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STRICTLY CONVEX RENORMINGS

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Abstract

A normed space X is said to be strictly convex if $x = y$ whenever $\|(x+y)/2\| = \|x\| = \|y\|$, in other words, when the unit sphere of X does not contain non-trivial segments. Our aim in this paper is the study of those normed spaces which admit an equivalent strictly convex norm. We present a characterization in linear topological terms of the normed spaces which are strictly convex renormable. We consider the class of all solid Banach lattices made up of bounded real functions on some set Γ. This class contains the Mercourakis space $c_1(\Sigma' \times \Gamma)$ and all duals of Banach spaces with unconditional uncountable bases. We characterize the elements of this class which admit a pointwise strictly convex renorming.

Introduction

A normed space X is said to be strictly *convex* if $x = y$ whenever $\|(x+y)/2\| = \|x\| = \|y\|$, in other words, when the unit sphere of X does not contain non-trivial segments. There are few results devoted to strictly convex renormings, most of them are based on the following simple observation. Let Y be a strictly convex normed space and $T : X \to Y$ a linear one-to-one bounded operator; then $|||x||| = \|x\| + \|Tx\|$, $x \in X$, is an equivalent strictly convex norm. Day (see, for example, [**4**, pp. 94–100]) constructed in $c_0(\Gamma)$ an equivalent strictly convex norm introducing in $c_0(\Gamma)$ a norm of Lorentz sequence space type. Another strictly convex norm in $c_0(\Gamma)$ can be found in [**3**, p. 282]. Using the fact that $c_0(\Gamma)$ admits a strictly convex norm and the norm $|||\cdot|||$ defined above, it was obtained that every weakly compact generated space and its dual (in particular every separable space and its dual) admits a strictly convex renorming. Dashiell and Lindenstrauss [**2**] defined a class of subspaces X of $\ell^\infty([0,1])$, which are strictly convex renormable and do not admit a one-to-one linear bounded operator into $c_0(\Gamma)$ for any Γ. Mercourakis (see, for example, [**3**, pp. 248, 286]) introduced the space $c_1(\Sigma' \times \Gamma)$, which is strictly convex renormable but does not admit a one-to-one linear bounded operator to $c_0(\Gamma)$ for any Γ. However, the strictly convex norm in the class defined in [**2**, p. 337] and in $c_1(\Sigma' \times \Gamma)$ is based on Day's strictly convex norm in $c_0(\Gamma)$. In [**1**], a quite wide class of dual strictly convex renormable Banach spaces, which are conjugate of Banach spaces, with unconditional basis is introduced. A characterization of strictly convex renormable spaces $C(K)$, when K is a tree or totally ordered, is obtained in [**5**] and [**7**], respectively. Quite recently, Smith [**10**] has characterized those trees K for which $C^*(K)$ admits a dual strictly convex norm.

Day (see, for example, [**4**, p. 123]) proved that the space $\ell^\infty_c(\Gamma)$ of all bounded functions with countable support does not admit a strictly convex renorming if Γ is uncountable. Other examples of subspaces of $\ell^\infty_c(\Gamma)$, which are not strictly convex renormable can be found in [**2**, **1**]. Haydon [**6**], using Baire category arguments, found some classes of spaces K for which $C(K)$ does not admit strictly convex renormings.

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Our aim in this paper is the study of those normed spaces which admit an equivalent strictly convex norm.

In Section 1 we present a characterization in linear topological terms of the normed spaces that are strictly convex renormable.

In Section 2 we consider the class of all solid Banach lattices made up of bounded real functions on some set Γ. This class contains the Mercourakis space $c_1(\Sigma' \times \Gamma)$ and all dual Banach spaces with unconditional uncountable bases. We characterize the elements of this class which admit a pointwise strictly convex renorming.

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1. *A characterization of strictly convex renormable spaces*

For a set A, we denote the diagonal of A^2 by $\Delta_2(A)$, that is, $\Delta_2(A) = \{(x,x) : x \in A\}$. Throughout the paper, given a linear space X, we denote by $D : X^2 \to X$ the map defined by the formula

$$D(x,y) = \frac{x+y}{2}. \tag{1.1}$$

Definition 1.1. Let X be a linear topological space. A subset M of X^2 is said to be *quasi-diagonal* if it is symmetric (that is, if $(x,y) \in M$ then $(y,x) \in M$) and if $x = y$ whenever $(x,y) \in M$ and x, $y \in \overline{\text{conv}}(DM)$. We say that M is *sigma quasi-diagonal* if M is a countable union of quasi-diagonal sets.

Theorem 1.2. *Let X be a normed space and F a subspace of X^* which is* 1*-norming for X. The following are equivalent:*

(i) *S_X^2 is a sigma quasi-diagonal set with respect to $(X, \sigma(X,F))$;*
(ii) *X^2 is a sigma quasi-diagonal set with respect to $(X, \sigma(X,F))$;*
(iii) *X admits an equivalent $\sigma(X,F)$ lower semicontinuous strictly convex norm.*

In particular, X admits an equivalent $\sigma(X,F)$ lower semicontinuous strictly convex norm if, and only if, X^2 is sigma quasi-diagonal with respect to $(X, \sigma(X,F))$.

Before proving Theorem 1.2 we need some assertions.

Lemma 1.3. *Let X be a normed space and F a subspace of X^* which is* 1*-norming for X. For $q > 0$ and $n \in \mathbb{N}$ the set*

$$L_{n,q} = \left\{ (x,y) \in X^2 : \|x\|, \|y\| \in [q, q(1+n^{-1})], \left\| \frac{x+y}{2} \right\| \leqslant (1-n^{-1}) \frac{\|x\| + \|y\|}{2} \right\}$$

is a quasi-diagonal set with respect to $(X, \sigma(X,F))$.

