[0,1]$ witnessing that Birkhoff integrability lies *strictly* between Bochner and McShane integrability (Theorem 4.8).

Section 5 presents a ZFC example of a scalarly null function defined on a Radon probability space which is not McShane integrable (Theorem 5.8). This provides a negative answer to the aforementioned question of Musial, without using additional set-theoretic axioms. The existence of such a function comes as an application of Fremlin's work [15] on measure-additive coverings together with the fact that every McShane integrable $\ell^1(I)$-valued function (where I is *any* set) is strongly measurable (Proposition 5.4).

2. Preliminaries

All unexplained terminology can be found in our standard references [9] and [21]. The cardinality of a set S is denoted by $\mathrm{card}(S)$. The continuum, $\mathrm{card}(\mathbb{R})$, is denoted by $\mathfrak{c}$. The density character of a topological space T, denoted by $\mathrm{dens}(T)$, is the minimal cardinality of a dense subset of T. Our Banach spaces X are assumed to be real. We sometimes write $\|\cdot\|_X$ to denote the norm of X if it is needed explicitly. By a 'subspace' of X we mean a closed linear subspace. As usual, B_X stands for the closed unit ball of X and X^* denotes the topological dual of X. The symbol w^* stands for the weak* topology on X^*. A set $\Gamma \subset X^*$ is said to be *total* if it separates the points of X (i.e. for each $x \in X \setminus \{0\}$ there is $x^* \in \Gamma$ such that $x^*(x) \neq 0$). A *Markushevich basis* of X is a family $(x_i, x_i^*)_{i\in I}$, where $x_i \in X$ and $x_i^* \in X^*$, such that $x_i^*(x_j) = \delta_{i,j}$ (the Kronecker symbol) for every $i, j \in I$, $\overline{\mathrm{span}}\{x_i : i \in I\} = X$ and $\{x_i^* : i \in I\}$ is total. By an 'operator' between Banach spaces we mean a linear continuous mapping.

After Enflo's theorem (cf. [4, Theorem 4.1, p. 149]), a Banach space X is *super-reflexive* if and only if it admits an equivalent uniformly convex norm. The basic example of super-reflexive space is $L^p(\nu)$ for $1 < p < \infty$ (where ν is any measure). A Banach space X is called *Hilbert generated* (resp. *super-reflexive generated*) if there exist a Hilbert (resp. super-reflexive Banach) space Y and an operator $T : Y \to X$ such that its range $T(Y)$ is dense in X. In particular, such an X is always WCG. Every separable Banach space is Hilbert generated, cf. [4, Lemma 2.5, p. 47]. Other examples of Hilbert generated spaces are $c_0(I)$ (for any set I) and $L^1(\nu)$ (for any probability measure ν): just consider the "identity" operators $\ell^2(I) \to c_0(I)$ and $L^2(\nu) \to L^1(\nu)$, respectively.

For an arbitrary Banach space the following implications hold:

$$\text{Hilbert generated} \Longrightarrow \text{super-reflexive generated} \Longrightarrow$$
$$\Longrightarrow \text{subspace of a Hilbert generated Banach space,}$$

and no one of these arrows can be reversed in general, see [12], Theorem 1 and Section 4. Subspaces of Hilbert generated Banach spaces can be nicely characterized via Markushevich bases as follows, see [13, Theorem 6] (cf. [21, Theorem 6.30]).

Theorem 2.1. *For a Banach space X the following statements are equivalent:*

(1) *X is a subspace of a Hilbert generated Banach space.*
(2) *There is a Markushevich basis $(x_i, x_i^*)_{i\in I}$ of X with $x_i \in B_X$ for all $i \in I$ satisfying the following property: for each $\varepsilon > 0$ there is a decomposition $I = \bigcup_{n\in\mathbb{N}} I_n^\varepsilon$ such that*

$$\text{for all } x^* \in B_{X^*} \text{ and all } n \in \mathbb{N}, \quad \mathrm{card}(\{i \in I_n^\varepsilon : |x^*(x_i)| > \varepsilon\}) \le n.$$

Moreover, in this case, the property mentioned in (2) holds for any Markushevich basis $(x_i, x_i^*)_{i\in I}$ of X such that $x_i \in B_X$ for all $i \in I$.

Equivalently, a Banach space X is a subspace of a Hilbert generated one if and only if (B_{X^*}, w^*) is uniform Eberlein compact (i.e. it is homeomorphic to a weakly compact subset of a Hilbert space) if and only if X admits an equivalent uniformly Gâteaux smooth norm (cf. [21, Theorem 6.30]).

Given a Banach space X, a complete probability space (Ω, Σ, μ) and a function $f : \Omega \to X$, recall that f is called:

- *scalarly null* if for each $x^* \in X^*$ the composition x^*f vanishes μ-a.e. (the exceptional set depending on x^*);
- *scalarly measurable* if x^*f is measurable for every $x^* \in X^*$;
- *scalarly bounded* if there is $M > 0$ such that for each $x^* \in B_{X^*}$ we have $|x^*f| \le M$ μ-a.e. (the exceptional set depending on x^*);
- *strongly measurable* if it is scalarly measurable and there is $E \in \Sigma$ with $\mu(E) = 1$ such that $f(E)$ is separable; equivalently, f is the μ-a.e. limit of a sequence of simple functions, cf. [9, Theorem 2, p. 42];
- *Bochner integrable* if it is strongly measurable and $\int_\Omega \|f(\cdot)\|_X \, d\mu < \infty$;
- *Pettis integrable* if x^*f is integrable for every $x^* \in X^*$ and for each $E \in \Sigma$ there is a vector $\int_E f \, d\mu \in X$ (the *Pettis integral* of f over E) such that

$$\int_E x^*f \, d\mu = x^*\Big(\int_E f \, d\mu\Big) \quad \text{for all } x^* \in X^*.$$

Clearly, every scalarly null function is Pettis integrable. Recall also that a function $g : \Omega \to X$ is *scalarly equivalent* to f if $f - g$ is scalarly null.

