POINTWISE LIMITS OF BIRKHOFF INTEGRABLE FUNCTIONS

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Abstract. We study the Birkhoff integrability of pointwise limits of sequences of Birkhoff integrable Banach space-valued functions, as well as the convergence of the corresponding integrals. Both norm and weak convergence are considered. We discuss the role that equi-Birkhoff integrability and the Bourgain property play in these problems. Incidentally, a convergence theorem for the Pettis integral with respect to the norm topology is presented.

1. Introduction

The Birkhoff integral [3] for functions taking values in Banach spaces plays an important role within the modern theory of vector integration, as it has been noticed recently in [1], [4], [5], [7], [16], [17], [18], [19], [20], [21] and [22] among others. An intriguing point concerns the validity of the classical convergence theorems of Lebesgue's integration theory for the case of Birkhoff integrable functions. We pointed out in [17] that the analogue of Lebesgue's dominated convergence theorem for the Birkhoff integral fails in general. Indeed, we gave an example of a uniformly bounded sequence of Birkhoff integrable functions $f_n : [0,1] \to c_0(\mathfrak{c})$ converging pointwise to a non Birkhoff integrable function (here $\mathfrak{c}$ stands for the cardinality of the continuum). In [16] a similar example is constructed, where the Banach space in the range is now any super-reflexive space with density character greater than or equal to $\mathfrak{c}$. As regards "positive" results, we have shown in [16] that Vitali's convergence theorem holds whenever the Banach space in the range is isomorphic to a subspace of ℓ_∞. On the other hand, M. Balcerzak and M. Potyrała [1] have provided conditions ensuring the interchange of the operations of limit and Birkhoff integral which involve the notions of equi-Birkhoff integrability and almost uniform convergence.

In this paper we try to go a bit further when studying the Birkhoff integrability of the pointwise limit of a sequence of Birkhoff integrable functions, and the convergence of the corresponding integrals. We address the problem for the norm and weak topologies. Throughout this paper (Ω, Σ, μ) is a complete finite measure space and X is a Banach space. Given a sequence of functions $f_n : \Omega \to X$ converging pointwise *in norm* to $f : \Omega \to X$, we consider the function

$$F : \Omega \to X_c, \quad F(t) := (f_n(t)),$$

where $X_c = (X \oplus X \oplus \dots)_c$ is the Banach space of all norm convergent sequences in X, equipped with the supremum norm. We analyze the relationship between certain properties of the sequence (f_n) and the function F. This new approach allows us to show that, *when X is isomorphic to a subspace of ℓ_∞, the collection $\{f_n : n \in \mathbb{N}\}$ is equi-Birkhoff integrable if and only if each f_n is Birkhoff integrable*

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and the family of compositions

$$\{x^* \circ f_n : \ x^* \in B_{X^*}, \ n \in \mathbb{N}\} \subset \mathbb{R}^\Omega \tag{1}$$

is uniformly integrable (Theorem 2.3). This result relies on the fact that, under such assumptions, the family appearing in (1) has the so-called Bourgain property (see Lemma 2.2). The Bourgain property [15] of a family of real-valued functions has been applied successfully by B. Cascales and the author [5, 20] to characterize the Birkhoff integrability of vector-valued functions (see Theorems 1.1 and 1.2 below). In general, the conclusion of Theorem 2.3 is not valid if the additional hypothesis on X is dropped, as we make clear in Example 2.10. Incidentally, our views also give new information on the Pettis integral theory: we prove that *if each* f_n *is Pettis integrable and the family in* (1) *is uniformly integrable, then* f *is Pettis integrable and* $\int_A f_n \, d\mu \to \int_A f \, d\mu$ *in norm for all* $A \in \Sigma$ (Theorem 2.8). This result can be seen as a "Vitali-type" convergence theorem for the Pettis integral with respect to the *norm* topology.

In the last part of the paper we deal with sequences of functions $f_n : \Omega \to X$ converging pointwise *in the weak topology of* X to a function $f : \Omega \to X$. In this case we study the associated function

$$F : \Omega \to X_{\ell_\infty}, \quad F(t) := (f_n(t)),$$

where $X_{\ell_\infty} = (X \oplus X \oplus \dots)_{\ell_\infty}$ is the Banach space of all bounded sequences in X, equipped with the supremum norm. It turns out that *if the collection* $\{f_n : n \in \mathbb{N}\}$ *is equi-Birkhoff integrable, then* f *is Birkhoff integrable and* $\int_A f_n \, d\mu \to \int_A f \, d\mu$ *weakly for all* $A \in \Sigma$ (Theorem 2.12). Finally, we also show that the analogue of Theorem 2.3 for the weak topology fails in general (see Theorem 2.14).

Terminology and preliminaries. All unexplained terminology can be found in our standard references [6], [9] and [23]. Our Banach spaces X are assumed to be real. We write $\|\cdot\|$ to denote the norm of X if it is needed explicitly. By a 'subspace' of X we mean a closed linear subspace. As usual, B_X stands for the closed unit ball of X and X^* denotes the topological dual of X. A set $B \subset B_{X^*}$ is *norming* if $\|x\| = \sup\{|x^*(x)| : x^* \in B\}$ for all $x \in X$.

