On locally uniformly rotund renormings in C(K) spaces

J. F. Martínez* A. Moltó† J. Orihuela‡ S. Troyanski §

November 8, 2007

Abstract

A characterization of the Banach spaces of type $C(K)$ which admit an equivalent locally uniformly rotund norm is obtained, and a method to apply it to concrete spaces is developed. As an application the existence of such renorming is deduced when K is a Namioka–Phelps compact or for some particular class of Rosenthal compacta, results recently obtained in [3] and [6] that were originally proved with methods developed ad hoc.

2006 Mathematics Subject Classification: 46B03, 46B20.
Key words: Renorming theory; Locally uniformly rotund norms; Namioka–Phelps compacta, Rosenthal compacta.

1 Introduction

The class of Banach spaces which admit an equivalent locally uniformly rotund norm (**LUR** for short) has been extensively studied and some characterizations of such spaces have already been obtained in terms of linear–topological conditions [2]. The **LUR** renorming techniques for a Banach space developed until now, which are free of martingale techniques, are based in two different approaches. In the first one, enough convex functions on the Banach space are constructed to apply Deville's lemma [1], see the decomposition method in Chapter 7, Lemma 1.1, sometimes adding an iteration processes and Banach's Contraction Mapping Theorem, to finally get an equivalent **LUR** norm, [1]. In the second one the existence of such norm is deduced from the

*The first author has been supported by Generalitat Valenciana CTBPRB/2002/292 and by MTM2007–64521 from MEC.

†The second author has been supported by MTM2007–64521 from MEC.

‡The third author has been supported by Fundación Séneca CARM: 00690/PI/04 and by MTM2005–08379 from MEC.

§The fourth author has been supported by Project of Institute of Mathematics and Informatics, Bulgarian Academy of Sciences and grant MM−1401/04 of Bulgarian NSF; by Projects MTM 2005 – 08379 from MEC, Spain and Fundación Séneca 00690/PI/04 CARM (Spain)

existence of a σ–slicely isolated network of the norm topology in $C(K)$, introducing a countable family of equivalent pointwise lower semicontinuous norms such that, roughly speaking, the **LUR** condition on a fixed sequence (x_n) and a point x controls whether the segments $[x_n, x]$ live inside small slices, this gives the equivalent **LUR** norm [8, 10]. In this paper, taking ideas of both approaches, a characterization of the existence of **LUR** norms on $C(K)$ type spaces is presented and applied to the very recent cases obtained in [3, 6] by means of the first method. More applications of this method can be found in [7] where the cases when K is a R. Haydon tree [4] or a totally ordered compact space [5] are discussed.

Theorem 1.1. *Let K be a compact space. The Banach space $C(K)$ admits an equivalent pointwise lower semicontinuous and* **LUR** *norm if, and only if, there is a countable family of subsets $\{C_n : n \in \mathbb{N}\}$ in $C(K)$ such that, for every $x \in C(K)$ and every $\varepsilon > 0$, there are $q \in \mathbb{N}$, a pointwise open half space H with $x \in H \cap C_q$ together with a finite covering $\mathcal{L}$ of K such that*

$$|y(s) - y(t)| < \varepsilon \quad \textit{whenever } s, t \in L,\ y \in H \cap C_q, \textit{ and } L \in \mathcal{L}.$$

Some compacta K which are relevant in this field are not defined by *internal* topological properties but in terms of their immersion in a product of real lines, $K \subset \mathbb{R}^\Gamma$ where the elements of Γ can be viewed as *coordinates* of the elements of K, this happens for instance in [6]. Therefore when we deal with this sort of compacta it is easier to apply Corollary 1.2 below that may be understood as a version of Theorem 1.1, where sets of *controlling coordinates* play the role that coverings have in Theorem 1.1.

Let us recall that any compact Hausdorff space can be embedded in a cube $[0, 1]^\Gamma$ for some Γ. So let $K \subset [0, 1]^\Gamma$ and let $x \in C(K)$, since any continuous function on K is uniformly continuous, given $\varepsilon > 0$ there must exist a finite set $T \subset \Gamma$ and $\delta > 0$ such that

$$s, t \in K,\ \sup_{\gamma \in T} |s(\gamma) - t(\gamma)| < \delta \implies |x(s) - x(t)| < \varepsilon. \tag{1}$$

Following [8], we say that T *$\varepsilon-$controls x with δ* whenever (1) holds.

Corollary 1.2. *Let K be a compact space. The Banach space $C(K)$ admits an equivalent pointwise lower semicontinuous and* **LUR** *norm if, and only if, there is a countable family of subsets $\{C_n : n \in \mathbb{N}\}$ in $C(K)$ such that, for every $x \in C(K)$ and every $\varepsilon > 0$, there are $q \in \mathbb{N}$, a pointwise open half space H with $x \in H \cap C_q$ together with a finite set $T \subset \Gamma$ and $\delta > 0$ such that T $\varepsilon-$controls every $y \in H \cap C_q$ with δ.*

Therefore, roughly speaking, the existence of a **LUR** renorming in $C(K)$ is equivalent to describing *regularly* the members of a finite covering of K on which each $x \in C(K)$ has arbitrarily small oscillation, alternatively a finite set of coordinates that ε–controls it. The *regularity* of these *descriptions* is based, like in the case of [2], on the existence of half spaces with certain properties, this is the motivation for developing, in our Section 3, a method to obtain such half spaces. The characterization and the method together allow us to deduce in Section 4 a unified approach to prove two new results in this field, on the one hand the existence of a **LUR** norm in a Banach space X such that X^* admits a **LUR** dual norm, result due to R. Haydon [3]; on the other hand the existence of such renorming in $C(K)$ spaces where K belongs to a particular class of Rosenthal compacta [6].