In order to recall the definition of the McShane integral we need to introduce some terminology. A *quasi-Radon* probability space [17, Chapter 41] is a quadruple $(\Omega, \mathfrak{T}, \Sigma, \mu)$, where (Ω, Σ, μ) is a complete probability space and $\mathfrak{T} \subset \Sigma$ is a topology on Ω such that μ is inner regular with respect to the collection of all closed sets, and $\mu(\bigcup \mathcal{G}) = \sup\{\mu(G) : G \in \mathcal{G}\}$ for every upwards directed family $\mathcal{G} \subset \mathfrak{T}$. For instance, every Radon probability space is quasi-Radon, see [17, 416A]. A *generalized McShane partition* of Ω is a sequence $(E_i, t_i)_{i\in\mathbb{N}}$ where the E_i's are pairwise disjoint measurable sets such that $\mu(\Omega \setminus \bigcup_{i\in\mathbb{N}} E_i) = 0$ and $t_i \in \Omega$ for every $i \in \mathbb{N}$. A *partial McShane partition* of Ω is a countable (maybe finite) collection $(E_i, t_i)_{i\in I}$ where the E_i's are pairwise disjoint measurable sets and $t_i \in \Omega$ for every $i \in I$. A *gauge* on Ω is a function $\delta : \Omega \to \mathfrak{T}$ such that $t \in \delta(t)$ for every $t \in \Omega$; a partial McShane partition $(E_i, t_i)_{i\in I}$ of Ω is *subordinate* to δ if $E_i \subset \delta(t_i)$ for every $i \in I$. For every gauge on Ω there is a generalized McShane partition of Ω subordinate to it, see [16, 1B(d)]. A function f defined on Ω and taking values in a Banach space X is called *McShane integrable*, with McShane integral $x \in X$, see [16, 1A], if for every $\varepsilon > 0$ there is a gauge δ on Ω such that

$$\limsup_{n\to\infty} \Big\| \sum_{i=1}^{n} \mu(E_i) f(t_i) - x \Big\|_X \leq \varepsilon$$

for every generalized McShane partition $(E_i, t_i)_{i\in\mathbb{N}}$ of Ω subordinate to δ. In this case f is also Pettis integrable (and the respective integrals coincide), see [16, 1Q].

We should point out that for functions defined on $[0, 1]$ (with the Lebesgue measure λ) the McShane integral can be defined in a simpler way by using 'finite partitions into non-overlapping closed subintervals'. More precisely, a function $f : [0, 1] \to X$ is McShane integrable, with integral $x \in X$, if and only if for every $\varepsilon > 0$ there is a positive function δ on $[0, 1]$ such that the inequality $\| \sum_{i=1}^{p} \lambda(I_i) f(t_i) - x\|_X \leq \varepsilon$ holds for every finite collection $I_1, \dots, I_p$ of non-overlapping closed intervals with $[0, 1] = \bigcup_{i=1}^{p} I_i$ and every choice of points $t_i \in [0, 1]$ with $I_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$ for all $1 \leq i \leq p$; see [16, 1F-1G].

3. Equivalence of McShane and Pettis integrability in subspaces of Hilbert generated Banach spaces

The following two lemmas will be frequently used throughout this section. The first one is an immediate consequence of the Saks-Henstock lemma [16, 2B]. The second one can be proved as [6, Lemma 2(2)] or can be deduced straightforwardly from Fremlin's convergence theorem for the McShane integral, see [16, 4A].

Lemma 3.1. *Let X be a Banach space, $(\Omega, \mathfrak{T}, \Sigma, \mu)$ a quasi-Radon probability space and $f : \Omega \to X$ a function. Then f is scalarly null and McShane integrable if and only if for every $\varepsilon > 0$ there is a gauge δ on Ω such that*

$$\Big\| \sum_{j=1}^{p} \mu(E_j) f(t_j) \Big\|_X \leq \varepsilon$$

for every partial McShane partition $(E_j, t_j)_{1\leq j\leq p}$ of Ω subordinate to δ.

Lemma 3.2. *Let X be a Banach space, $(\Omega, \mathfrak{T}, \Sigma, \mu)$ a quasi-Radon probability space and $f_n : \Omega \to X$ a sequence of scalarly null McShane integrable functions converging pointwise to a function $f : \Omega \to X$. Then f is scalarly null and McShane integrable.*

Given a subspace Y of a Banach space X, it is easy to check that a Y-valued function is Pettis (resp. McShane) integrable if and only if it is Pettis (resp. McShane) integrable when considered as an X-valued function. Thus, in order to prove that Pettis and McShane integrability coincide in *subspaces* of Hilbert generated Banach spaces (Theorem 3.7 below) it would be sufficient to check the case of Hilbert generated Banach spaces. However, our techniques apply directly to the general case. The proof of Theorem 3.7 is divided into several lemmas.

Given two sets $A \subset B$, a Banach space X and a function $f : B \to X$, we write $f\chi_A$ to denote the X-valued function on B which agrees with f on A and vanishes on $B \setminus A$.

Lemma 3.3. *Let X be a subspace of a Hilbert generated Banach space and consider a Markushevich basis $(x_i, x_i^*)_{i\in I}$ of X. Let $(\Omega, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon probability space and $f : \Omega \to X$ a scalarly null function. If $f(\Omega) \subset \{\lambda x_i : i \in I, \lambda \in \mathbb{R}\}$, then f is McShane integrable.*

Proof. We assume without loss of generality (normalize!) that $x_i \in B_X$ for all $i \in I$. For each $t \in \Omega$ we can write $f(t) = h(t)x_{i(t)}$ for some $h(t) \in \mathbb{R}$ and $i(t) \in I$.

Claim. We can assume without loss of generality that $h : \Omega \to \mathbb{R}$ is bounded. Indeed, for each $n \in \mathbb{N}$, set $A_n := \{t \in \Omega : |h(t)| \leq n\}$. Then (A_n) is an increasing sequence of subsets covering Ω, hence $f\chi_{A_n} \to f$ pointwise as $n \to \infty$. Clearly, each $f\chi_{A_n}$ is scalarly null and admits a representation of the form $f\chi_{A_n}(t) = h_n(t)x_{i(t)}$, where $h_n = h\chi_{A_n}$ is bounded. In view of Lemma 3.2, f is McShane integrable if each $f\chi_{A_n}$ is. This proves the claim.