A function $f : \Omega \to X$ is *Birkhoff integrable*, with integral $\int_\Omega f \, d\mu \in X$, if for every $\varepsilon > 0$ there is a countable partition (A_m) of Ω in Σ such that, for any choice of points $t_m \in A_m$, the series $\sum_m \mu(A_m) f(t_m)$ converges unconditionally in X and $\|\sum_m \mu(A_m) f(t_m) - \int_\Omega f \, d\mu\| \leq \varepsilon$. In this case, f is also Pettis integrable and the respective integrals coincide.

Given a function $h : \Omega \to \mathbb{R}$ and $A \in \Sigma$, we write

$$\operatorname{osc}(h|_A) = \sup\{|h(t) - h(t')| : \ t, t' \in A\}.$$

A family $\mathcal{H} \subset \mathbb{R}^\Omega$ has the *Bourgain property* if for every $\varepsilon > 0$ and every $A \in \Sigma$ with $\mu(A) > 0$ there are $A_1, \dots, A_n \in \Sigma$, $A_i \subset A$ with $\mu(A_i) > 0$, having the following property: for each $h \in \mathcal{H}$ there is at least one $1 \leq i \leq n$ such that $\operatorname{osc}(h|_{A_i}) \leq \varepsilon$. Every family with the Bourgain property is made up of measurable functions, cf. [15, Theorem 11].

The characteristic function of a set $A \subset \Omega$ is denoted by χ_A.

A family $\mathcal{G}$ of real-valued integrable functions on Ω is called *uniformly integrable* if it is $\|\cdot\|_1$-bounded and for every $\varepsilon > 0$ there is $\delta > 0$ such that $\int_A |g| \, d\mu \leq \varepsilon$ for all $g \in \mathcal{G}$ whenever $A \in \Sigma$ satisfies $\mu(A) \leq \delta$.

The following results connect the Bourgain property and the Birkhoff integral. For a function $f : \Omega \to X$ and a norming set $B \subset B_{X^*}$, we write

$$Z_f := \{x^* \circ f : \ x^* \in B_{X^*}\} \subset \mathbb{R}^\Omega \quad \text{and} \quad Z_{f,B} := \{x^* \circ f : \ x^* \in B\} \subset \mathbb{R}^\Omega.$$

Recall that Z_f is always uniformly integrable whenever $f : \Omega \to X$ is Pettis integrable, cf. [23, Theorem 4-2-2].

Theorem 1.1 ([5]). *Let $f : \Omega \to X$ be a function.*

(i) *f is Birkhoff integrable if and only if Z_f is uniformly integrable and has the Bourgain property.*
(ii) *If f is bounded, then f is Birkhoff integrable if and only if there is a norming set $B \subset B_{X^*}$ such that $Z_{f,B}$ has the Bourgain property.*

Theorem 1.2 ([20]). *Suppose X is isomorphic to a subspace of ℓ_∞. Let $f : \Omega \to X$ be a function. Then f is Birkhoff integrable if and only if there is a norming set $B \subset B_{X^*}$ such that $Z_{f,B}$ is uniformly integrable and has the Bourgain property.*

2. The results

Following [1], we say that a collection $\{f_n : n \in \mathbb{N}\}$ of Birkhoff integrable functions from Ω to X is *equi-Birkhoff integrable* if for every $\varepsilon > 0$ there is a countable partition (A_m) of Ω in Σ such that, for any choice of points $t_m \in A_m$, we have:

- For each $\delta > 0$ there is $k \in \mathbb{N}$ such that $\|\sum_{m\in M} \mu(A_m) f_n(t_m)\| \leq \delta$ for every finite set $M \subset \mathbb{N}$ disjoint from $\{1, \ldots, k\}$ and for all $n \in \mathbb{N}$. (In particular, each series $\sum_m \mu(A_m) f_n(t_m)$ converges unconditionally in X.)
- $\|\sum_m \mu(A_m) f_n(t_m) - \int_\Omega f_n \, d\mu\| \leq \varepsilon$ for all $n \in \mathbb{N}$.

Proposition 2.1. *Let $f_n : \Omega \to X$ be a sequence of functions converging pointwise in norm to a function $f : \Omega \to X$. Then $\{f_n : n \in \mathbb{N}\}$ is equi-Birkhoff integrable if and only if the function*

$$F : \Omega \to X_c, \quad F(t) := (f_n(t)),$$

is Birkhoff integrable. In this case:

(i) *the family*

$$\bigcup_{n\in\mathbb{N}} Z_{f_n} = \{x^* \circ f_n : \ x^* \in B_{X^*}, \ n \in \mathbb{N}\}$$

is uniformly integrable and has the Bourgain property;

(ii) *f is Birkhoff integrable and, for each $A \in \Sigma$, we have*

$$\int_A f_n \, d\mu \to \int_A f \, d\mu \quad \textit{in norm.}$$

Proof. Observe that X_c is a subspace of X_{ℓ_∞} and so we can also look at F as an X_{ℓ_∞}-valued function. For each $n \in \mathbb{N}$, let $\pi_n : X_{\ell_\infty} \to X$ be the nth-coordinate projection, which is linear and continuous, with $\pi_n \circ F = f_n$.