Proof. Given $(x,y) \in L_{n,q}$ we find $f \in F$, $\|f\| = 1$, such that

$$f(x) \geqslant \|x\| - \frac{q}{2n^2}.$$

Since $\|x\| \geqslant q$ we get

$$f(x) \geqslant q\left(1 - \frac{1}{2n^2}\right). \tag{1.2}$$

On the other hand, for $(u,v) \in L_{n,q}$ we have

$$f\left(\frac{u+v}{2}\right) \leqslant \left\|\frac{u+v}{2}\right\| \leqslant (1-n^{-1})\frac{\|u\|+\|v\|}{2} \leqslant (1-n^{-1})q(1+n^{-1}) = (1-n^{-2})q.$$

Then using (1.2) we get

$$\sup_{DL_{n,q}} f \leqslant (1-n^{-2})q \leqslant f(x) - \frac{q}{2n^2},$$

where D is the map defined in (1.1). So $x \notin \overline{\operatorname{conv}(DL_{n,q})}^{\sigma(X,F)}$. □

LEMMA 1.4. *Let X be a normed space and F a subspace of X^*, which is 1-norming for X. Let $0<q<r$ and M, $N \subset X$ such that*

$$N \subset qB_X, \quad M \subset (2r-q)B_X, \quad M \cap rB_X = \emptyset.$$

Then the set $L = (M\times N)\cup(N\times M)$ is quasi-diagonal with respect to $(X, \sigma(X,F))$.

Proof. Pick $(x,y) \in L$. Assume that $x \in M$, $y \in N$. Since $\|x\| > r$ there exists $f \in F$, $\|f\| = 1$, such that $f(x) > r$. For $(u,v) \in L$ we have

$$f\left(\frac{u+v}{2}\right) \leqslant \frac{\|u\|+\|v\|}{2} \leqslant \frac{1}{2}(q+2r-q) = r.$$

So $\sup_{DL} f \leqslant r < f(x)$, where D is the map defined by (1.1). Hence $x \notin \overline{\operatorname{conv}(DL)}^{\sigma(X,F)}$. □

COROLLARY 1.5. *Let X be a normed space and F a subspace of X^*, which is 1-norming for X. Then the set $P = \{(x,y)\in X^2 : \|x\| \neq \|y\|\}$ is $\sigma(X,F)$ quasi-diagonal.*

Proof. For q, $r \in \mathbb{Q}$, $0<q<r$, we set

$$P_{r,q} = (qB_X \times ((2r-q)B_X \setminus rB_X)) \cup (((2r-q)B_X \setminus rB_X) \times qB_X).$$

From Lemma 1.4, we get that $P_{r,q}$ are quasi-diagonal sets. We show that

$$P = \bigcup_{q,r} P_{q,r}.$$

Pick $(x,y) \in P$. We can find q, $r \in \mathbb{Q}$ such that

$$\min(\|x\|,\|y\|) < q < r < \max(\|x\|,\|y\|) < 2r-q.$$

Then we have $(x,y) \in P_{q,r}$. □

We say that a set $M \subset X$ is positively homogeneous if $\lambda x \in M$ whenever $\lambda > 0$ and $x \in M$.

PROPOSITION 1.6. *Let $L \subset X^2$ be a positively homogeneous $\sigma(X,F)$ quasi-diagonal set, where F is a norming subspace for X. Then X admits an equivalent $\sigma(X,F)$-lower semicontinuous norm $\|\cdot\|_L$ such that $x=y$ whenever $(x,y)\in L$, $\|x\|_L = \|y\|_L = \|(x+y)/2\|_L$.*

Proof. Let L_n, $n=1, 2, \ldots,$ be quasi-diagonal sets covering L. Without loss of generality we may assume that $\{L_n\}_{n=1}^\infty$ are bounded. Otherwise we can replace $\{L_n\}_{n=1}^\infty$ by $\{L_n \cap pB_{X^2}\}_{n,p=1}^\infty$. Pick $z_n \in L_n$ and denote by $\|\cdot\|_{m,n}$ the Minkowski functional of $-z_n + \overline{M_{m,n}}^{\sigma(X^2,F^2)}$, where

$$M_{m,n} = \operatorname{conv}(L_n) + m^{-1}B_{X^2}.$$

We choose $a_{m,n} > 0$ in such a way that the function $\varphi : X^2 \to \mathbb{R}$ defined by the formula

$$\varphi(z) = \sum_{m,n=1}^{\infty} a_{m,n} \|z - z_n\|^2_{m,n}, \quad z \in X^2,$$

is bounded on B_{X^2}. Clearly φ is a convex, uniformly norm continuous function on bounded sets. Set for $w \in X^2$

$$|||w||| = \inf\left\{\lambda > 0 : \varphi\left(\frac{w}{\lambda}\right) + \varphi\left(-\frac{w}{\lambda}\right) \leqslant 2c\right\}$$

where $c = \sup_{B_{X^2}} \varphi$. It is easy to see that $|||\cdot|||$ is an equivalent norm on X^2. Clearly $\|\cdot\|_{m,n}$ are $\sigma(X^2, F^2)$-lower semicontinuous. Hence φ and $|||\cdot|||$ are $\sigma(X^2, F^2)$-lower semicontinuous too. For $x \in X$ we set $\|x\|_L = |||(x, x)|||$. Pick x, $y \in X$ such that $(x, y) \in L$ and $\|x\|_L = \|y\|_L = \|(x + y)/2\|_L$. Since L is positively homogeneous, without loss of generality we can assume that $\|x\|_L = 1$. Set $u = (x, x)$ and $v = (y, y)$. We have

$$\varphi(u) + \varphi(-u) = \varphi(v) + \varphi(-v) = \varphi\left(\frac{u+v}{2}\right) + \varphi\left(-\frac{u+v}{2}\right) = 2c.$$