As usual we denote by $(K, \mathcal{T})$ a compact Hausdorff topological space and by $C(K)$ the Banach space of real–valued continuous functions on K, endowed with the supremum norm $\|x\|_\infty = \sup\{|x(t)| : t \in K\}$. Let us remember that a norm $\|\cdot\|$ on a normed space X is said to be locally uniformly rotund (**LUR**) if

$$\lim_{n\to\infty} \|x - x_n\| = 0 \quad \text{whenever} \quad \lim_{n\to\infty} (2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2) = 0.$$

Let us recall that given a bounded set A of a normed space X, the *Kuratowski index of non–compactness* of A, $\alpha(A)$, is defined by $\alpha(A) = \inf_{n\in\mathbb{N}} \alpha(A, n)$ where

$$\alpha(A, n) = \inf\{\varepsilon > 0 : \ A \text{ can be covered by } n \text{ sets of diameter less than } \varepsilon\}.$$

The characterization (iii) of Theorem 1.3 of Banach spaces that admit a **LUR** norm is obtained in [2] and we shall need it here:

Theorem 1.3 ([2], [8], [10])**.** *Let X be a Banach space and let F be a norming linear subspace of X^*. The following assertions are equivalent:*

*(i) X admits an equivalent $\sigma(X, F)-$lower semicontinuous **LUR** norm;*

(ii) there exists a decomposition $X = \bigcup_{n\in\mathbb{N}} X_n$ in such a way that given $\varepsilon > 0$, $n \in \mathbb{N}$ and $x \in X_n$ there exists a $\sigma(X, F)-$open half space H containing x such that diam $(H \cap X_n) < \varepsilon$;

(iii) there exists a decomposition $X = \bigcup_{n\in\mathbb{N}} X_n$ in such a way that given $\varepsilon > 0$, $n \in \mathbb{N}$ and $x \in X_n$ there exists a $\sigma(X, F)-$open half space H containing x such that $\alpha(H \cap X_n) < \varepsilon$.

2 A characterization

Proof of Theorem 1.1. Assume that $C(K)$ admits a pointwise lower semicontinuous **LUR** norm. If $C(K) = \bigcup_{n\in\mathbb{N}} C_n$ is the decomposition of Theorem 1.3.(ii), given $x \in C(K)$ and $\varepsilon > 0$ let n be a natural number and let H be a pointwise open half space such that $x \in C_n \cap H$ and

$$\text{diam } (H \cap C_n) < \varepsilon/3. \tag{2}$$

Since x is continuous, by compactness we get a finite covering $\mathcal{L}$ of K such that the oscillation of x in every $L \in \mathcal{L}$ is

$$\text{osc}\,(x, L) < \varepsilon/3. \tag{3}$$

From (2) and (3) it follows that $\mathrm{osc}(y, L) < \varepsilon$ for every $y \in H \cap C_n$ and every $L \in \mathcal{L}$.

Conversely let $C(K) = \bigcup_{n=1}^{+\infty} C_n$ be the decomposition of the statement. Then given $\varepsilon > 0$, $n \in \mathbb{N}$ and $x \in C_n$ there are a pointwise open half space H containing x and a finite covering $\mathcal{L}$ of K such that $osc(y, L) < \varepsilon/9$ for every $y \in H \cap C_n$ and every $L \in \mathcal{L}$. If $C_{n,M} = \{x \in C_n : \|x\|_\infty \leq M\}$ then $C_n = \bigcup_{M \in \mathbb{N}} C_{n,M}$.

Following [6, Proposition 5], take $x \in C_{n,M}$ and set $\{I_j\}_{j=1}^{\ell}$ a finite family of open real intervals of length less than $\varepsilon/9$ satisfying $[-M, M] \subset \bigcup_{j=1}^{\ell} I_j$. If $\mathcal{L} = \{L_i\}_{i=1}^{m}$ for some $m \in \mathbb{N}$, then for every $1 \leq i \leq m$ choose a point $s_i \in L_i$ and for every map $\pi : \{1, \ldots, m\} \longrightarrow \{1, \ldots, \ell\}$ fix a function $x_\pi \in H \cap C_{n,M}$ satisfying $x_\pi(s_i) \in I_{\pi(i)}$ for all $1 \leq i \leq m$ (whenever this is possible). We claim that for these x_π we have $H \cap C_{n,M} \subset \bigcup_\pi B(x_\pi, \varepsilon/3)$. Indeed, if $y \in H \cap C_{n,M}$ then for every $1 \leq i \leq m$ there exists $1 \leq k_i \leq \ell$ such that $y(s_i) \in I_{k_i}$. Denote by π the map $i \longmapsto k_i$. If $s \in K$ there exists $1 \leq i \leq m$ such that $s \in L_i$ so that

$$\begin{aligned} |y(s) - x_\pi(s)| &\leq |y(s) - y(s_i)| + |y(s_i) - x_\pi(s_i)| + |x_\pi(s_i) - x_\pi(s)| < \\ &< osc(y, L_i) + \mathrm{length}(I_{\pi(i)}) + osc(x_\pi, L_i) < \varepsilon/3. \end{aligned}$$