Suppose first that $\{f_n : n \in \mathbb{N}\}$ is equi-Birkhoff integrable. Given $\varepsilon > 0$, let (A_m) be a countable partition of Ω in Σ fulfilling the requirements in the definition of equi-Birkhoff integrability for this ε, and take any choice $t_m \in A_m$. Then for every $\delta > 0$ there is $k \in \mathbb{N}$ such that $\|\sum_{m\in M} \mu(A_m) F(t_m)\|_{X_c} \leq \delta$ for every finite set $M \subset \mathbb{N}$ disjoint from $\{1, \ldots, k\}$, that is, the series $\sum_m \mu(A_m) F(t_m)$ converges unconditionally in X_c. Its sum satisfies

$$\Big\|\pi_n\Big(\sum_m \mu(A_m) F(t_m)\Big) - \int_\Omega f_n \, d\mu\Big\| = \Big\|\sum_m \mu(A_m) f_n(t_m) - \int_\Omega f_n \, d\mu\Big\| \leq \varepsilon$$

for all $n \in \mathbb{N}$, and so the sequence $\varphi := (\int_\Omega f_n \, d\mu)$ belongs to X_{ℓ_∞} and satisfies

$$\Big\|\sum_m \mu(A_m) F(t_m) - \varphi\Big\|_{X_{\ell_\infty}} \leq \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, F is Birkhoff integrable as X_{ℓ_∞}-valued function, with integral φ. Since F takes its values in X_c, it follows at once that $\varphi \in X_c$ and that F is Birkhoff integrable as X_c-valued function.

Conversely, assume that F is Birkhoff integrable. For each $n \in \mathbb{N}$, the composition $\pi_n \circ F = f_n$ is Birkhoff integrable and $\pi_n(\int_\Omega F \, d\mu) = \int_\Omega f_n \, d\mu$ (bear in mind that π_n is linear and continuous). The equi-Birkhoff integrability of $\{f_n : n \in \mathbb{N}\}$ follows straightforwardly from the Birkhoff integrability of F.

In order to prove the last part of the proposition, notice that the set

$$B := \{x^* \circ \pi_n|_{X_c} : \; x^* \in B_{X^*}, \; n \in \mathbb{N}\} \subset B_{X_c^*} \tag{2}$$

is norming and satisfies $\bigcup_{n\in\mathbb{N}} Z_{f_n} = Z_{F,B}$. Assume that F is Birkhoff integrable. Clearly, statement (i) follows from Theorem 1.1 (i). Let us turn to the proof of (ii). Let $L : X_c \to X$ be the linear and continuous mapping which sends each sequence to its limit. Then, since F is Birkhoff integrable, the same holds for $L(F) = f$ and, for each $A \in \Sigma$, we have

$$\lim_n \int_A f_n \, d\mu = L\Big(\Big(\int_A f_n \, d\mu\Big)\Big) = L\Big(\int_A F \, d\mu\Big) = \int_A f \, d\mu.$$

The proof is finished. □

Statement (ii) in the previous theorem has been proved in [1, Theorem 6] by a different method.

Lemma 2.2. *Suppose X is isomorphic to a subspace of ℓ_∞. Let $f_n : \Omega \to X$ be a sequence of functions converging pointwise in norm to a function $f : \Omega \to X$. The following statements are equivalent:*

(i) *Z_{f_n} has the Bourgain property for every $n \in \mathbb{N}$;*
(ii) *$\bigcup_{n\in\mathbb{N}} Z_{f_n}$ has the Bourgain property.*

Proof. (ii)⇒(i) is obvious and it only remains to prove (i)⇒(ii). Without loss of generality, we can assume that X is isometric to a subspace of ℓ_∞, which is equivalent to saying that B_{X^*} is weak*-separable. Therefore, since the f_n's and f are scalarly measurable, all the real-valued functions $t \mapsto \|f_n(t) - f(t)\|$ are measurable (cf. [20, Corollary 4.6]). Fix $\varepsilon > 0$ and $A \in \Sigma$ with $\mu(A) > 0$. By Egorov's theorem, we have $\|f_n(\cdot) - f(\cdot)\| \to 0$ almost uniformly and, in particular, there exist $A' \subset A$, $A' \in \Sigma$ with $\mu(A') > 0$, and $N \in \mathbb{N}$ such that

$$\sup_{t\in A', \; n>N} \|f_n(t) - f(t)\| \leq \varepsilon. \tag{3}$$

By [16, Lemma 2.3], the family Z_f has the Bourgain property. Set $f_0 := f$. Since $\bigcup_{n=0}^N Z_{f_n}$ has the Bourgain property, there exist $B_1, \dots, B_k \subset A'$, $B_i \in \Sigma$ with $\mu(B_i) > 0$, such that

$$\sup_{0\leq n\leq N, \; x^*\in B_{X^*}} \min_{1\leq i\leq k} \operatorname{osc}(x^* \circ f_n|_{B_i}) \leq \varepsilon.$$