By convexity of φ we get

$$\frac{\varphi(u) + \varphi(v)}{2} - \varphi\left(\frac{u+v}{2}\right) = 0.$$

So

$$\sum_{m,n=1}^{\infty} a_{m,n} \left(\frac{\|u - z_n\|^2_{m,n} + \|v - z_n\|^2_{m,n}}{2} - \left\|\frac{u+v}{2} - z_n\right\|^2_{m,n}\right) = 0.$$

Again by convex arguments [**3**, p. 45] we get for m, $n = 1, 2, \ldots$

$$\|u - z_n\|_{m,n} = \|v - z_n\|_{m,n} = \left\|\frac{u+v}{2} - z_n\right\|_{m,n}. \tag{1.3}$$

Pick $n \in \mathbb{N}$ such that $(x, y) \in L_n$. Since L_n is symmetric we obtain that $(y, x) \in L_n$ too. So $(u + v)/2 = ((x, y) + (y, x))/2 \in \operatorname{conv}(L_n)$. Hence $\|((u + v)/2) - z_n\|_{m,n} \leqslant 1$ for all $m = 1, 2, \ldots$. From (1.3) we obtain that $\|u - z_n\|_{m,n} = \|v - z_n\|_{m,n} \leqslant 1$ for all $m = 1, 2, \ldots$. So

$$u - z_n, v - z_n \in \bigcap_{m=1}^{\infty} \left(-z_n + \overline{M_{m,n}}^{\sigma(X^2,F^2)}\right),$$

that is,

$$u, v \in \bigcap_{m=1}^{\infty} \overline{M_{m,n}}^{\sigma(X^2,F^2)}. \tag{1.4}$$

We show that

$$u, v \in \overline{\operatorname{conv}(L_n)}^{\sigma(X^2,F^2)}. \tag{1.5}$$

Assume now that $u \notin \overline{\operatorname{conv}(L_n)}^{\sigma(X^2,F^2)}$. According to the Hahn–Banach theorem there exists $f \in F^2$ and $b \in \mathbb{R}$ such that

$$f(u) > b > \sup_{L_n} f. \tag{1.6}$$

Set $H = \{w \in X^2 : f(w) \leqslant b\}$. We can find $m \in \mathbb{N}$ such that $M_{m,n} \subset H$. Since H is $\sigma(X^2, F^2)$-closed we obtain that $\overline{M_{m,n}}^{\sigma(X^2,F^2)} \subset H$. From (1.6) we obtain that $u \notin H$. Hence $u \notin \overline{M_{m,n}}^{\sigma(X^2,F^2)}$, which contradicts (1.4). So (1.5) is proved. From (1.5) we obtain that x, $y \in \overline{\operatorname{conv}(DL_n)}^{\sigma(X,F)}$, where D is the map defined in (1.1). Since L_n is quasi-diagonal we get $x = y$. $\square$

Proof of Theorem 1.2. (i)⇒(ii): We have $S_X^2 = \bigcup_{n=1}^{\infty} L_n$, where each L_n is quasi-diagonal with respect to $(X, \sigma(X, F))$. Given $n \in \mathbb{N}$ and $q, r \in \mathbb{Q}^+$, let $L_{n,q,r}$ be the set of all $(x, y) \in X^2$ such that $x \neq y$, $\underline{(x, y) \in} \|x\| L_n$, $\|x\| = \|y\| \in]q, r[$ and, either $(x, x) \in X^2 \setminus [q, r]\overline{\operatorname{conv} L_n}$, or $(y, y) \in X^2 \setminus [q, r]\overline{\operatorname{conv} L_n}$. Clearly each $L_{n,q,r}$ is symmetric and it is easy to prove that

$$\{(x, y) \in X^2 : x \neq y, \|x\| = \|y\|\} = \bigcup\{L_{n,q,r} : n \in \mathbb{N}, q, r \in \mathbb{Q}^+\}.$$

Moreover, since $\overline{\operatorname{conv} L_{n,q,r}} \subset [q, r]\overline{\operatorname{conv} L_n}$ the sets $L_{n,q,r}$ are quasi-diagonal, so X^2 is sigma quasi-diagonal.

(ii)⇒(iii): This follows from Proposition 1.6.

(iii)⇒(ii): This is a consequence of Lemma 1.3 and Corollary 1.5.

(ii)⇒(i): This is obvious. □

As a consequence of Theorem 1.2 we get the following.

PROPOSITION 1.7 (Talagrand [**3**, p. 313]). *There is no equivalent strictly convex dual norm in $C([0, \omega_1])^*$.*

Proof. Indeed, otherwise, according to Theorem 1.2 we have $C([0, \omega_1])^* \times C([0, \omega_1])^* = \bigcup_{n=1}^{\infty} M_n$, where every M_n is quasi-diagonal. For $n \in \mathbb{N}$, let S_n be the set of all $(s, t) \in [0, \omega_1[\times [0, \omega_1[$ such that $(\delta_s, \delta_t) \in M_n$. Then $[0, \omega_1[\times [0, \omega_1[= \bigcup_{n=1}^{\infty} S_n$. Moreover, since $(s, s) \in \overline{S_n}$ implies $\delta_s \in \overline{\{D(x, y) : (x, y) \in M_n\}}$ we conclude that the set S_n has the following property:

$$(s, t) \in S_n, \ \ (s, s), (t, t) \in \overline{S_n} \Longrightarrow s = t \qquad \text{for all } s, t \in [0, \omega_1[. \tag{1.7}$$

Set $\pi_i : [0, \omega_1[\times [0, \omega_1[\to [0, \omega_1[$, $\pi_i(\alpha_1, \alpha_2) = \alpha_i$, $i = 1, 2$. Let A be the (possibly empty) set made up by all $n \in \mathbb{N}$ for which there exists $\alpha_n \in [0, \omega_1[$ such that $S_n \subset ([0, \alpha_n] \times [0, \omega_1[) \cup ([0, \omega_1[\times [0, \alpha_n])$. If $A \neq \emptyset$ and $\alpha := \sup_A \alpha_n$ we have $\alpha < \omega_1$ and

$$S_n \cap ([\alpha, \omega_1[\times [\alpha, \omega_1[) = \emptyset \quad \text{for all } n \in A. \tag{1.8}$$