Since the Kuratowski index of non–compactness of $H \cap C_{n,M}$ is less than ε, from Theorem 1.3.(iii) we conclude that $C(K)$ admits an equivalent pointwise lower semicontinuous **LUR** norm. ■

Proof of Corollary 1.2. Assume that $C(K)$ admits a pointwise lower semicontinuous **LUR** norm. Take C_n, $x \in C(K)$, $\varepsilon > 0$, H, $x \in H \cap C_n$ as in the proof of Theorem 1.1 satisfying (2). If the finite set $T \subset \Gamma$ $\varepsilon/3$–controls x with $\delta > 0$ it is easy to check that T ε–controls every $y \in H \cap C_n$ with $\delta > 0$.

Conversely let $\{C_n : n \in \mathbb{N}\}$ in $C(K)$ satisfying the assertion of the corollary. Given $\varepsilon > 0$ let $q \in \mathbb{N}$, a pointwise open half space H with $x \in H \cap C_q$ and a finite set $T \subset \Gamma$ and $\delta > 0$ such that

(4) $$T \ \varepsilon\text{–controls every } y \in H \cap C_q \text{ with } \delta > 0.$$

Given $s \in K$ let V_s the open neighbourhood of s made up by all $t \in K$ such that $\sup_{\gamma \in T} |s(\gamma) - t(\gamma)| < \delta/2$. The compactness of K yields a finite covering $\mathcal{L}$ of K such that $\sup_{\gamma \in T} |s(\gamma) - t(\gamma)| < \delta$ for each s, $t \in L$ and each $L \in \mathcal{L}$. This and (4) show that $|y(s) - y(t)| < \varepsilon$ whenever s, $t \in L$, $y \in H \cap C_q$ and $L \in \mathcal{L}$. To finish the proof it is enough to apply Theorem 1.1. ■

Remark. Recall that a subset A of X is said to be *radial* if for every $x \in X$ there exists $\rho > 0$ such that $\rho x \in A$. A linear subspace F of X^* is called *norming* whenever $|x| = \sup\{|f(x)| : f \in B_{X^*} \cap F\}$, $x \in X$, is an equivalent norm on X. Theorem 1.1 and Corollary 1.2 hold if we replace $C(K)$ by any linear subspace X of it, if we change pointwise lower semicontinuous by $\sigma(C(K), F)$–lower semicontinuous, where F is a norming subspace of $C(K)^*$, and if we change X by any of its *radial* subsets. This observation may be of some use to apply the above characterizations to spaces $C_0(L)$ where L is a locally compact space. Moreover it shows that to apply the characterizations above it is enough to decompose the unit ball $B_{C(K)} = \bigcup C_n$ instead of the whole space $C(K)$ as it will be done below.

To apply Theorem 1.1 it is enough to show that for every $\varepsilon > 0$, there is a countable family of subsets $\{C_{n,\varepsilon} : n \in \mathbb{N}\}$ in $C(K)$ such that, for every $x \in C(K)$ there is $q \in \mathbb{N}$ and a pointwise open half space H with $x \in H \cap C_{q,\varepsilon}$ together with a finite covering $\mathcal{L}$ of K such that $|y(s) - y(t)| < \varepsilon$ whenever $s, t \in L$, $y \in H \cap C_{q,\varepsilon}$, and $L \in \mathcal{L}$. Indeed, the (countable) family $\{C_{n,1/m} : n, m \in \mathbb{N}\}$ satisfies the requirements of Theorem 1.1. A similar remark can be done about Corollary 1.2.

3 A method to construct half spaces

To apply Theorem 1.1 and Corollary 1.2 to concrete compact spaces K, it is necessary to obtain a *method* to associate to each $x \in C(K)$ and each $\varepsilon > 0$ a finite covering $\mathcal{L}$ of K such that $osc(x, L) < \varepsilon$ for every $L \in \mathcal{L}$ or, alternatively, a finite set of coordinates that $\varepsilon-$controls x. Often this method gives a decomposition of $C(K)$ that fulfills this requirement where H is not a pointwise open half space but a finite intersection of pointwise open half spaces. In general it is not possible to obtain a refinement of this decomposition for which the above characterization holds [4]. In the lemma below a method is developed to get necessary conditions to get such half spaces, specialists will recognize in (7) the rigidity condition.

Lemma 3.1. *Let φ_k be, $1 \leq k \leq n$, convex and lower semicontinuous maps on a convex set B of a locally convex space X. Let $A_0 \subset B$ for which*

$$osc(\varphi_k, A_0) \leq 1 \text{ for all } 1 \leq k \leq n. \tag{5}$$

Let δ and θ be such that $0 < 4\delta^{1/n} \leq \theta \leq 1$. Fix $x \in A_0$ and for every $1 \leq k \leq n$ set $A_k = \{y \in A_{k-1} : \ \varphi_k(x) - \varphi_k(y) < \delta\}$. Suppose that for every $1 \leq k \leq n$ we have

$$\varphi_k(x) \geq \sup\{\varphi_k(y) : \ y \in A_{k-1}\} - \delta, \quad \text{and} \tag{6}$$

$$\{y \in A_{k-1} : \quad \delta \leq \varphi_k(x) - \varphi_k(y) < \theta\} = \emptyset. \tag{7}$$

Then there exists a continuous linear map f on X such that $\{y \in A_0 : \ f(x - y) < 1\} \subset A_n$.