Bearing in mind (3), it follows that

$$\sup_{n\in\mathbb{N}, \; x^*\in B_{X^*}} \min_{1\leq i\leq k} \operatorname{osc}(x^* \circ f_n|_{B_i}) \leq 3\varepsilon.$$

This shows that $\bigcup_{n\in\mathbb{N}} Z_{f_n}$ has the Bourgain property. □

Theorem 2.3. *Suppose X is isomorphic to a subspace of ℓ_∞. Let $f_n : \Omega \to X$ be a sequence of Birkhoff integrable functions converging pointwise in norm to a function $f : \Omega \to X$. The following statements are equivalent:*

(i) *$\{f_n : n \in \mathbb{N}\}$ is equi-Birkhoff integrable;*
(ii) *$\bigcup_{n\in\mathbb{N}} Z_{f_n}$ is uniformly integrable.*

Proof. By Proposition 2.1 it only remains to prove (ii)⇒(i). Let $F : \Omega \to X_c$ be as in Proposition 2.1. Then $Z_{F,B} = \bigcup_{n\in\mathbb{N}} Z_{f_n}$ is uniformly integrable, where $B \subset B_{X_c^*}$ is the norming set defined in (2). On the other hand, for each $n \in \mathbb{N}$ the function f_n is Birkhoff integrable and so Z_{f_n} has the Bourgain property (Theorem 1.1 (i)). By Lemma 2.2, we conclude that the family $Z_{F,B}$ has the Bourgain property. Observe also that X_c is isomorphic to a subspace of ℓ_∞ because X is. Now Theorem 1.2 applied to F ensures that it is Birkhoff integrable, which is equivalent to saying that $\{f_n : n \in \mathbb{N}\}$ is equi-Birkhoff integrable (Proposition 2.1). The proof is over. □

Corollary 2.4. *Suppose X is separable. Let $f_n : \Omega \to X$ be a sequence of functions converging pointwise in norm to a function $f : \Omega \to X$. Then $\{f_n : n \in \mathbb{N}\}$ is equi-Birkhoff integrable if and only if $\bigcup_{n\in\mathbb{N}} Z_{f_n}$ is uniformly integrable.*

Proof. Fix $n \in \mathbb{N}$. Since X is separable, f_n is Birkhoff integrable if and only if it is Pettis integrable (see [14, Corollary 5.11]), if and only if Z_{f_n} is uniformly integrable (cf. [12, Theorem 5.2]). The result now follows from Theorem 2.3. □

Corollary 2.5. *Let $f_n : \Omega \to \mathbb{R}$ be a sequence of functions converging pointwise to $f : \Omega \to \mathbb{R}$. Then $\{f_n : n \in \mathbb{N}\}$ is equi-Birkhoff integrable if and only if it is uniformly integrable.*

Remark 2.6. The analogue of the previous corollary for the notion of "equi-McShane integrability" is known, see [11]. In fact, our methods can be adapted easily to deduce this result as well. For detailed information on the McShane integral theory, we refer the reader to [2], [8] and [10].

As we have mentioned in the introduction, the analogue of Lebesgue's dominated convergence theorem for the Birkhoff integral does not hold in general. In [16] and [17] one can find examples of uniformly bounded sequences of Birkhoff integrable functions converging pointwise to non Birkhoff integrable functions. It turns out that the difficulty of interchanging the operations of limit and Birkhoff integral relies on the integrability character of the limit function, rather than on the behavior of the sequence of integrals, see Theorem 2.8 below. We first recall a "Vitali-type" theorem for the Pettis integral due to K. Musiał, cf. [12, Theorem 8.1]. For more information on convergence theorems for the Pettis integral, see [12] and [13].

Theorem 2.7 (Musiał)**.** *Let $f_n : \Omega \to X$ be a sequence of Pettis integrable functions and $f : \Omega \to X$ a function such that:*

- *for each $x^* \in X^*$, we have $x^* \circ f_n \to x^* \circ f$ μ-a.e.;*
- *$\bigcup_{n\in\mathbb{N}} Z_{f_n}$ is uniformly integrable.*

Then f is Pettis integrable and, for each $A \in \Sigma$, we have $\int_A f_n \, d\mu \to \int_A f \, d\mu$ weakly in X.