Therefore $\mathbb{N} \setminus A \neq \emptyset$, so it makes sense to take $\varphi : \mathbb{N} \to \mathbb{N} \setminus A$ which is onto and

$$\varphi^{-1}(\{n\}) \text{ is infinite for all } n \in \mathbb{N} \setminus A. \tag{1.9}$$

Now according to the choice of A we can define by induction two maps λ, $\mu : \mathbb{N} \to [0, \omega_1[$ such that

$$(\lambda(n), \mu(n)) \in S_{\varphi(n)}, \quad n \in \mathbb{N}; \tag{1.10}$$

$$\max\{\lambda(n), \mu(n)\} < \min\{\lambda(n+1), \mu(n+1)\}, \quad n \in \mathbb{N}; \tag{1.11}$$

$$\alpha < \min\{\lambda(1), \mu(1)\}. \tag{1.12}$$

From (1.11) it follows that

$$\lim_n \lambda(n) = \lim_n \mu(n). \tag{1.13}$$

Let $\beta = \lim_n \lambda(n) = \lim_n \mu(n)$. From (1.12) we get $\beta > \alpha$. Moreover, from (1.9), (1.10) and (1.13)

$$(\beta, \beta) \in \bigcap\{\overline{S_n} : n \in \mathbb{N} \setminus A\}. \tag{1.14}$$

Once more we can define by induction two maps η, $\rho : \mathbb{N} \to [0, \omega_1[$ for which (1.10)–(1.12) hold when we replace λ, μ and α by η, ρ and β. Then let $\gamma = \lim_n \eta(n) = \lim_n \rho(n)$, we have

$$\gamma > \beta > \alpha \quad \text{and} \quad (\gamma, \gamma) \in \bigcap\{\overline{S_n} : n \in \mathbb{N} \setminus A\}. \tag{1.15}$$

From (1.8) and (1.15) it follows that $(\beta, \gamma) \notin S_n$ for any $n \in A$. So, let $n_0 \in \mathbb{N} \setminus A$ such that $(\beta, \gamma) \in S_{n_0}$. This, (1.14) and (1.15) contradict (1.7). □

2. *Strict convexity in a lattice*

In this section $(X, \|\cdot\|_X)$ will be a solid Banach lattice of real functions on some set Γ such that $\|y\|_\infty \leqslant \|y\|_X \leqslant \|x\|_X$ whenever $|y(\gamma)| \leqslant |x(\gamma)|$ for all $\gamma \in \Gamma$ for some $x \in X$. Let $\|\cdot\|$ be an equivalent pointwise lower semicontinuous norm

$$\|x\|_\infty \leqslant \|x\| \leqslant \|x\|_X \leqslant K\|x\|.$$

From now until Lemma 2.5 the symbol $\|\cdot\|$ will denote this norm.

LEMMA 2.1. *If* $\operatorname{supp} x = \{\gamma \in \Gamma : x(\gamma) \neq 0\}$ *is uncountable for some* $x \in X$*, then* X *contains a lattice isomorphic copy of* $\ell^\infty(\Lambda)$ *for some uncountable set* Λ.

Proof. Set $\Lambda_n = \{\gamma \in \Gamma : |x(\gamma)| \geqslant n^{-1}\}$. We have

$$\operatorname{supp} x = \bigcup_{n=1}^{\infty} \Lambda_n.$$

So for some n the set Λ_n is uncountable. If $y \in \ell^\infty(\Lambda_n)$ we have for every $\gamma \in \Lambda_n$

$$\frac{|y(\gamma)|}{n\|y\|_\infty} \leqslant |x(\gamma)|.$$

So for all $\gamma \in \Gamma$ we have $|z_y(\gamma)| \leqslant n\|y\|_\infty |x(\gamma)|$ for all $\gamma \in \Gamma$, where $z_y(\gamma) = y(\gamma)$ if $\gamma \in \Lambda_n$ and $z_y(\gamma) = 0$ if $\gamma \notin \Lambda_n$. From our assumption it follows that $z_y \in X$ and

$$\|y\|_\infty = \|z_y\|_\infty \leqslant \|z_y\|_X \leqslant n\|x\|_X \|y\|_\infty.$$

Hence $\ell^\infty(\Lambda_n)$ is isomorphic to a subspace of X. □

In X we introduce a new norm. Let $G = \{-1, 1\}^\Gamma$, and let μ be the Haar translation invariant measure on the Abelian group G. For $s = \{s_\gamma\}_{\gamma \in \Gamma} \in G$ and $x \in X$ we set

$$x^s(\gamma) = s_\gamma x(\gamma).$$

Clearly for a fixed $x \in X$ the function of s, x^s is continuous on G when we consider in X the pointwise convergence topology. As we have already mentioned we shall assume that the norm $\|\cdot\|$ is pointwise lower semicontinuous. So in this case the function on s $\|x^s\|$ is pointwise lower semicontinuous on G for a fixed $x \in X$. We set

$$|||x||| = \left(\int_G \|x^s\|^2 \, d\mu(s) \right)^{1/2}. \tag{2.1}$$

Clearly $|||\cdot|||$ is an equivalent norm on X. Since μ is translation invariant we obtain for every $x \in X$ and every $s \in G$

$$|||x^s||| = |||x|||. \tag{2.2}$$

From convex arguments we obtain that

$$\frac{|||x|||^2 + |||y|||^2}{2} - \left|\left|\left| \frac{x+y}{2} \right|\right|\right|^2 > 0 \tag{2.3}$$

if and only if

$$\mu\left(\left\{ s \in G : \frac{\|x^s\|^2 + \|y^s\|^2}{2} - \left\| \frac{x^s + y^s}{2} \right\|^2 > 0 \right\} \right) > 0.$$

For $\Lambda \subset \Gamma$ and $x \in X$ we set

$$P_\Lambda x(\gamma) = \begin{cases} x(\gamma) & \text{if } \gamma \in \Lambda; \\ 0 & \text{otherwise.} \end{cases}$$

For a $\beta \in \Gamma$ we shall write P_β instead of $P_{\{\beta\}}$.