Proof. Set $q = 4/\theta$ and $\varphi = \sum_{i=1}^{n} q^{n+1-i}\varphi_i$. Since φ is a convex and lower semicontinuous function on B there must exist an $\varepsilon-$subdifferential at x for every $\varepsilon > 0$ [9, p. 48]. Then there exists a continuous linear map g on X such that $\varphi(x) - \varphi(y) < g(x - y) + \theta/6$ for every $y \in B$. Set $S = \{y \in A_0 : \ 6\, g(x - y) < \theta\}$. We will show by induction that $S \subset A_k$ for every $1 \leq k \leq n$. Clearly $S \subset A_0$. Assume that for some k, $1 \leq k \leq n$, we have $S \subset A_{k-1}$ and pick $y \in S$. Since $A_0 \supseteq A_1 \supseteq \ldots \supseteq A_{k-1} \supseteq S$ from (6) we get $\varphi_i(x) - \varphi_i(y) \geq -\delta$ for every $1 \leq i \leq k - 1$. From this and (5) we have

$$
\begin{aligned}
q^{n+1-k}\left(\varphi_k(x)-\varphi_k(y)\right) &= \varphi(x)-\varphi(y) \\
&- \sum_{1\le i\le k-1} q^{n+1-i}\left(\varphi_i(x)-\varphi_i(y)\right) - \sum_{k+1\le i\le n} q^{n+1-i}\left(\varphi_i(x)-\varphi_i(y)\right) < \\
&< \; g(x-y)+\frac{\theta}{6}+\delta \sum_{1\le i\le k-1} q^{n+1-i} + \sum_{k+1\le i\le n} q^{n+1-i}.
\end{aligned}
$$

Then

$$q^{n+1-k}\left(\varphi_k(x)-\varphi_k(y)\right) < \frac{\theta}{3}+\frac{\delta q^{n+1}}{q-1}+\frac{q^{n+1-k}}{q-1}.$$

Taking into account that $q = 4/\theta$, the above inequality yields

$$\varphi_k(x)-\varphi_k(y) < \frac{\theta}{3}+\theta\frac{\delta(4/\theta)^n+1}{4-\theta} < \frac{\theta}{3}+\frac{2\theta}{4-\theta} \le \theta.$$

Since $y \in A_{k-1}$, from (7) we deduce that $\varphi_k(x)-\varphi_k(y) < \delta$ so $y \in A_k$. ■

4 Some applications

4.1 Namioka–Phelps compacta

In this section we will deduce from Theorem 1.1 and Lemma 3.1 the existence of an equivalent locally uniformly rotund norm on $C(K)$ when K is a *Namioka–Phelps* compact. The existence of a **LUR** renorming in $C(K)$ for such compacta K was proved by R. Haydon in [3], deducing that a Banach space X has an equivalent **LUR** norm whenever X^* has a **LUR** dual norm. This class of compacta was introduced by M. Raja in [11] proving that (B_{X^*}, ω^*) belongs to this class whenever X^* is a dual Banach space with a **LUR** dual norm.

Let us recall that a family $\mathcal{H} = \{H_i : \; i \in I\}$ of subsets of a topological space $(X, \mathcal{T})$ is said to be $\mathcal{T}-$*isolated* if for every $i \in I$

$$H_i \cap \overline{\bigcup\{H_j : \; j \in I, \; j \neq i\}}^{\mathcal{T}} = \emptyset.$$

We will say that $\mathcal{H}$ is a $\mathcal{T}-\sigma-$*isolated* family if $\mathcal{H}$ is a countable union of $\mathcal{T}-$isolated families.

A collection $\mathcal{N}$ of subsets of a topological space $(X, \mathcal{T})$ is said to be a *network* for the topology $\mathcal{T}$ if for every $U \in \mathcal{T}$ and every $x \in U$ there exists $N \in \mathcal{N}$ such that $x \in N \subset U$.

Definition 4.1. [11] A compact Hausdorff space $(K, \mathcal{T})$ is said to be a *Namioka–Phelps compact* if there is a $\mathcal{T}-$lower semicontinuous metric ρ on K such that the metric topology induced by ρ has a network which is $\mathcal{T}-\sigma-$isolated.

In this section, using ideas of [3], we will deduce from our characterization and our method of constructing half spaces that $C(K)$ admits an equivalent pointwise lower semicontinuous **LUR** norm when K is a Namioka–Phelps compact. As in [3], we first show that there exists a $\sigma-$isolated covering of K with some special properties (see Theorem 4.3 below). Then we associate to each $x \in C(K)$ and each $\varepsilon > 0$ a finite covering $\mathcal{L}$ of K such that

$$osc(x, L) < \varepsilon \text{ for every } L \in \mathcal{L}, \tag{8}$$

then using Lemma 3.1 we will deduce that Theorem 1.1 holds.