Theorem 2.8. *Let $f_n : \Omega \to X$ be a sequence of Pettis integrable functions converging pointwise in norm to a function $f : \Omega \to X$. The following statements are equivalent:*

(i) *$\bigcup_{n\in\mathbb{N}} Z_{f_n}$ is uniformly integrable;*
(ii) *f is Pettis integrable and, for each $A \in \Sigma$, we have*

$$\int_A f_n \, d\mu \to \int_A f \, d\mu \quad \textit{in norm.}$$

Proof. (ii)⇒(i) follows from the Nikodým boundedness theorem (cf. [6, Theorem 1, p. 14]) and the Vitali-Hahn-Saks theorem (cf. [6, Corollary 10, p. 24]) applied to the sequence of vector measures $\nu_n : \Sigma \to X$ defined by $\nu_n(E) = \int_E f_n \, d\mu$, because their semivariation is given by $\|\nu_n\|(A) = \sup_{x^*\in B_{X^*}} \int_A |x^* \circ f_n| \, d\mu$ for all $A \in \Sigma$.

(i)⇒(ii): The Pettis integrability of f follows from Theorem 2.7. Since Z_f and $\bigcup_{n\in\mathbb{N}} Z_{f_n}$ are uniformly integrable, the same holds for the family $\bigcup_{n\in\mathbb{N}} Z_{f_n-f}$. Thus, we can suppose without loss of generality that $f=0$. Consider the function

$$F:\Omega\to X_{c_0},\quad F(t):=(f_n(t)),$$

where $X_{c_0}=(X\oplus X\oplus\dots)_{c_0}$ is the subspace of X_c made up of all sequences converging to 0 in norm. For each $k\in\mathbb{N}$, the function

$$F_k:\Omega\to X_{c_0},\quad F_k(t)=(f_1(t),\dots,f_k(t),0,0,\dots),$$

is Pettis integrable, as can be easily seen. Observe that $F_k\to F$ pointwise for the norm topology of X_{c_0}. *We claim that $\bigcup_{k\in\mathbb{N}} Z_{F_k}$ is uniformly integrable.* Indeed, notice that $X_{c_0}^*$ can be identified with the Banach space $E:=(X^*\oplus X^*\oplus\dots)_{\ell_1}$ of all sequences (x_n^*) in X^* such that $\sum_n\|x_n^*\|<\infty$, equipped with the norm $\|(x_n^*)\|_E:=\sum_n\|x_n^*\|$, the duality being

$$\langle (x_n^*),(x_n)\rangle=\sum_n x_n^*(x_n).$$

Therefore, each $h\in\bigcup_{k\in\mathbb{N}} Z_{F_k}$ can be written as $h=\sum_{i=1}^k x_i^*\circ f_i$, where $x_i^*\in X^*$ and $\sum_{i=1}^k\|x_i^*\|\le 1$; taking $y_i^*\in B_{X^*}$ with $\|x_i^*\|\cdot y_i^*=x_i^*$, we have

$$h=\sum_{i=1}^k\|x_i^*\|\cdot(y_i^*\circ f_i)\in \operatorname{aco}\Big(\bigcup_{n\in\mathbb{N}} Z_{f_n}\Big),$$

where the symbol "aco" stands for "absolutely convex hull" in $\mathbb{R}^\Omega$. It follows that $\bigcup_{k\in\mathbb{N}} Z_{F_k}\subset \operatorname{aco}(\bigcup_{n\in\mathbb{N}} Z_{f_n})$. Since $\bigcup_{n\in\mathbb{N}} Z_{f_n}$ is uniformly integrable, the same holds for its absolutely convex hull and the claim is proved.

Theorem 2.7 applied to the sequence (F_k) now ensures that the X_{c_0}-valued function F is Pettis integrable. For each $n\in\mathbb{N}$, let $p_n:X_{c_0}\to X$ be the nth-coordinate projection (which is linear and continuous). Then, for each $A\in\Sigma$, we have

$$\int_A f_n\,d\mu=\int_A p_n\circ F\,d\mu=p_n\Big(\int_A F\,d\mu\Big)\to 0\quad\text{in norm.}$$

The proof is over. □

Combining Proposition 2.1 and Theorems 2.3 and 2.8, we arrive at:

Corollary 2.9. *Suppose X is isomorphic to a subspace of ℓ_∞. Let $f_n:\Omega\to X$ be a sequence of Birkhoff integrable functions converging pointwise in norm to a function $f:\Omega\to X$. The following statements are equivalent:*

- (i) *$\{f_n:n\in\mathbb{N}\}$ is equi-Birkhoff integrable;*
- (ii) *$\bigcup_{n\in\mathbb{N}} Z_{f_n}$ is uniformly integrable;*
- (iii) *f is Birkhoff integrable and, for each $A\in\Sigma$, we have*

$$\int_A f_n\,d\mu\to\int_A f\,d\mu\quad\textit{in norm.}$$

The situation can change dramatically if the assumption "X is isomorphic to a subspace of ℓ_∞" is dropped, as the following example shows. We take some ideas of [20, Example 3.1] (due to D. H. Fremlin). In our example the unit interval $[0,1]$ is equipped with the Lebesgue measure on the σ-algebra of all Lebesgue measurable subsets. Recall that a function $g:\Omega\to X$ *scalarly null* if for each $x^*\in X^*$ the composition $x^*\circ g$ is negligible, that is, $x^*\circ g=0$ μ-a.e.