Lemma 2.2. *For every non-empty $\Lambda \subset \Gamma$ we have $|||P_\Lambda||| = 1$.*

Proof. Using (2.2) we get for every $x \in X$

$$\begin{aligned}|||P_\Lambda x||| &= \tfrac{1}{2}(|||(P_\Lambda x + P_{\Gamma\setminus\Lambda}x) + (P_\Lambda x - P_{\Gamma\setminus\Lambda}x)|||)\\ &\leqslant \tfrac{1}{2}(|||P_\Lambda x + P_{\Gamma\setminus\Lambda}x||| + |||P_\Lambda x - P_{\Gamma\setminus\Lambda}x|||) = |||x|||.\end{aligned}$$

□

Lemma 2.3. *The norm $|||\cdot|||$ is pointwise lower semicontinuous whenever $\|\cdot\|$ is pointwise lower semicontinuous and provided X does not contain isomorphic copies of $\ell^\infty(\Lambda)$ for an uncountable set Λ.*

Proof. Pick $x_\alpha \in X$ such that $\lim_\alpha x_\alpha = x$ for some $x \in X$ in the topology of pointwise convergence. Set $\Lambda = \operatorname{supp} x$. From lemma 2.1 we obtain that $\#\Lambda \leqslant \aleph_0$. So there exists a sequence $\{\alpha_k\}_{k=1}^\infty$ such that

$$\lim_k |||x_{\alpha_k}||| = \liminf_\alpha |||x_\alpha||| \tag{2.4}$$

and

$$\lim_k P_\Lambda x_{\alpha_k}(\gamma) = x(\gamma) \quad \text{for all } \gamma \in \Gamma.$$

Since $\|\cdot\|$ is pointwise lower semicontinuous we obtain

$$\liminf_k \|P_\Lambda x^s_{\alpha_k}\| \geqslant \|x^s\| \quad \text{for all } s \in G. \tag{2.5}$$

Taking into account Fatou's Lemma we get

$$\int_G \liminf_k \|P_\Lambda x^s_{\alpha_k}\|^2 \, d\mu(s) \leqslant \liminf_k \int_G \|P_\Lambda x^s_{\alpha_k}\|^2 \, d\mu(s).$$

This inequality, (2.5), (2.4) and Lemma 2.2 imply

$$|||x||| = \left(\int_G \|x^s\|^2 \, d\mu(s)\right)^{1/2} \leqslant \liminf_k |||P_\Lambda x_{\alpha_k}||| \leqslant \lim_k |||x_{\alpha_k}||| = \liminf_\alpha |||x_\alpha|||.$$

□

Lemma 2.4. *The norm $|||\cdot|||$ is a lattice norm provided $\#\operatorname{supp} x \leqslant \aleph_0$ for every $x \in X$.*

Proof. Let $\mathcal{A}$ be the family of all finite subsets A of Γ partially ordered by inclusion. Then for every $z \in X$ we have $\lim_A P_A z = z$ in the topology of pointwise convergence. Since $|||\cdot|||$ is pointwise lower semicontinuous we get

$$\liminf_A |||P_A z||| \geqslant |||z|||.$$

On the other hand, Lemma 2.2 gives us $|||P_A z||| \leqslant |||z|||$ so

$$\lim_A |||P_A z||| = |||z|||. \tag{2.6}$$

Pick now x, $y \in X$ with $|x| \leqslant |y|$. For every finite set $A \subset \Gamma$ we can find $\lambda_\sigma \geqslant 0$, $\sigma \in \{-1,1\}^A \times \{1\}^{\Gamma\setminus A}$, such that $\sum_\sigma \lambda_\sigma = 1$ and $P_A x = \sum_\sigma \lambda_\sigma P_A y^\sigma$. From (2.2) we have $|||P_A y^\sigma||| = |||P_A y|||$ for all σ. Hence $|||P_A x||| \leqslant |||P_A y|||$. Having in mind (2.6) we get $|||x||| \leqslant |||y|||$.

□

Lemma 2.5. *For every $p \in (1,2]$ there exists a positive number c_p such that for every x, $y \in \ell_p$ we have*

$$(\|x\|^p + \|y\|^p)^{(2/p)-1} \left(\frac{\|x\|^p + \|y\|^p}{2} - \left\|\frac{x+y}{2}\right\|^p\right) \geqslant c_p \|x-y\|^2.$$

This inequality is a homogeneous version of the uniform convexity inequality for ℓ_p, $1 < p \leqslant 2$. For the proof see, for example, [8] or [9].

By $\mathrm{con}\{\delta_\gamma : \gamma \in \Gamma\}$, we denote the positive cone generated by the Dirac measures δ_γ, $\gamma \in \Gamma$.