Definition 4.2. Given a compact space $(K, \mathcal{T})$, a family $\mathcal{I}$ of subsets of K and a subset H of K, we say that $\mathcal{I}$ is *rigidly finite at* H when

(i) the family $\{I \in \mathcal{I} : \ I \cap H \neq \emptyset\}$ is finite, nonempty and

(ii) $H \cap \overline{\bigcup\{I \in \mathcal{I} : \ I \cap H = \emptyset\}} = \emptyset$.

We start by proving the following result, which is based on [3, Lemma 3.3] and it is essential to associate to each $x \in C(K)$ and each $\varepsilon > 0$ a finite covering $\mathcal{L}$ of K for which (8) holds; moreover it plays a key role to fulfil the requirements of Lemma 3.1, see Proposition 4.5.(iii)–(iv) and (10) below.

Theorem 4.3. *Let K be a compact space and let $\mathcal{I}$ be a $\sigma-$isolated covering of K. Then there exists another covering $\mathcal{J}$ of K such that $\mathcal{J} = \bigcup_{i\in\mathbb{N}} \mathcal{J}(i)$, where each family $\mathcal{J}(i)$ is isolated and*

(i) for every nonempty closed subset H of K there exists $i \in \mathbb{N}$ such that $\mathcal{J}(i)$ is rigidly finite at H;

(ii) for every $J \in \mathcal{J}$ there is $I \in \mathcal{I}$ such that $J \subset \overline{I}$.

As usual given a family of sets $\mathcal{J}$ the symbol $\bigcup \mathcal{J}$ stands for the union of all the elements of $\mathcal{J}$.

Lemma 4.4. *Let K be a compact space, let H be a closed subset of K and $\mathcal{J}$ an isolated family in K. If $\mathcal{J}$ is not rigidly finite at H then either $H \cap \overline{\bigcup \mathcal{J}} = \emptyset$ or $H \cap \overline{\bigcup \mathcal{J}} \setminus \bigcup \mathcal{J} \neq \emptyset$.*

Proof. Set $\mathcal{M} = \{J \in \mathcal{J} : \ J \cap H \neq \emptyset\}$, let us distinguish three possibilities:

a) $\mathcal{M}$ is empty. Then $H \cap \bigcup \mathcal{J} = \emptyset$ and $H \cap \overline{\bigcup \mathcal{J}} = H \cap \overline{\bigcup \mathcal{J}} \setminus \bigcup \mathcal{J}$.

b) $\mathcal{M}$ is infinite. Then there exists a family $\{J_i : \ i \in \mathbb{N}\} \subset \mathcal{J}$ with $J_i \neq J_k$ for $i \neq k$, satisfying $J_i \cap H \neq \emptyset$ for every $i \in \mathbb{N}$. Let $t_i \in J_i \cap H$, $i \in \mathbb{N}$, choose an accumulation point t of the set $\{t_i : \ i \in \mathbb{N}\}$ then $t \in H \cap \overline{\bigcup \mathcal{J}}$ and $t \notin \bigcup \mathcal{J}$ since $\mathcal{J}$ is isolated.

c) $\mathcal{M}$ is a nonempty finite set. If $\mathcal{J}$ is not rigidly finite at H then

$$H \cap \overline{\bigcup\{J \in \mathcal{J} : \ J \cap H = \emptyset\}} \neq \emptyset.$$

Since $\mathcal{J}$ is isolated we get

$$\overline{\bigcup\{J \in \mathcal{J} : J \cap H = \emptyset\}} \subset \overline{\bigcup \mathcal{J}} \setminus \bigcup \mathcal{M} = \left(\overline{\bigcup \mathcal{J}} \setminus \bigcup \mathcal{J}\right) \cup \left(\bigcup (\mathcal{J} \setminus \mathcal{M})\right).$$

Hence $H \cap \overline{\bigcup\{J \in \mathcal{J} : J \cap H = \emptyset\}} \subset H \cap \overline{\bigcup \mathcal{J}} \setminus \bigcup \mathcal{J}$.

■

Proof of Theorem 4.3. Let $\mathcal{I}$ be a $\sigma-$isolated covering of K. Then $K = \bigcup\{I : I \in \mathcal{I}\}$ and for every $i \in \mathbb{N}$ there is an isolated family $\mathcal{I}(i)$ such that $\mathcal{I}$ is the family of all sets that belong to some $\mathcal{I}(i)$ for some $i \in \mathbb{N}$. The proof is divided into three steps.

<u>*Step 1.*</u> We can assume that $\overline{\bigcup \mathcal{I}(i)} \setminus \bigcup \mathcal{I}(i)$ is closed for every $i \in \mathbb{N}$.