Example 2.10. *There is a sequence of scalarly null Birkhoff integrable functions $f_n:[0,1]\to\ell_\infty(\mathfrak{c})$ converging pointwise to 0 in norm such that $\bigcup_{n\in\mathbb{N}} Z_{f_n}$ fails the Bourgain property and $\{f_n:n\in\mathbb{N}\}$ is not equi-Birkhoff integrable.*

Proof. Let $\{\mathcal{E}_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of all non-empty finite collections of Borel subsets of $[0,1]$ with positive Lebesgue measure. Then there is a family $(J_\alpha)_{\alpha<\mathfrak{c}}$ of pairwise disjoint finite subsets of $[0,1]$ such that $J_\alpha \cap E \neq \emptyset$ for every $\alpha < \mathfrak{c}$ and every $E \in \mathcal{E}_\alpha$ (see the proof of [20, Example 3.1]). For each $n \in \mathbb{N}$, set

$$\mathcal{H}_n := \{n\chi_{\{t\}} : \ t \in J_\alpha, \ \alpha < \mathfrak{c}, \ \#(J_\alpha) = n\} \subset \mathbb{R}^{[0,1]}$$

and $\mathcal{F}_n := \mathrm{co}(\mathcal{H}_n)$ (the convex hull of $\mathcal{H}_n$ in $\mathbb{R}^{[0,1]}$), where the symbol $\#(S)$ denotes the cardinality of a set S. Take $\mathcal{F} := \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ and observe that $\#(\mathcal{F}) = \mathfrak{c}$. For each $n \in \mathbb{N}$, define $f_n : [0,1] \to \ell_\infty(\mathcal{F})$ by

$$f_n(t)(h) := \begin{cases} h(t) & \text{if } h \in \mathcal{F}_n, \\ 0 & \text{if } h \in \mathcal{F} \setminus \mathcal{F}_n. \end{cases}$$

We will check that the sequence (f_n) satisfies the required properties.

✓ *Each f_n is Birkhoff integrable.* For each $h \in \mathcal{F}$, let $e_h \in B_{\ell_\infty(\mathcal{F})^*}$ be the functional given by $e_h(x) := x(h)$. The set $B = \{e_h : h \in \mathcal{F}\} \subset B_{\ell_\infty(\mathcal{F})^*}$ is norming and

$$Z_{f_n,B} = \{e_h \circ f_n : \ h \in \mathcal{F}\} = \{0\} \cup \mathcal{F}_n.$$

The family $\mathcal{H}_n$ has the Bourgain property; indeed, given a measurable set $A \subset [0,1]$ with positive measure, we can find disjoint measurable sets $A_1, A_2 \subset A$ with positive measure and, clearly, each $h \in \mathcal{H}_n$ vanishes on either A_1 or A_2. Since, in addition, $\mathcal{H}_n$ is uniformly bounded, it follows that its convex hull $\mathcal{F}_n$ has the Bourgain property too, see [20, p. 1304]. Therefore, $Z_{f_n,B}$ has the Bourgain property as well. Since f_n is bounded, we can apply Theorem 1.1 (ii) to conclude that f_n is Birkhoff integrable, as required.

✓ *Each f_n is scalarly null.* Since $Z_{f_n,B}$ is made up of negligible functions, given any $A \in \Sigma$ we have $e_h(\int_A f_n \, d\mu) = \int_A e_h \circ f_n \, d\mu = 0$ for all $h \in \mathcal{F}$, that is, $\int_A f_n \, d\mu = 0$. Therefore, Z_{f_n} is also made up of negligible functions.

✓ *$f_n \to 0$ pointwise for the norm topology of $\ell_\infty(\mathcal{F})$.* Fix $t \in [0,1]$. If t does not belong to $\bigcup_{\alpha<\mathfrak{c}} J_\alpha$, then $h(t) = 0$ for all $h \in \mathcal{F}$ and so $f_n(t) = 0$ for all $n \in \mathbb{N}$. Suppose, on the contrary, that $t \in J_\alpha$ for some $\alpha < \mathfrak{c}$. Set $n_0 := \#(J_\alpha)$. Then $f_n(t) = 0$ for all $n > n_0$. Indeed, if $\beta < \mathfrak{c}$ satisfies $\#(J_\beta) = n$, then $\beta \neq \alpha$ and, since the $J_\gamma's$ are pairwise disjoint, we have $t \notin J_\beta$. Thus $h(t) = 0$ for all $h \in \mathcal{H}_n$ and so $h(t) = 0$ for all $h \in \mathcal{F}_n$. It follows that $f_n(t) = 0$, as claimed.