So, we assume that $M := \sup_{t\in\Omega} |h(t)| < \infty$. Fix $\varepsilon > 0$. Since the Markushevich basis $(x_i, x_i^*)_{i\in I}$ satisfies the property mentioned in Theorem 2.1(2), there is a decomposition $I = \bigcup_{n\in\mathbb{N}} I_n^\varepsilon$ such that

$$\text{(1)} \qquad \text{for all } x^* \in B_{X^*} \text{ and all } n \in \mathbb{N}, \quad \operatorname{card}(\{i \in I_n^\varepsilon : |x^*(x_i)| > \varepsilon\}) \leq n.$$

For each $i \in I$ there is a unique $n(i) \in \mathbb{N}$ such that $i \in I_{n(i)}^\varepsilon$; since $x_i^* f$ vanishes μ-a.e., we can find a gauge δ_i on Ω such that

$$\text{(2)} \qquad \Big|\sum_{j=1}^{p} \mu(E_j)x_i^* f(t_j)\Big| \leq \frac{\varepsilon}{2^{n(i)} \cdot n(i)}$$

for every partial McShane partition $(E_j, t_j)_{1\leq j\leq p}$ of Ω subordinate to δ_i (apply Lemma 3.1 to the real-valued function $x_i^* f$).

Define a gauge δ on Ω by $\delta(t) := \delta_{i(t)}(t)$. Take any partial McShane partition $(E_j, t_j)_{1\leq j\leq p}$ of Ω subordinate to δ. Define

$$J(i) := \{1 \leq j \leq p : \ i(t_j) = i\} \quad \text{for all } i \in I.$$

Fix $x^* \in B_{X^*}$ and set

$$A := \{i \in I : \ |x^*(x_i)| \leq \varepsilon\} \quad \text{and} \quad B_n := \{i \in I_n^\varepsilon : \ |x^*(x_i)| > \varepsilon\} \ \text{ for all } n \in \mathbb{N}.$$

We can write

$$\text{(3)} \quad \sum_{j=1}^{p} \mu(E_j)f(t_j) = \sum_{i\in A}\Big(\sum_{j\in J(i)} \mu(E_j)f(t_j)\Big) + \sum_{n\in\mathbb{N}}\Big(\sum_{i\in B_n}\Big(\sum_{j\in J(i)} \mu(E_j)f(t_j)\Big)\Big).$$

On the one hand, since the $J(i)$'s are pairwise disjoint, we have

$$\begin{aligned}(4)\quad \Big|x^*\Big(\sum_{i\in A}\Big(\sum_{j\in J(i)}\mu(E_j)f(t_j)\Big)\Big)\Big| &= \Big|\sum_{i\in A}\Big(\sum_{j\in J(i)}\mu(E_j)x^*f(t_j)\Big)\Big| \leq \\ \leq \sum_{i\in A}\Big(\sum_{j\in J(i)}\mu(E_j)|x^*f(t_j)|\Big) &= \sum_{i\in A}\Big(\sum_{j\in J(i)}\mu(E_j)|h(t_j)|\Big)\cdot|x^*(x_i)| \leq \\ \leq M\cdot\Big(\sum_{i\in A}\Big(\sum_{j\in J(i)}\mu(E_j)\Big)\Big)\cdot\varepsilon &= M\cdot\mu\Big(\bigcup_{i\in A}\bigcup_{j\in J(i)}E_j\Big)\cdot\varepsilon \leq M\cdot\varepsilon.\end{aligned}$$

On the other hand, for each $i\in I$ with $J(i)\neq\emptyset$, we have $x^*f(t_j)=x_i^*f(t_j)x^*(x_i)$ whenever $j\in J(i)$, and therefore (2) yields

$$\begin{aligned}(5)\quad \Big|x^*\Big(\sum_{j\in J(i)}\mu(E_j)f(t_j)\Big)\Big| &= \Big|\sum_{j\in J(i)}\mu(E_j)x^*f(t_j)\Big| = \\ &= \Big|\sum_{j\in J(i)}\mu(E_j)x_i^*f(t_j)\Big|\cdot|x^*(x_i)| \leq \frac{\varepsilon}{2^{n(i)}\cdot n(i)},\end{aligned}$$

because $x_i\in B_X$ and $(E_j,t_j)_{j\in J(i)}$ is a partial McShane partition of Ω subordinate to δ_i. Bearing in mind (3), (4), (5) and the fact that $\mathrm{card}(B_n)\leq n$ for all $n\in\mathbb{N}$ (by (1)), it follows that

$$\begin{aligned}&\Big|x^*\Big(\sum_{j=1}^{p}\mu(E_j)f(t_j)\Big)\Big| \leq \\ &\leq \Big|x^*\Big(\sum_{i\in A}\Big(\sum_{j\in J(i)}\mu(E_j)f(t_j)\Big)\Big)\Big| + \sum_{n\in\mathbb{N}}\Big(\sum_{i\in B_n}\Big|x^*\Big(\sum_{j\in J(i)}\mu(E_j)f(t_j)\Big)\Big|\Big) \leq \\ &\leq M\cdot\varepsilon + \sum_{n\in\mathbb{N}}\mathrm{card}(B_n)\cdot\frac{\varepsilon}{2^n\cdot n} \leq (M+1)\cdot\varepsilon.\end{aligned}$$

As $x^*\in B_{X^*}$ is arbitrary, we have $\|\sum_{j=1}^{p}\mu(E_j)f(t_j)\|_X\leq(M+1)\cdot\varepsilon$. As $\varepsilon>0$ is arbitrary, Lemma 3.1 tells us that f is McShane integrable. □

In the following lemma we use the fact that every subspace of a Hilbert generated Banach space belongs to the class of *weakly Lindelöf determined* (*WLD* for short) spaces (cf. [21, Theorem 6.13]). Recall that a Banach space X is WLD if (B_{X^*},w^*) is a Corson compact, i.e. it is homeomorphic to a set $S\subset[-1,1]^I$ (endowed with the product topology) such that $\{i\in I: s(i)\neq 0\}$ is countable for every $s\in S$.