Indeed, for every $i \in \mathbb{N}$ we define the family

$$\widetilde{\mathcal{I}}(i) := \left\{ \overline{I} \setminus \overline{\bigcup \mathcal{I}(i) \setminus I} : I \in \mathcal{I}(i) \right\}.$$

It is clear that $\widetilde{\mathcal{I}}(i)$ is an isolated family and that each $\overline{\bigcup \widetilde{\mathcal{I}}(i)} \setminus \bigcup \widetilde{\mathcal{I}}(i)$ is just the set of all points t in K such that each neighbourhood of t meets at least two members of $\widetilde{\mathcal{I}}(i)$, then $\overline{\bigcup \widetilde{\mathcal{I}}(i)} \setminus \bigcup \widetilde{\mathcal{I}}(i)$ is closed. Since each $\mathcal{I}(i)$ is isolated we have $I \subset \overline{I} \setminus \overline{\bigcup \mathcal{I}(i) \setminus I}$ for every $I \in \mathcal{I}(i)$, therefore the family of all sets that belong to some $\widetilde{\mathcal{I}}(i)$ for some $i \in \mathbb{N}$ is a $\sigma-$isolated covering of K. From now on we will write $\mathcal{I}(i)$ instead of $\widetilde{\mathcal{I}}(i)$ for $i \in \mathbb{N}$.

<u>*Step 2.*</u> The construction of $\mathcal{J}$.

Following [3] we define recursively isolated families $\mathcal{I}(i_1, \ldots, i_n)$ for $(i_1, \ldots, i_n) \in \mathbb{N}^{<\omega}$, the family of all finite sequences of natural numbers. If $\mathcal{I}(i_1, \ldots, i_n)$ has been defined set

$$I(i_1, \ldots, i_n) = \bigcup \mathcal{I}(i_1, \ldots, i_n) \text{ and } J(i_1, \ldots, i_n) = \overline{I(i_1, \ldots, i_n)} \setminus I(i_1, \ldots, i_n).$$

Given $j \in \mathbb{N}$ let $\mathcal{I}(i_1, \ldots, i_n, j)$ be the (isolated) family

$$\mathcal{I}(i_1, \ldots, i_n, j) := \{J(i_1, \ldots, i_n) \cap I : I \in \mathcal{I}(j)\}.$$

From step 1 it follows that each $J(i_1, \ldots, i_n)$ is closed and $I(i_1, \ldots, i_n) = \emptyset$ if $(i_1, \ldots, i_n)$ has repeated terms. Then set $\mathcal{J}$ as the family of all sets in any $\mathcal{I}(i_1, \ldots, i_n)$ for some $(i_1, \ldots, i_n) \in \mathbb{N}^{<\omega}$. From the choice of $\mathcal{J}$ we have that *(ii)* holds. Let us show condition *(i)*.

<u>*Step 3.*</u> For every nonempty closed subset H of K there is $(i_1, \ldots, i_n) \in \mathbb{N}^{<\omega}$ such that $\mathcal{I}(i_1, \ldots, i_n)$ is rigidly finite at H.

Indeed, otherwise from Lemma 4.4 there exists a non empty closed set H such that

$$H \cap \overline{I(i_1, \dots, i_n)} = \emptyset \text{ or } H \cap J(i_1, \dots, i_n) \neq \emptyset \text{ for any } (i_1, \dots, i_n) \in \mathbb{N}^{<\omega}. \tag{9}$$

Since $\mathcal{I}$ is a covering of K, there is $i \in \mathbb{N}$ such that $H \cap I(i) \neq \emptyset$. If we write $i_1 = \min\{i \in \mathbb{N} : H \cap \overline{I(i)} \neq \emptyset\}$ then $H \cap \overline{I(i_1)} \neq \emptyset$ and from (9) it follows that $H \cap J(i_1) \neq \emptyset$. Thus, there is $j \in \mathbb{N}$ such that $\emptyset \neq H \cap J(i_1) \cap I(j) = H \cap I(i_1, j)$ and we can set $i_2 = \min\{i \in \mathbb{N} : H \cap \overline{I(i_1, i)} \neq \emptyset\}$. Note that $i_2 \neq i_1$ since otherwise $I(i_1, i_2)$ is empty. Proceeding recursively we can obtain a sequence of pairwise distinct natural numbers $(i_n)_{n\geq 1}$ satisfying

(a) $H \cap \overline{I(\ell)} = \emptyset$ if $\ell < i_1$ but $H \cap \overline{I(i_1)} \neq \emptyset$;

(b) $H \cap \overline{I(i_1, \dots, i_{n-1}, \ell)} = \emptyset$ if $\ell < i_n$ and $n \geq 2$ and $H \cap \overline{I(i_1, \dots, i_n)} \neq \emptyset$.

We claim that $(i_n)_{n\geq 1}$ is strictly increasing. Indeed, from $H \cap \overline{I(i_1, i_2)} \subset H \cap \overline{I(i_2)}$ and (a) we get $i_2 > i_1$. Let $n \geq 2$, since each set $J(i_1, \dots, i_n)$ is closed we have

$$\begin{aligned} I(i_1, \dots, i_{n+1}) &= J(i_1, \dots, i_n) \cap I(i_{n+1}) \subset \overline{I(i_1, \dots, i_n)} \cap I(i_{n+1}) \\ &\subset J(i_1, \dots, i_{n-1}) \cap I(i_{n+1}) = I(i_1, \dots, i_{n-1}, i_{n+1}) \end{aligned}$$

which implies $H \cap \overline{I(i_1, \dots, i_{n+1})} \subset H \cap \overline{I(i_1, \dots, i_{n-1}, i_{n+1})}$. From (b) we deduce that $i_{n+1} > i_n$.