✓ *$\bigcup_{n\in\mathbb{N}} Z_{f_n}$ fails the Bourgain property.* Given $\alpha < \mathfrak{c}$ and taking $n := \#(J_\alpha)$, we observe that

$$\chi_{J_\alpha} = \sum_{t\in J_\alpha} \frac{1}{n} \cdot n\chi_{\{t\}} \in \mathrm{co}(\mathcal{H}_n) = \mathcal{F}_n.$$

It follows that $\{\chi_{J_\alpha} : \alpha < \mathfrak{c}\} \subset \mathcal{F}$. It was noticed in [20, Example 3.1] that the family $\{\chi_{J_\alpha} : \alpha < \mathfrak{c}\}$ fails the Bourgain property; this follows from the inner regularity of the Lebesgue measure with respect to the Borel σ-algebra of $[0,1]$ and the fact that, given finitely many Borel sets $A_1, \dots, A_n \subset [0,1]$ with positive measure, there is $\alpha < \mathfrak{c}$ such that $\mathcal{E}_\alpha = \{A_1, \dots, A_n\}$, and therefore $\mathrm{osc}(\chi_{J_\alpha}|_{A_i}) = 1$ for all $1 \leq i \leq n$. Since

$$\bigcup_{n\in\mathbb{N}} Z_{f_n} \supset \bigcup_{n\in\mathbb{N}} Z_{f_n,B} \supset \mathcal{F} \supset \{\chi_{J_\alpha} : \ \alpha < \mathfrak{c}\},$$

we conclude that $\bigcup_{n\in\mathbb{N}} Z_{f_n}$ fails the Bourgain property.

✓ *$\{f_n : n \in \mathbb{N}\}$ is not equi-Birkhoff integrable.* Indeed, this follows from Proposition 2.1 bearing in mind that $\bigcup_{n\in\mathbb{N}} Z_{f_n}$ fails the Bourgain property. □

We next deal with a sequence of functions $f_n : \Omega \to X$ converging pointwise to a function $f : \Omega \to X$ for the *weak topology* of X. Observe that, by the Banach-Steinhaus theorem, such a sequence is *pointwise bounded*, that is, for each $t \in \Omega$ the sequence $(f_n(t))$ is bounded in X.

Proposition 2.11. *Let $f_n : \Omega \to X$ be a pointwise bounded sequence of functions.*

(i) *Then $\{f_n : n \in \mathbb{N}\}$ is equi-Birkhoff integrable if and only if the function*

$$F : \Omega \to X_{\ell_\infty}, \quad F(t) := (f_n(t)),$$

is Birkhoff integrable. In this case, $\bigcup_{n\in\mathbb{N}} Z_{f_n}$ is uniformly integrable and has the Bourgain property.

(ii) *Suppose X is isomorphic to a subspace of ℓ_∞. Then $\{f_n : n \in \mathbb{N}\}$ is equi-Birkhoff integrable if and only if $\bigcup_{n\in\mathbb{N}} Z_{f_n}$ is uniformly integrable and has the Bourgain property.*

Proof. (i) can be deduced in the same way as Proposition 2.1. Part (ii) follows from Theorem 1.2 bearing in mind that $\bigcup_{n\in\mathbb{N}} Z_{f_n} = Z_{F,B}$, where B is the norming set

$$B := \{x^* \circ \pi_n : \ x^* \in B_{X^*}, \ n \in \mathbb{N}\} \subset B_{X^*_{\ell_\infty}}$$

and $\pi_n : X_{\ell_\infty} \to X$ denotes the nth-coordinate projection. □

Theorem 2.12. *Let $f_n : \Omega \to X$ be a sequence of functions converging pointwise in the weak topology to a function $f : \Omega \to X$. If $\{f_n : n \in \mathbb{N}\}$ is equi-Birkhoff integrable, then f is Birkhoff integrable and, for each $A \in \Sigma$, we have*

$$\int_A f_n \, d\mu \to \int_A f \, d\mu \quad \textit{weakly.}$$

Proof. By Proposition 2.11 (i), the family $\bigcup_{n\in\mathbb{N}} Z_{f_n}$ is uniformly integrable. Then we can apply Theorem 2.7 to infer that f is Pettis integrable and that the sequence $(\int_A f_n \, d\mu)$ converges weakly to $\int_A f \, d\mu$ for all $A \in \Sigma$. It only remains to show that f is Birkhoff integrable. Observe that Z_f is contained in the pointwise closure of $\bigcup_{n\in\mathbb{N}} Z_{f_n}$ in $\mathbb{R}^\Omega$. Since the Bourgain property is preserved by taking pointwise closures (cf. [15, Theorem 11]) and $\bigcup_{n\in\mathbb{N}} Z_{f_n}$ has the Bourgain property (Proposition 2.11 (i)), the family Z_f also has the Bourgain property and an appeal to Theorem 1.1 (i) ensures that f is Birkhoff integrable. The proof is complete. □

We finish the paper by showing that the analogues of Lemma 2.2, Theorem 2.3 and Corollary 2.4 *for the weak topology* are not valid in general. To this end, we need the following fact due to D. H. Fremlin (personal communication) which is included here with his kind permission.

Lemma 2.13 (Fremlin)**.** *Suppose $\mu(\Omega) = 1$ and let (A_n) be an independent sequence of measurable subsets such that*

$$\sum_{n\in\mathbb{N}} \mu(A_n)(1 - \mu(A_n)) = \infty.$$

Then the family $\{\chi_{A_n} : n \in \mathbb{N}\}$ fails the Bourgain property.