Lemma 3.4. *Let X be a subspace of a Hilbert generated Banach space and consider a Markushevich basis $(x_i,x_i^*)_{i\in I}$ of X. Let $(\Omega,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon probability space and $f:\Omega\to X$ a scalarly null function. Let $g:\Omega\to X$ be a function such that*

$$g(t)\in\mathrm{span}\{(x_i^*f(t))x_i:\, i\in I\}\quad \textit{for all } t\in\Omega.$$

Then g is scalarly null and McShane integrable.

Proof. For each $t\in\Omega$, we can find $n(t)\in\mathbb{N}$, real numbers $a_1(t),\dots,a_{n(t)}(t)$ and a set $\{i_1(t),\dots,i_{n(t)}(t)\}\subset I$ such that

$$g(t)=\sum_{n=1}^{n(t)}a_n(t)x_{i_n(t)}$$

with the additional property that $a_n(t) = 0$ whenever $x^*_{i_n(t)} f(t) = 0$. For each $n \in \mathbb{N}$ we define a function $g_n : \Omega \to X$ by the formula

$$g_n(t) = \begin{cases} a_n(t) x_{i_n(t)} & \text{if } n(t) \geq n, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, we have $g = \sum_{n=1}^{\infty} g_n$ pointwise and, therefore, in order to finish the proof we only have to check that each g_n is scalarly null and McShane integrable (by Lemma 3.2). To this end, fix $n \in \mathbb{N}$. Since the range of g_n is contained in $\{\lambda x_i : i \in I, \lambda \in \mathbb{R}\}$, in order to prove that g_n is McShane integrable it suffices to show that g_n is scalarly null (by Lemma 3.3).

Fix $x^* \in X^*$. Since X is WLD, the set $\{i \in I : x^*(x_i) \neq 0\}$ is countable, say $\{i_1, i_2, \dots\}$ (cf. [21, Lemma 5.35]). For each $k \in \mathbb{N}$, set

$$B_k := \{t \in \Omega : \ n(t) \geq n, \ i_n(t) = i_k\}$$

and observe that for each $t \in B_k$ we have $x^* g_n(t) = a_n(t) x^*(x_{i_k})$, with $a_n(t) = 0$ whenever $x^*_{i_k} f(t) = 0$; since f is scalarly null, the function $x^*_{i_k} f$ vanishes μ-a.e. and so the same holds for $x^* g_n \chi_{B_k}$. Writing $B := \bigcup_{k \in \mathbb{N}} B_k$, it follows that $x^* g_n \chi_B$ vanishes μ-a.e. On the other hand, $x^* g_n(t) = 0$ for all $t \in \Omega \setminus B$. Therefore $x^* g_n = 0$ μ-a.e. As $x^* \in X^*$ is arbitrary, g_n is scalarly null and the proof is over. □

A Markushevich basis $(x_i, x_i^*)_{i \in I}$ of a Banach space X is called *strong* if every $x \in X$ belongs to $\overline{\text{span}}\{x_i^*(x) x_i : i \in I\}$. A striking result of Terenzi [31] (cf. [21, Theorem 1.36]) states that every separable Banach space admits a strong Markushevich basis. This result can be extended to more general classes of Banach spaces by means of projectional resolutions of the identity and a standard transfinite induction argument. For instance, every WLD Banach space admits a strong Markushevich basis, see [21, Corollary 5.2]. In particular, *every subspace of a Hilbert generated Banach space admits a strong Markushevich basis.*

Lemma 3.5. *Let X be a Banach space admitting a strong Markushevich basis $(x_i, x_i^*)_{i \in I}$. Let Ω be a set and $f : \Omega \to X$ a function. Then for every $\varepsilon > 0$ there is a function $g : \Omega \to X$ such that*

$$g(t) \in \text{span}\{(x_i^* f(t)) x_i : i \in I\} \quad \textit{and} \quad \|f(t) - g(t)\|_X \leq \varepsilon \quad \textit{for all } t \in \Omega.$$

Proof. Straightforward. □

Lemma 3.6. *Let X be a subspace of a Hilbert generated Banach space, $(\Omega, \mathfrak{T}, \Sigma, \mu)$ a quasi-Radon probability space and $f : \Omega \to X$ a scalarly null function. Then f is McShane integrable.*

Proof. As we have already mentioned, X admits a strong Markushevich basis $(x_i, x_i^*)_{i \in I}$. In view of Lemmas 3.5 and 3.4, there is a sequence of scalarly null McShane integrable X-valued functions defined on Ω converging uniformly to f. An appeal to Lemma 3.2 ensures that f is McShane integrable. □

We are now ready to prove our main result in this section.

Theorem 3.7. *Let X be a subspace of a Hilbert generated Banach space, $(\Omega, \mathfrak{T}, \Sigma, \mu)$ a quasi-Radon probability space and $f : \Omega \to X$ a function. Then f is McShane integrable if and only if it is Pettis integrable.*

Proof. It only remains to prove the *if* part. Since X is WLD, it is weakly Lindelöf (cf. [21, Theorem 5.37]) and so weakly measure-compact. This fact and the scalar measurability of f ensure the existence of a strongly measurable function $g : \Omega \to X$ which is scalarly equivalent to f (see [10, Proposition 5.4]). Since $h := f - g$ is scalarly null, Lemma 3.6 applies to infer that h is McShane integrable. On the other hand, $g = f - h$ is Pettis integrable and strongly measurable, hence it is McShane integrable (see [16, 4C]). It follows that $f = g + h$ is McShane integrable. □

All Banach spaces considered in the following corollary are subspaces of Hilbert generated spaces. Observe that the separable case (i) has been used in the proof of Theorem 3.7. Cases (ii) and (iii) are due to Di Piazza and Preiss [6]. Case (iv) was recently proved by the second named author [27].