THEOREM 2.6. *Let X be a solid Banach lattice of real functions on some set Γ such that $\|\cdot\|_\infty \leqslant \|\cdot\|_X$. The following assertions are equivalent:*

(i) *X admits a pointwise lower semicontinuous strictly convex norm;*

(ii) *X admits a lattice pointwise lower semicontinuous strictly convex norm;*

(iii) *X admits a pointwise lower semicontinuous strictly lattice norm (that is, $\|x\| < \|y\|$ whenever $|x| < |y|$);*

(iv) *the set Z of all pairs $z = (x, y) \in X^2$, $0 < x < y$, can be written $Z = \bigcup_{n\in\mathbb{N}} Z_n$ in such a way that for every $z = (x, y) \in Z_n$ there exists $f \in \mathrm{con}\{\delta_\gamma : \gamma \in \Gamma\}$ with*

$$f(y) > \sup\left\{ f\left(\frac{u+v}{2}\right) : (u, v) \in Z_n \right\}.$$

Proof. We use the diagram

$$\text{(i)} \Longrightarrow \text{(ii)} \Longrightarrow \text{(iii)} \Longrightarrow \text{(iv)} \Longrightarrow \text{(iii)} \Longrightarrow \text{(ii)} \Longrightarrow \text{(i)}.$$

(i) $\Rightarrow$ (ii): Assume that $\|\cdot\|$ is a pointwise lower semicontinuous strictly convex norm. Let $|||\cdot|||$ be the norm obtained from $\|\cdot\|$ by (2.1). Since $\|\cdot\|$ is a strictly convex norm, X does not contain isomorphic copies of $\ell^\infty(\Lambda)$ for an uncountable set Λ (see, for example, [4, p. 123]). Then from Lemma 2.3 and Lemma 2.4 we obtain that $|||\cdot|||$ is a lattice pointwise lower semicontinuous norm. From (2.3), it follows that $|||\cdot|||$ is strictly convex.

(ii) $\Rightarrow$ (iii): Pick x, $y \in X$, $0 < x < y$. Let $\|\cdot\|$ be a lattice strictly convex norm. Then $\|x\| \leqslant \|y\|$. Assume that $\|x\| = \|y\|$. Since $x < (x + y)/2 < y$ we get $\|x\| \leqslant \|(x + y)/2\| \leqslant \|y\|$. Hence $\|(x + y)/2\| = \|x\| = \|y\|$. Since $\|\cdot\|$ is strictly convex we have $x = y$.

(iii) $\Rightarrow$ (iv): This follows directly from the proof of Corollary 1.5.

(iv) $\Rightarrow$ (iii): Denote by $\widetilde{Z}$ (respectively, $\widetilde{Z}_n$) the set of all $z = (x, y) \in X^2$ such that either $(|x|, |y|)$ or $(|y|, |x|)$ belongs to Z (respectively, Z_n). Let us see that $\widetilde{Z}_n$ is quasi-diagonal with respect to $(X, \text{pointwise})$. Indeed let $(|x|, |y|) \in Z_n$, $f = \sum a_\gamma \delta_\gamma$, $a_\gamma > 0$ and

$$f(|y|) > \sup\left\{ f\left(\frac{u+v}{2}\right) : (u, v) \in Z_n \right\}.$$

Set $g = \sum a_\gamma \,\mathrm{sign}\, y(\gamma)\delta_\gamma$. Then $g(y) = f(|y|)$ and $g(u + v) \leqslant f(|u| + |v|)$ for any $(u, v) \in X^2$. Hence $\widetilde{Z}_n$ is pointwise quasi-diagonal and $\widetilde{Z}$ is sigma pointwise quasi-diagonal. According to Proposition 1.6 there exists on X a pointwise lower semicontinuous equivalent norm $\|\cdot\|$ such that $x = y$ whenever $(x, y) \in \widetilde{Z}$ and $\|x\| = \|y\| = \|(x + y)/2\|$. Let us show that $\#\mathrm{supp}\, u \leqslant \aleph_0$ for every $u \in X$. Indeed, otherwise, from Lemma 2.1 it follows that there exists $\Lambda \subset \Gamma$, $\#\Lambda > \aleph_0$, such that every bounded function v on Λ with $\mathrm{supp}\, v \subset \Lambda$ belongs to X. Then, from a slight adaptation of Day's proof that $\ell^\infty(\Lambda)$ does not admit a strictly convex norm, it follows that there exist x, $y \in X$, $0 < x < y$, with $\|x\| = \|y\| = \|(x + y)/2\|$ (see, for example, [4, p. 123]). For the sake of completeness we include a proof of this assertion.

Let $\ell^\infty_c(\Gamma)$ be the subspace of $\ell^\infty(\Gamma)$ made up by all $x \in \ell^\infty(\Gamma)$ with countable support. Let S be the unit sphere of $\ell^\infty_c(\Gamma)$ (for the supremum norm). For each $x \in S$, $x > 0$, let $F_x := \{y \in S : y\restriction_{\mathrm{supp}\, x} = x\restriction_{\mathrm{supp}\, x}, y > 0\}$, $m_x := \inf\{\|y\| : y \in F_x\}$ and $M_x := \sup\{\|y\| : y \in F_x\}$. The assertion will be proved as soon as we find $x \in S$ with $x > 0$ such that

$$M_x = m_x. \tag{2.7}$$

This will follow from two observations. First, for any $x \in S$ with $x > 0$ we have

$$\|x\| \leqslant \frac{M_x + m_x}{2}. \tag{2.8}$$

Indeed, given $\varepsilon > 0$, take $y \in F_x$ such that $\|y\| - \varepsilon < m_x$. Then for any $y' \in F_x$ we have $2x \leqslant y + y'$, therefore, $2\|x\| \leqslant \|y\| + \|y'\| \leqslant m_x + M_x + \varepsilon$ and (2.8) follows.

On the other hand, let us observe that for any sequence $\{x_n\}_{n=1}^{\infty}$, $x_n \in S$, $x_n > 0$, such that $x_{n+1} \in F_{x_n}$ for $n \in \mathbb{N}$, the bounded sequences $\{M_{x_n}\}_{n=1}^{\infty}$ and $\{m_{x_n}\}_{n=1}^{\infty}$ are monotone, therefore convergent. Then (2.7) will be proved if we show that they converge to the same limit. For this purpose we take $\{x_n\}_{n=1}^{\infty}$ in such a way that $M_{x_n} - \|x_{n+1}\| < 2^{-n-1}$; then from (2.8) we obtain that

$$\frac{M_{x_{n+1}} - m_{x_{n+1}}}{2} = M_{x_{n+1}} - \frac{M_{x_{n+1}} + m_{x_{n+1}}}{2} \leqslant M_{x_n} - \|x_{n+1}\| < 2^{-n-1}.$$

Thus

$$M_{x_{n+1}} - m_{x_{n+1}} < 2^{-n},$$

which implies that $\lim_{n\to\infty} m_{x_n} = \lim_{n\to\infty} M_{x_n}$ and (2.7) is proved.