Proof. Since the sequence (A_n) is independent, the same happens with $(\Omega \setminus A_n)$ and, therefore, $(A_n \times (\Omega \setminus A_n))$ is an independent sequence in the product probability space $(\Omega \times \Omega, \Sigma \otimes \Sigma, \mu \times \mu)$ such that

$$\sum_{n\in\mathbb{N}} (\mu \times \mu)\big((A_n \times (\Omega \setminus A_n)\big) = \sum_{n\in\mathbb{N}} \mu(A_n)(1 - \mu(A_n)) = \infty.$$

Fix $B_1, \dots, B_m \in \Sigma$ with positive measure. We claim that there is $n \in \mathbb{N}$ such that

$$(B_i \times B_i) \cap (A_n \times (\Omega \setminus A_n)) \neq \emptyset \quad \text{for all } 1 \leq i \leq m. \tag{4}$$

Indeed, suppose not. Then $\mathbb{N} = \bigcup_{i=1}^m P_i$, where

$$P_i := \{n \in \mathbb{N} : \ (B_i \times B_i) \cap (A_n \times (\Omega \setminus A_n)) = \emptyset\}.$$

There is some $1 \leq i \leq m$ such that $\sum_{n \in P_i} (\mu \times \mu)\big((A_n \times (\Omega \setminus A_n)\big) = \infty$. Write $P_i = \{n_1 < n_2 < \dots\}$. By the Borel-Cantelli lemma, we have

$$(\mu \times \mu)\Big(\bigcap_{k=1}^{\infty} \bigcup_{m \geq k} \big(A_{n_m} \times (\Omega \setminus A_{n_m})\big)\Big) = 1.$$

Since $(\mu \times \mu)(B_i \times B_i) > 0$, we have

$$(B_i \times B_i) \cap \Big(\bigcap_{k=1}^{\infty} \bigcup_{m \geq k} \big(A_{n_m} \times (\Omega \setminus A_{n_m})\big)\Big) \neq \emptyset,$$

which contradicts the definition of P_i.

Therefore, (4) holds for some $n \in \mathbb{N}$. Thus, $\operatorname{osc}(\chi_{A_n}|_{B_i}) = 1$ for all $1 \leq i \leq m$. It follows that $\{\chi_{A_n} : n \in \mathbb{N}\}$ fails the Bourgain property. $\square$

Recall that a Banach space has the *Schur property* if every weakly convergent sequence is norm convergent.

Theorem 2.14. *Suppose X fails the Schur property and μ is an atomless probability. Then there is a uniformly bounded sequence of simple (hence Birkhoff integrable) functions $f_n : \Omega \to X$ converging pointwise in the weak topology to 0 such that $\bigcup_{n \in \mathbb{N}} Z_{f_n}$ fails the Bourgain property and $\{f_n : n \in \mathbb{N}\}$ is not equi-Birkhoff integrable.*

Proof. Since X fails the Schur property, there is a weakly convergent sequence in X, say (x_n), which is not norm convergent. We can assume without loss of generality that $x_n \to 0$ weakly and $\|x_n\| = 1$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, fix $x_n^* \in B_{X^*}$ such that $x_n^*(x_n) = 1$.

Since μ is atomless, we can find an independent sequence (A_n) in Σ such that $\mu(A_n) = 1/n$ for all $n \in \mathbb{N}$ (cf. [9, 272X(a)]). Then the family $\{\chi_{A_n} : n \in \mathbb{N}\}$ fails the Bourgain property (by Lemma 2.13). Moreover, since $\sum_{n \in \mathbb{N}} \mu(A_n) = \infty$, the Borel-Cantelli lemma ensures that the set $E := \bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} A_m \in \Sigma$ satisfies $\mu(E) = 1$.

For each $n \in \mathbb{N}$, we consider the simple function

$$f_n : \Omega \to X, \quad f_n := x_n \chi_{A_n \cap E}.$$

On the one hand, we have $f_n \to 0$ pointwise for the weak topology of X. Indeed, given $t \in \Omega$ we have, for each $n \in \mathbb{N}$, that either $f_n(t) = 0$ or $f_n(t) = x_n$; since $x_n \to 0$ weakly, it follows at once that $f_n(t) \to 0$ weakly as well. On the other hand, observe that $x_m^* \circ f_m = \chi_{A_m \cap E} \in \bigcup_{n \in \mathbb{N}} Z_{f_n}$ for all $m \in \mathbb{N}$. Since $\{\chi_{A_n} : n \in \mathbb{N}\}$ fails the Bourgain property and $\mu(E) = 1$, the family $\{\chi_{A_n \cap E} : n \in \mathbb{N}\}$ fails the Bourgain property too. It follows that $\bigcup_{n \in \mathbb{N}} Z_{f_n}$ does not have the Bourgain property. The fact that $\{f_n : n \in \mathbb{N}\}$ is not equi-Birkhoff integrable is now a consequence of Proposition 2.11 (i). The proof is complete. $\square$

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