Corollary 3.8. *Let X be a Banach space satisfying any of the following conditions:*

(i) *X is separable.*
(ii) *$X = c_0(I)$ (for any set I).*
(iii) *X is super-reflexive.*
(iv) *$X = L^1(\nu)$ (for any probability measure ν).*

Let $(\Omega, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon probability space and $f : \Omega \to X$ a function. Then f is McShane integrable if and only if it is Pettis integrable.

A Markushevich basis $(x_i, x_i^*)_{i\in I}$ of a Banach space X is called an *unconditional Schauder basis* of X if, for each $x \in X$, the family $(x_i^*(x)x_i)_{i\in I}$ is summable in X and $x = \sum_{i\in I} x_i^*(x)x_i$; in this case, the basis is called *symmetric* if, for each bijection $\pi : I \to I$ and each $a_i \in \mathbb{R}$, $i \in I$, the family $(a_i x_i)_{i\in I}$ is summable if and only if $(a_i x_{\pi(i)})_{i\in I}$ is summable. A result of Troyanski [32] (cf. [21, Theorem 7.54]) states that, if a non-separable Banach space X has a symmetric unconditional Schauder basis and does not admit an equivalent uniformly Gâteaux smooth norm, then X is isomorphic to $\ell^1(I)$ for some set I. As we mentioned in Section 2, the property of admitting an equivalent uniformly Gâteaux smooth norm characterizes the subspaces of Hilbert generated Banach spaces. Bearing in mind Theorem 3.7 and the fact that every Pettis integrable $\ell^1(\mathfrak{c})$-valued function is strongly measurable (see Remark 5.3 in Section 5 below), we arrive at the following:

Corollary 3.9. *Let X be a Banach space with $\mathrm{dens}(X) \leq \mathfrak{c}$ having a symmetric unconditional Schauder basis. Let $(\Omega, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon probability space and $f : \Omega \to X$ a function. Then f is McShane integrable if and only if it is Pettis integrable.*

In the statement of the previous corollary, the restriction on the density character cannot be removed in general, see Theorem 5.8 in Section 5.

4. Comparing the Bochner, Birkhoff and McShane integrals in super-reflexive generated Banach spaces

Given a complete probability space (Ω, Σ, μ) and a Banach space X, a function $f : \Omega \to X$ is called *Birkhoff integrable*, with Birkhoff integral $x \in X$, if for every $\varepsilon > 0$ there is a countable partition (A_n) of Ω in Σ such that, for any choice of points $t_n \in A_n$, the series $\sum_n \mu(A_n)f(t_n)$ converges unconditionally in X and

$$\Big\| \sum_n \mu(A_n)f(t_n) - x \Big\|_X \leq \varepsilon.$$

This notion of integrability plays an interesting role in vector integration, see for instance [3, 14, 23, 25, 26, 29, 28]. For a function $f : \Omega \to X$ we always have

$$\text{Bochner integrable} \Longrightarrow \text{Birkhoff integrable} \Longrightarrow \text{Pettis integrable}$$

and the respective integrals coincide. No one of these implications can be reversed in general, but Birkhoff and Pettis integrability are still equivalent for functions taking values in a separable Banach space (cf. [3, 23]).

It is known that, in general, every Birkhoff integrable function defined on a quasi-Radon probability space is McShane integrable, see [14, Proposition 4] (cf. [25]). The converse holds for functions taking values in subspaces of ℓ^∞, see [14, Theorem 10]. Examples of McShane integrable functions which are not Birkhoff integrable can be found in [14] and [23].

The main purpose of this section is to prove that, for any super-reflexive generated Banach space X with $\mathrm{dens}(X) \geq \mathfrak{c}$, we can always construct X-valued functions defined on $[0, 1]$ witnessing that

$$\text{Bochner integrable} \not\Longleftarrow \text{Birkhoff integrable} \not\Longleftarrow \text{McShane integrable},$$

see Theorem 4.8 below. The particular case of super-reflexive spaces was considered by the second named author in [23, 29]. We need some previous work.

Lemma 4.1. *Let $T : Y \to X$ be a one-to-one operator between Banach spaces, where Y is WLD. Then* $\mathrm{dens}(Y) \leq \mathrm{dens}(X^*, w^*)$.

Proof. Let $\Gamma \subset X^*$ be a total set with $\mathrm{card}(\Gamma) = \mathrm{dens}(X^*, w^*)$. The fact that T is one-to-one implies that the set of compositions $\{x^* \circ T : x^* \in \Gamma\} \subset Y^*$ is total and, therefore, we have $\mathrm{dens}(Y^*, w^*) \leq \mathrm{card}(\Gamma) = \mathrm{dens}(X^*, w^*)$. The conclusion now follows from the equality $\mathrm{dens}(Y^*, w^*) = \mathrm{dens}(Y)$, which holds because Y is WLD (cf. [21, Proposition 5.40]). □

Lemma 4.2. *Let (Ω, Σ, μ) be a complete probability space, X a Banach space and $f : \Omega \to X$ a Pettis integrable function. Suppose there is a total set $\Gamma \subset X^*$ such that, for each $x^* \in \Gamma$, we have $x^* f = 0$ μ-a.e. Then f is scalarly null.*

Proof. Given any $A \in \Sigma$, we have $x^*(\int_A f \, d\mu) = \int_A x^* f \, d\mu = 0$ for all $x^* \in \Gamma$, hence $\int_A f \, d\mu = 0$. As $A \in \Sigma$ is arbitrary, f is scalarly null. □

Strong and scalar measurability, as well as scalar nullity, are preserved when composing with operators. In Proposition 4.3 we discuss the inverse problem for one-to-one operators. Recall that the class of Banach spaces X for which (X^*, w^*) is *angelic* (i.e. for any bounded set $K \subset X^*$, every point in its w^*-closure is the w^*-limit of a sequence in K) contains all WLD spaces, cf. [21, Proposition 5.27].