Finally, by compactness there exists a point $t \in \bigcap_{n=1}^{+\infty} H \cap J(i_1, \dots, i_n)$ and there is $i \in \mathbb{N}$ such that $t \in I(i)$. For every $n \in \mathbb{N}$ it follows that $t \in H \cap J(i_1, \dots, i_n) \cap I(i) = H \cap I(i_1, \dots, i_n, i)$ and by minimality we get $i \geq i_{n+1}$ for all $n \in \mathbb{N}$, a contradiction. ■

We denote by K a (Namioka–Phelps) compact space, by $\mathcal{T}$ its topology and by ρ a $\mathcal{T}-$lower semicontinuous metric on K such that the metric topology induced by ρ has a network $\mathcal{D}$ which is $\mathcal{T}-\sigma-$isolated. From now on and unless otherwise stated, all the closures are taken with respect to $\mathcal{T}$. Theorem 4.3 applied to the $\mathcal{T}-\sigma-$isolated family $\mathcal{D}^\ell = \{A \in \mathcal{D} : \rho-diam(A) \leq 1/\ell\}$ yields a covering $\mathcal{I}^\ell$ of K made up by all the sets that belong to any $\mathcal{I}^\ell(i)$ for $i \in \mathbb{N}$, where each family $\mathcal{I}^\ell(i)$ is $\mathcal{T}-$isolated, satisfying

(a) each set of $\mathcal{I}^\ell$ is included in the closure of some set of $\rho-$diameter at most $1/\ell$;

(b) for every nonempty closed subset H of K there is $i \in \mathbb{N}$ such that $\mathcal{I}^\ell(i)$ is rigidly finite at H.

Following [3], for every $x \in C(K)$ and every $\varepsilon > 0$, we are going to describe a method to split up every closed subset L of K where the oscillation of x on L is bigger than ε, in such a way that Theorem 1.1 gives a pointwise lower semicontinuous renorming.

Given $\mathcal{M} \subset \mathcal{I}^\ell(i)$ such that the cardinal of $\mathcal{M}$ is $\#\mathcal{M} = m$ let

$$\Phi(x, L, \mathcal{M}) = \frac{1}{m} \sum_{M \in \mathcal{M}} \sup x_{\restriction L \cap \overline{M}} \quad \text{and} \quad \Psi(x, L, \mathcal{M}) = \frac{1}{m} \sum_{M \in \mathcal{M}} \inf x_{\restriction L \cap \overline{M}}.$$

Proposition 4.5 ([3]). *Let $x \in C(K)$ be and let $\varepsilon > 0$. Then there exists $\ell \in \mathbb{N}$ such that if L is a closed subset of K with $osc(x, L) \geq \varepsilon$ then there are $m, n, i, j \in \mathbb{N}$ and a pair $(\mathcal{M}, \mathcal{N})$ satisfying*

(i) $\mathcal{M} \subset \mathcal{I}^\ell(i)$, $\mathcal{N} \subset \mathcal{I}^\ell(j)$, $\#\mathcal{M} = m$, $\#\mathcal{N} = n$ *and* $A \cap L \neq \emptyset$ *for every* $A \in \mathcal{M} \cup \mathcal{N}$;

(ii) $\overline{\bigcup \mathcal{M}} \bigcap \overline{\bigcup \mathcal{N}} = \emptyset$;

(iii) $\Phi(x, L, \mathcal{M}) > (1 - 1/m) \sup x_{\restriction L} + 1/m \ \sup x_{\restriction L \cap \overline{\bigcup (\mathcal{I}^\ell(i) \setminus \mathcal{M})}}$;

(iv) $\Psi(x, L, \mathcal{N}) < (1 - 1/n) \inf x_{\restriction L} + 1/n \ \inf x_{\restriction L \cap \overline{\bigcup (\mathcal{I}^\ell(j) \setminus \mathcal{N})}}$.

Proof. It is easy to see that x is $\rho-$uniformly continuous then there exists $\ell \in \mathbb{N}$ such that $|x(s) - x(t)| < \varepsilon/3$ whenever $\rho(s, t) \leq 1/\ell$ with $s, t \in K$. Suppose that $osc(x, L) \geq \varepsilon$ for some closed subset L of K and let $H_1 = \{t \in L : x(t) = \sup x_{\restriction L}\}$. According to the properties of $\mathcal{I}^\ell$ there exists $i \in \mathbb{N}$ such that the family $\mathcal{M} := \{I \in \mathcal{I}^\ell(i) : I \cap H_1 \neq \emptyset\}$ is finite and nonempty, say $\#\mathcal{M} = m$ for some $m \in \mathbb{N}$, and $H_1 \cap \overline{\bigcup(\mathcal{I}^\ell(i) \setminus \mathcal{M})} = \emptyset$. This clearly implies

$$\sup x_{\restriction L \cap \overline{\bigcup (\mathcal{I}^\ell(i) \setminus \mathcal{M})}} < \sup x_{\restriction L}. \tag{10}$$

Note that $\Phi(x, L, \mathcal{M}) = \sup x_{\restriction L}$ since $\sup x_{\restriction L} = \sup x_{\restriction L \cap \overline{M}}$ for all $M \in \mathcal{M}$. A similar argument with $H_2 = \{t \in L : x(t) = \inf x_{\restriction L}\}$ gives a $j \in \mathbb{N}$ such that the set $\mathcal{N} := \{I \in \mathcal{I}^\ell(j) : I \cap H_2 \neq \emptyset\}$ is finite and nonempty, say $\#\mathcal{N} = n$ for some $n \in \mathbb{N}$, and $\Psi(x, L, \mathcal{N}) = \inf x_{\restriction L} < \inf x_{\restriction L \cap \overline{\bigcup (\mathcal{I}^\ell(j) \setminus \mathcal{N})}}$. Hence, *(i)*, *(iii)* and *(iv)* hold.