Now we introduce $|||\cdot|||$ by (2.1). According to Lemma 2.4 $|||\cdot|||$ is a lattice norm. We show that $|||\cdot|||$ is a strictly lattice norm. Indeed let $|x| < |y|$. Then $(x^s, y^s) \in \widetilde{Z}$ for all $s \in G$, therefore

$$\frac{\|x^s\|^2 + \|y^s\|^2}{2} - \left\|\frac{x^s + y^s}{2}\right\| > 0.$$

From (2.3) we obtain that $|||x||| < |||y|||$.

(iii) $\Rightarrow$ (ii): We set $\delta_\gamma(x) = x(\gamma)$ for $x \in X$ and $\gamma \in \Gamma$. Since the norm $\|\cdot\|$ is pointwise lower semicontinuous, span $\{\delta_\gamma\}_{\gamma\in\Gamma} \subset X^*$ must be 1-norming for X. For $p \geqslant 1$ set

$$\|x\|_p = \sup\left\{\left(\sum_{\gamma\in\Gamma} |a_\gamma x(\gamma)|^p\right)^{1/p} : \left\|\sum_{\gamma\in\Gamma} a_\gamma \delta_\gamma\right\| \leqslant 1\right\}.$$

Having in mind that span $\{\delta_\gamma\}_{\gamma\in\Gamma}$ is 1-norming for X we get

$$\|x\| = \|x\|_1 \geqslant \|x\|_p. \tag{2.9}$$

From the choice of $\|\cdot\|_p$ it follows that it is pointwise lower semicontinuous for every $p > 1$. It is easy to see that

$$\lim_{p\to 1} \|x\|_p = \|x\|. \tag{2.10}$$

Claim. *For every $z \in X$ and every $\varepsilon > 0$ there exists $p_{z,\varepsilon} > 1$ such that*

$$|a_\beta z(\beta)|^p > \varepsilon \tag{2.11}$$

whenever

$$1 \leqslant p \leqslant p_{z,\varepsilon}, \quad \|z\| - \|z - P_\beta z\| \geqslant 3\varepsilon, \tag{2.12}$$

$$\left\|\sum_{\gamma\in\Gamma} a_\gamma \delta_\gamma\right\| \leqslant 1, \quad \sum_{\gamma\in\Gamma} |a_\gamma z(\gamma)|^p > \|z\|_p^p - \varepsilon.$$

Proof. For $\tau \geqslant 0$ and $p \geqslant 0$ we set $\mu_p(\tau) = \max\{\tau^p, \tau\}$. It is easy to see that for $\sigma \in [0, \tau]$ and $p \geqslant 1$ the inequality

$$\mu_p(\tau) \geqslant \tau - \sigma + \sigma^p \tag{2.13}$$

holds. Pick $z \in X$ and $\varepsilon > 0$. From (2.9) and (2.10) it follows that there exists $p_{z,\varepsilon} > 1$ such that for all $p \in [1, p_{z,\varepsilon}]$

$$\|z\|_p^p \geqslant \mu_p(\|z\|) - \varepsilon. \tag{2.14}$$

We have $\|z\| - \|y\| \geqslant 3\varepsilon$, where $y = z - P_\beta z$. Since $\|y\| \leqslant \|z\|$ we can apply (2.13) for $\tau = \|z\|$ and $\sigma = \|y\|$. Taking into account that $\|y\|_p \leqslant \|y\|$ from (2.14) we obtain that for all $p \in [1, p_{z,\varepsilon}]$

$$\|z\|_p^p - \|y\|_p^p \geqslant \mu_p(\|z\|) - \varepsilon - \|y\|^p \geqslant \|z\| - \|y\| - \varepsilon \geqslant 2\varepsilon. \tag{2.15}$$

Pick $\sum_{\gamma\in\Gamma} a_\gamma \delta_\gamma$ and p satisfying (2.12). Then we have

$$\|y\|_p^p \geqslant \sum_{\gamma\in\Gamma} |a_\gamma z(\gamma)|^p - |a_\beta z(\beta)|^p > \|z\|_p^p - \varepsilon - |a_\beta z(\beta)|^p .$$

This together with (2.15) implies (2.11). □

Pick $p_n > 1$, $n = 1, 2, \dots$ such that $p_n \longrightarrow 1$ and set

$$\Phi(x) = \sum_{n=0}^{\infty} 2^{-n} \|x\|_{p_n}^{p_n},$$

where $p_0 = 1$. Let $|||\cdot|||$ be the Minkowski functional of Φ. Let us prove that $|||\cdot|||$ is strictly convex. Indeed, suppose that $|||x||| = |||y||| = |||(x+y)/2|||$. By convexity arguments we have

$$\frac{\|x\|_{p_n}^{p_n} + \|y\|_{p_n}^{p_n}}{2} - \left\|\frac{x+y}{2}\right\|_{p_n}^{p_n} = 0, \quad n = 0, 1, 2, \dots. \tag{2.16}$$

Assume that $x(\beta) \neq y(\beta)$ for some $\beta \in \Gamma$. We consider two cases. First let $x(\beta)y(\beta) < 0$. Then since $|x(\beta) + y(\beta)| < |x(\beta)| + |y(\beta)|$ and $\|\cdot\|$ is a strictly lattice norm we get

$$\|x+y\|_1 = \|x+y\| < \|\, |x| + |y|\, \| \leqslant \|x\| + \|y\| = \|x\|_1 + \|y\|_1$$

which contradicts (2.16) for $p_0 = 1$.