Proposition 4.3. *Let $T : Y \to X$ be a one-to-one operator between Banach spaces. Let (Ω, Σ, μ) be a complete probability space and $f : \Omega \to Y$ a function.*

(i) *If (Y^*, w^*) is angelic, then f is scalarly measurable if and only if $T \circ f$ is.*
(ii) *If Y is WLD, then f is strongly measurable if and only if $T \circ f$ is.*
(iii) *If (Y^*, w^*) is angelic, then f is scalarly null if and only if $T \circ f$ is.*

Proof. It only remains to prove the *if* parts.

(i) Suppose that (Y^*, w^*) is angelic and that $T \circ f$ is scalarly measurable. Since T is one-to-one, the set $\Lambda := \{x^* \circ T : x^* \in X^*\} \subset Y^*$ is total. Observe that f is Σ-$\sigma(\Lambda)$-measurable, where $\sigma(\Lambda)$ is the σ-algebra on Y generated by Λ. Since (Y^*, w^*)

is angelic, a result of Gulisashvili [20] says that $\sigma(\Lambda) = \sigma(Y^*)$ (the σ-algebra on Y generated by Y^*). Hence f is Σ-$\sigma(Y^*)$-measurable, i.e. scalarly measurable.

(ii) Suppose that Y is WLD and that $T \circ f$ is strongly measurable. Then there is $E \in \Sigma$ with $\mu(E) = 1$ such that $T(f(E))$ is separable. Set $Y_0 := \overline{\mathrm{span}}(f(E)) \subset Y$ and $X_0 := \overline{\mathrm{span}}(T(f(E))) \subset X$, and observe that $T(Y_0) \subset X_0$. Since every subspace of a WLD Banach space is also WLD (cf. [21, Corollary 5.43]), the space Y_0 is WLD. Lemma 4.1 applied to the restriction $T|_{Y_0} : Y_0 \to X_0$ ensures that Y_0 is separable, hence the same holds for $f(E)$. On the other hand, (Y^*, w^*) is angelic and $T \circ f$ is scalarly measurable, hence f is scalarly measurable as well (by (i)). It follows that f is strongly measurable.

(iii) Suppose that (Y^*, w^*) is angelic and that $T \circ f$ is scalarly null. Then f is scalarly measurable (by (i)) and so we can find a sequence (E_n) in Σ with $\Omega = \bigcup_{n\in\mathbb{N}} E_n$ such that, for each $n \in \mathbb{N}$, the function $f\chi_{E_n}$ is scalarly bounded, cf. [22, Proposition 3.1]. Fix $n \in \mathbb{N}$. Since every Banach space with w^*-angelic dual has the Pettis Integral Property [11], $f\chi_{E_n}$ is Pettis integrable. Moreover, for every $y^* \in \Lambda$ the composition $y^* f\chi_{E_n}$ vanishes μ-a.e. (since $T \circ f$ is scalarly null). Hence Lemma 4.2 can be applied to conclude that $f\chi_{E_n}$ is scalarly null. As $n \in \mathbb{N}$ is arbitrary, f is scalarly null and the proof is over. □

A similar result for "integrable" functions does not hold in general:

Remark 4.4. Let $T : Y \to X$ be an operator between Banach spaces such that $T(Y)$ is not closed and let (Ω, Σ, μ) be an atomless probability space. Then there is a non Pettis integrable function $f : \Omega \to Y$ such that $T \circ f$ is Bochner integrable.

Proof. Since $T(Y)$ is not closed, we can find a non convergent sequence (y_n) in Y such that $\|T(y_{n+1}) - T(y_n)\|_X \leq 2^{-n}$ for all $n \in \mathbb{N}$. Let (A_n) be a disjoint sequence in Σ with $\mu(A_n) > 0$ for all $n \in \mathbb{N}$. Define $f : \Omega \to Y$ by $f(t) := (y_{n+1} - y_n)/\mu(A_n)$ if $t \in A_n$ for some $n \in \mathbb{N}$, and $f(t) := 0$ if $t \notin \bigcup_{n\in\mathbb{N}} A_n$. Then the function $T \circ f$ is Bochner integrable, since $\sum_{n=1}^{\infty} \|T(y_{n+1} - y_n)\|_X < \infty$. However, f is not Pettis integrable because the series $\sum_{n=1}^{\infty}(y_{n+1} - y_n)$ does not converge in Y. □

The following examples show that in Proposition 4.3 the additional assumption on Y cannot be dropped in general.

Example 4.5. Let $T : \ell^1([0,1]) \to \ell^2([0,1])$ be the "identity" operator and let us consider the function $f : [0,1] \to \ell^1([0,1])$ given by $f(t) := e_t$ (where $e_t(s) = \delta_{t,s}$). Then $T \circ f$ is scalarly null but f is not scalarly measurable.

Example 4.6. Let D be the Banach space of all real-valued functions on $[0,1]$ which are right continuous and have left limits, equipped with the supremum norm. Then D^* is w^*-angelic but D is not WLD (observe that D is isomorphic to a non-separable subspace of ℓ^∞). Let us consider any one-to-one operator $T : D \to c_0$ and let $f : [0,1] \to D$ be any scalarly measurable function which is not strongly measurable (e.g. the one given by $f(t) := \chi_{[0,t)}$, cf. [10, Section 6]). Since c_0 is separable and $T \circ f$ is scalarly measurable, $T \circ f$ is strongly measurable.

In the proof of the next lemma we use the fact that a Banach space is super-reflexive if and only if its dual is super-reflexive, cf. [4, Corollary 4.6, p. 152].

Lemma 4.7. *Let X be a super-reflexive generated Banach space. Then there exist a super-reflexive Banach space Y with* $\mathrm{dens}(Y) = \mathrm{dens}(X)$ *and a one-to-one operator $T : Y \to X$ with dense range.*

Proof. There exist a super-reflexive Banach space Z and an operator $S : Z \to X$ with dense range. Since $(Z/\ker S)^*$ is isomorphic to a subspace of the super-reflexive space Z^*, the space $(Z/\ker S)^*$ is super-reflexive too and so the same holds for $Y := Z/\ker S$. Let $T : Y \to X$ be the one-to-one operator such that $S = T \circ \pi$, where $\pi : Z \to Y$ is the canonical quotient operator. Clearly, T has dense range and so $\mathrm{dens}(X) \leq \mathrm{dens}(Y)$. On the other hand, we also have $\mathrm{dens}(X) = \mathrm{dens}(X^*, w^*)$ (because X is WLD, cf. [21, Proposition 5.40]). Lemma 4.1 finishes the proof. □

We can now prove the chief result of this section.

Theorem 4.8. *Let X be a super-reflexive generated Banach space with density character* $\mathrm{dens}(X) \geq \mathfrak{c}$. *Then there exist:*

(i) *a bounded Birkhoff integrable function $h : [0,1] \to X$ which is not strongly measurable (hence not Bochner integrable);*

(ii) *a bounded scalarly null McShane integrable function $g : [0,1] \to X$ which is not Birkhoff integrable.*

Proof. (i) By Lemma 4.7, there exist a super-reflexive Banach space Y such that $\mathrm{dens}(Y) = \mathrm{dens}(X)$ and a one-to-one operator $T : Y \to X$. In [29] (proofs of Lemma 3.2 and Theorem 1.2) it is shown that, since Y is super-reflexive and $\mathrm{dens}(Y) \geq \mathfrak{c}$, there is a bounded Birkhoff integrable function $f : [0,1] \to Y$ which is not strongly measurable. Define $h : [0,1] \to X$ by $h := T \circ f$. Clearly, h is bounded and Birkhoff integrable. By Proposition 4.3(ii), h is not strongly measurable.

(ii) Since X is WLD and $\mathrm{dens}(X) \geq \mathfrak{c}$, there is a bounded scalarly null function $g : [0,1] \to X$ which is not Birkhoff integrable, see [23, Theorem 2.3]. On the other hand, X is a subspace of a Hilbert generated Banach space and therefore Lemma 3.6 can be applied to conclude that g is McShane integrable. □

Remark 4.9. The function f in the proof of Theorem 4.8(i) can be chosen Riemann integrable, see [29]. Of course, in this case h is Riemann integrable too. Moreover, for Hilbert generated Banach spaces, Theorem 4.8(i) can be proved in a simpler way with the help of the function $f : [0,1] \to \ell^2([0,1])$ given by $f(t) := e_t$.

5. A scalarly null function which is not McShane integrable

An operator between Banach spaces is called *absolutely summing* if it takes unconditionally convergent series to absolutely convergent ones. Of course, absolutely summing operators also improve the integrability properties of Banach space-valued functions, see e.g. [2, 7, 24]. In several situations the composition of a Pettis integrable function with an absolutely summing operator is Bochner integrable, but this is not always the case, see [24] for detailed information.

The main purpose of this section is to ensure the existence of a scalarly null Banach space-valued function (defined on a Radon probability space) which is not McShane integrable, see Theorem 5.8 below. To this end we will take into account the fact that the composition of a McShane integrable function with an absolutely summing operator is always Bochner integrable [24]; this result relies on the studies on the so-called variational McShane integral due to Di Piazza and Musial [5].

Lemma 5.1. *Let Y be a Banach space for which there is an absolutely summing one-to-one operator T from Y into another Banach space X. Let $(\Omega, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon probability space and $f : \Omega \to Y$ a scalarly null function. Then f is McShane integrable if and only if $f = 0$ μ-a.e.*

Proof. Since every Bochner integrable function is McShane integrable, it only remains to prove the *only if* part. So assume that f is McShane integrable. Then $g := T \circ f : \Omega \to X$ is Bochner integrable, see [24, Theorem 3.13]. Clearly, g is scalarly null. Since g is strongly measurable, we conclude that $g = 0$ μ-a.e. The injectivity of T now implies that $f = 0$ μ-a.e. □

Corollary 5.2. *Let Y be a Banach space satisfying any of the following conditions:*

(i) *$Y = \ell^1(I)$ (for any set I).*

(ii) *$Y = C(K)$, where K is a compact Hausdorff topological space supporting a strictly positive Radon measure.*

Let $(\Omega, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon probability space and $f : \Omega \to Y$ a scalarly null function. Then f is McShane integrable if and only if $f = 0$ μ-a.e.

Proof. The result follows from Lemma 5.1 taking into account that, in both cases, there is an absolutely summing one-to-one operator from Y into another Banach space. Indeed:

(i) The "identity" operator $\ell^1(I) \to \ell^2(I)$ is absolutely summing according to Grothendieck's theorem, cf. [8, Theorem 3.4].

(ii) Let ν be a Radon probability measure on K such that $\nu(G) > 0$ for every non-empty open set $G \subset K$. The operator $T : C(K) \to L^1(\nu)$ which sends each function to its equivalence class is one-to-one and absolutely summing (cf. [8, 2.9]). □

The following folk remark allows us to generalize Corollary 5.2(i) by saying that every McShane integrable $\ell^1(I)$-valued function is strongly measurable, for any set I (see Proposition 5.4 below).

Remark 5.3. Let (Ω, Σ, μ) be a complete probability space and $f : \Omega \to \ell^1(I)$ a Pettis integrable function (where I is any set). Since $\ell^1(I)$ has the Radon-Nikodým property (cf. [9, Corollary 8, p. 83]) and the $\ell^1(I)$-valued measure $A \rightsquigarrow \int_A f\, d\mu$ is μ-continuous and has σ-finite variation (cf. [22, Theorem 4.1]), it follows that f is scalarly equivalent to a strongly measurable function. Moreover, if $\mathrm{card}(I) \leq \mathfrak{c}$ then $\ell^1(I)^*$ is w^*-separable and so f is strongly measurable.