Assume now that $x(\beta)y(\beta) \geqslant 0$. Since $x(\beta) \neq y(\beta)$ we get $|x(\beta) + y(\beta)| > 0$. Set $z = (x+y)/2$, $3\varepsilon = \|z\| - \|z - P_\beta z\|$. Since $\|\cdot\|$ is a strictly lattice norm we have $\varepsilon > 0$. According to the claim, we can find $p_{z,\varepsilon} > 1$ such that (2.12) implies (2.11). Fix $n \in \mathbb{N}$ such that $1 < p_n < \min\{2, p_{z,\varepsilon}\}$. Set

$$\eta := 4c_{p_n} \varepsilon^{2/p_n} \left(\frac{x(\beta) - y(\beta)}{x(\beta) + y(\beta)}\right)^2 \Big/ \left(\|x\|_{p_n}^{p_n} + \|y\|_{p_n}^{p_n}\right)^{(2/p_n)-1}, \tag{2.17}$$

where c_{p_n} is from Lemma 2.5. We can find $f = \sum_\gamma a_\gamma \delta_\gamma$, $\|f\| \leqslant 1$ and

$$\sum_\gamma |a_\gamma z(\gamma)|^{p_n} > \|z\|_{p_n}^{p_n} - \min\{\varepsilon, \eta\}. \tag{2.18}$$

From (2.16) it follows that

$$\frac{1}{2}\sum_\gamma \left(|a_\gamma x(\gamma)|^{p_n} + |a_\gamma y(\gamma)|^{p_n}\right) - \sum_\gamma |a_\gamma z(\gamma)|^{p_n} \leqslant \left(\|x\|_{p_n}^{p_n} + \|y\|_{p_n}^{p_n}\right)/2 - \|z\|_{p_n}^{p_n} + \eta = \eta.$$

From Lemma 2.5 we get

$$\begin{aligned} c_{p_n}\left(\sum_\gamma |a_\gamma(x(\gamma) - y(\gamma))|^{p_n}\right)^{2/p_n} &\leqslant \eta\left(\sum_\gamma |a_\gamma x(\gamma)|^{p_n} + \sum_\gamma |a_\gamma y(\gamma)|^{p_n}\right)^{(2/p_n)-1} \\ &\leqslant \eta\left(\|x\|_{p_n}^{p_n} + \|y\|_{p_n}^{p_n}\right)^{(2/p_n)-1}. \end{aligned}$$

Hence

$$(a_\beta |x(\beta) - y(\beta)|)^2 < \frac{\eta\left(\|x\|_{p_n}^{p_n} + \|y\|_{p_n}^{p_n}\right)^{(2/p_n)-1}}{c_{p_n}}.$$

From (2.18) and the claim we deduce

$$\left|\frac{a_\beta(x(\beta)+y(\beta))}{2}\right|^{p_n} > \varepsilon.$$

Then

$$\left(\frac{2\varepsilon^{(1/p_n)}|x(\beta)-y(\beta)|}{|x(\beta)+y(\beta)|}\right)^2 < \frac{\eta\big(\|x\|_{p_n}^{p_n}+\|y\|_{p_n}^{p_n}\big)^{(2/p_n)-1}}{c_{p_n}},$$

which contradicts (2.17).

The implication (ii) $\Longrightarrow$ (i) is trivial. □

If X has an unconditional basis $\{e_\gamma\}_{\gamma\in\Gamma}$, then X^* can be identified with a lattice, which fulfils the lattice conditions at the beginning of this section. In [**11**], a Gâteaux smooth norm $|||\cdot|||$ is obtained on ℓ_1 with unconditional constant 1, the dual norm of which is not strictly lattice.

From Theorem 2.6 it follows that in a Banach space X with unconditional basis, the existence of a dual strictly convex norm in X^* implies the existence of a dual strictly lattice norm.

Corollary 2.7. *Let X and Γ be the Banach lattice and the set considered at the beginning of this section. Let $\{\Gamma_n\}_1^\infty$ be a sequence of subsets of Γ such that for every $x\in X$ and $\alpha\in\operatorname{supp} x$ there exists $a\in(0,|x(\alpha)|)$ and $m\in\mathbb{N}$ with $\alpha\in\Gamma_m$, $\#\{\gamma\in\Gamma_m : |x(\gamma)|>a\}<\infty$. Then X admits a pointwise lower semicontinuous strictly convex norm.*

Proof. For m, $n\in\mathbb{N}$ set

$$\|x\|_{m,n} = \sup\left\{\sum_{\gamma\in A}|x(\gamma)| : A\subset\Gamma_m, \#A\leqslant n\right\},$$

and

$$|||x||| = \|x\| + \sum_{m,n=1}^{\infty} 2^{-m-n}\|x\|_{m,n}.$$

Pick x, $y\in X$, such that $|x|>|y|$. We have $|x(\gamma)|\geqslant|y(\gamma)|$ for all $\gamma\in\Gamma$ and $|x(\alpha)|>|y(\alpha)|$ for some $\alpha\in\Gamma$. We can find $a\in(0,|x(\alpha)|)$ and $m\in\mathbb{N}$ such that $\alpha\in\Gamma_m$ and $\#\{\gamma\in\Gamma_m : |x(\gamma)|>a\}<\infty$. Set $A=\{\gamma\in\Gamma_m : |x(\gamma)|>a\}$ and $n=\#A$. Clearly, $\sup\{|y(\gamma)| : \gamma\in\Gamma_m\setminus A\}\leqslant\sup\{|x(\gamma)| : \gamma\in\Gamma_m\setminus A\}<|x(\alpha)|$. This implies $\|x\|_{m,n}>\|y\|_{m,n}$, so $|||x|||>|||y|||$. □

The Mercourakis space $c_1(\Sigma'\times\Gamma)$ satisfies the conditions of the above corollary; see [**3**, Remark 6.3, p. 249].

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