Proposition 5.4. *Let $(\Omega, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon probability space and consider a McShane integrable function $f : \Omega \to \ell^1(I)$ (where I is any set). Then f is strongly measurable.*

Proof. We already know that f is scalarly equivalent to a strongly measurable function $g : \Omega \to \ell^1(I)$ (Remark 5.3). Hence $h := f - g$ is scalarly null and g is Pettis integrable. Since g is strongly measurable, it follows that g is McShane integrable (see [16, 4C]). Therefore, h is McShane integrable as well. An appeal to Corollary 5.2(i) ensures that h vanishes μ-a.e., hence $f = g$ μ-a.e. and so f is strongly measurable. □

In order to find a scalarly null $\ell^1(I)$-valued function which is not McShane integrable other measure theoretic ingredients are needed. Given a complete probability space (Ω, Σ, μ), a family $(f_i)_{i \in I}$ of real-valued functions defined on Ω is called *measure-additive* (see [15, 11A]) if $\sum_{i \in I} |f_i(t)| < \infty$ for every $t \in \Omega$ and, for each set $J \subset I$, the function $\sum_{i \in J} f_i$ is measurable. The connection between the theory of measure-additive families of real-valued functions and vector integration is explained in the following simple lemma.

Lemma 5.5. *Let (Ω, Σ, μ) be a complete probability space and $(f_i)_{i\in I}$ a family of real-valued functions defined on Ω such that $\sum_{i\in I}|f_i(t)| < \infty$ for every $t \in \Omega$. Define $f : \Omega \to \ell^1(I)$ by $f(t) := (f_i(t))_{i\in I}$. Then:*

(i) *$f = 0$ μ-a.e. if and only if $\sum_{i\in I}|f_i|$ vanishes μ-a.e.*

(ii) *f is scalarly measurable if and only if $(f_i)_{i\in I}$ is measure-additive.*

(iii) *f is scalarly null if and only if, for each set $J \subset I$, the function $\sum_{i\in J} f_i$ vanishes μ-a.e.*

Proof. (i) Observe that $\|f(t)\|_{\ell^1(I)} = \sum_{i\in I}|f_i(t)|$ for all $t \in \Omega$.

(ii) The *only if* part follows at once from the fact that, for each $J \subset I$, the composition of f with the functional $\chi_J \in \ell^\infty(I) = \ell^1(I)^*$ coincides with $\sum_{i\in J} f_i$. Conversely, assume that $(f_i)_{i\in I}$ is measure-additive and fix $\varphi \in \ell^\infty(I)$. For each $m \in \mathbb{N}$ we can find a finite partition $\{J_1^m, \dots, J_{n_m}^m\}$ of I and real numbers $a_1^m, \dots, a_{n_m}^m$ such that $\|\varphi - \varphi_m\|_{\ell^\infty(I)} \le 1/m$, where $\varphi_m := \sum_{k=1}^{n_m} a_k^m \chi_{J_k^m}$. Notice that the composition $\varphi_m f = \sum_{k=1}^{n_m} a_k^m (\sum_{i\in J_k^m} f_i)$ is measurable. Since $\varphi_m f \to \varphi f$ pointwise as $m \to \infty$, we conclude that φf is measurable. The proof of (iii) is analogous. □

One of the striking applications of Fremlin's work [15] on measure-additive coverings reads as follows (see [15, 11D]): if $(f_i)_{i\in I}$ is a measure-additive family of real-valued functions defined on a Radon probability space such that each f_i vanishes a.e., then their sum $\sum_{i\in I} f_i$ also vanishes a.e. In view of Lemma 5.5(iii), this result can be restated in the following way:

Theorem 5.6 (Fremlin)**.** *Let $(\Omega, \mathfrak{T}, \Sigma, \mu)$ be a Radon probability space and $(f_i)_{i\in I}$ a measure-additive family of real-valued functions defined on Ω. Define $f : \Omega \to \ell^1(I)$ by $f(t) := (f_i(t))_{i\in I}$. Then f is scalarly null if and only if each f_i vanishes μ-a.e.*

Aniszczyk and Frankiewicz [1] (cf. [15, 11C]) proved that if $(f_\alpha)_{\alpha<\mathfrak{c}}$ is a measure-additive family of real-valued functions defined on a Radon probability space, then $\sum_{\alpha<\mathfrak{c}}|f_\alpha|$ vanishes a.e. whenever each f_α does. Moreover, they showed that the restriction on the cardinality cannot be dropped in general (cf. [15, 12H]). As usual, $\mathfrak{c}^+$ denotes the smallest cardinal greater than $\mathfrak{c}$.

Example 5.7 (Aniszczyk - Frankiewicz)**.** *There exist a Radon probability space $(\Omega, \mathfrak{T}, \Sigma, \mu)$ and a measure-additive family $(f_\alpha)_{\alpha<\mathfrak{c}^+}$ of real-valued functions defined on Ω such that each f_α vanishes μ-a.e. but $\sum_{\alpha<\mathfrak{c}^+}|f_\alpha|$ does not vanish μ-a.e.*

As an application of Theorem 5.6 and Example 5.7 (via Corollary 5.2(i) and Lemma 5.5(i)) we get the desired result:

Theorem 5.8. *There exist a Radon probability space $(\Omega, \mathfrak{T}, \Sigma, \mu)$ and a scalarly null function $f : \Omega \to \ell^1(\mathfrak{c}^+)$ which is not McShane integrable.*

To the best of our knowledge, it remains unknown whether there are ZFC examples of scalarly null Banach space-valued functions defined on $[0, 1]$ which are not McShane integrable.

Remark 5.9. Lemma 5.1, Corollary 5.2 and Proposition 5.4 still remain valid if McShane integrability is replaced by Talagrand integrability [30] or Birkhoff integrability on an arbitrary complete probability space. This is because the composition of any Talagrand (resp. Birkhoff) integrable function with an absolutely summing operator is always Bochner integrable [24]. In particular, the function given in Theorem 5.8 is neither Talagrand integrable.

Perhaps it is worth mentioning here another nice application of Theorem 5.6, also due to Fremlin: *every* $\ell^1(I)$*-valued scalarly measurable function defined on a Radon probability space is scalarly equivalent to a strongly measurable function, for any set* I, see [15, 11E].

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