To prove *(ii)* observe that given $M \in \mathcal{M}$ there exists $A \in \mathcal{D}$ such that $M \subset \overline{A}$ and $\rho - diam(A) \leq 1/\ell$. By $\rho-$uniform continuity of x it follows that $osc(x, \overline{M}) \leq \varepsilon/3$ and since $M \cap H_1 \neq \emptyset$ we get $x(t) \geq \sup x_{\restriction L} - \varepsilon/3$ for every $t \in \overline{M}$. Similarly, $x(t) \leq \inf x_{\restriction L} + \varepsilon/3$ for every $N \in \mathcal{N}$ and every $t \in \overline{N}$. Condition *(ii)* follows from the fact that $osc(x, L) \geq \varepsilon$. ■

We say that a pair $(\mathcal{M}, \mathcal{N})$ satisfying *(i)–(iv)* of Proposition 4.5 is a *good choice* of x of type (ℓ, m, n, i, j) on L. From condition *(ii)*, for every good choice $(\mathcal{M}, \mathcal{N})$ we can and do fix a pair of closed sets, $X(\mathcal{M}, \mathcal{N})$ and $Y(\mathcal{M}, \mathcal{N})$, such that $K = X(\mathcal{M}, \mathcal{N}) \cup Y(\mathcal{M}, \mathcal{N})$ and $\bigcup \mathcal{N} \cap X(\mathcal{M}, \mathcal{N}) = \bigcup \mathcal{M} \cap Y(\mathcal{M}, \mathcal{N}) = \emptyset$.

Observe that given $x \in C(K)$ and a closed subset L of K such that $osc(x, L) \geq \varepsilon$ for some $\varepsilon > 0$, there exists a good choice $(\mathcal{M}, \mathcal{N})$ of x on L of some type and we can split L up into $L \cap X(\mathcal{M}, \mathcal{N})$ and $L \cap Y(\mathcal{M}, \mathcal{N})$. Proposition 4.5.(iii)–(iv) enables us to prove next lemma which shows that the good choice of x on L is unique, if we fix its type, and that a *suitable rigidity condition* holds; from this it will be deduced a *rule* to decompose every closed set on which a continuous function has an oscillation not less than ε. Set

$$\mathcal{B}(L, \ell, m, i) = \{\mathcal{M} \subset \mathcal{I}^\ell(i) : \#\mathcal{M} = m, \ M \cap L \neq \emptyset \text{ for all } M \in \mathcal{M}\}.$$

Lemma 4.6 ([3]). *If $(\mathcal{M}, \mathcal{N})$ is a good choice of $x \in C(K)$ on a closed subset L of K of type (ℓ, m, n, i, j) then* *(i)* $\sup\{\Phi(x, L, \mathcal{M}') : \mathcal{M}' \in \mathcal{B}(L, \ell, m, i),\ \mathcal{M}' \neq \mathcal{M}\} < \Phi(x, L, \mathcal{M})$;

(ii) $\inf\{\Psi(x, L, \mathcal{N}') : \mathcal{N}' \in \mathcal{B}(L, \ell, n, j),\ \mathcal{N}' \neq \mathcal{N}\} > \Psi(x, L, \mathcal{N})$.

Proof. If $\mathcal{M}' \in \mathcal{B}(L, \ell, m, i)$ and $\mathcal{M}' \neq \mathcal{M}$ then there exists $M_0 \in \mathcal{M}' \setminus \mathcal{M} \subset \mathcal{I}^\ell(i) \setminus \mathcal{M}$ so that

$$\Phi(x, L, \mathcal{M}') = \frac{1}{m}\left(\sum_{M \in \mathcal{M}' \setminus \{M_0\}} \sup x_{\restriction_{L \cap \overline{M}}}\right) + \frac{1}{m} \sup x_{\restriction_{L \cap \overline{M_0}}}$$

$$\leq \left(1 - \frac{1}{m}\right) \sup x_{\restriction_L} + \frac{1}{m} \sup x_{\restriction L \cap \overline{\bigcup(\mathcal{I}^\ell(i) \setminus \mathcal{M})}}.$$

The proof of (ii) is similar. ■

Given $x \in C(K)$ with $osc(x, K) \geq \varepsilon$, we are going to iterate the above decomposition to get a covering $\mathcal{L}$ fulfilling the requirements of Theorem 1.1. Such a covering should be finite, so this iterative process is going to be defined in such a way that it finishes after a finite number of steps.

In order to cope with this requirement, fix a map τ from the nonnegative integers into $\mathbb{N}^5$ with the property that, for every $(\ell, m, n, i, j) \in \mathbb{N}^5$, the set $\tau^{-1}(\ell, m, n, i, j)$ is infinite. Let $\mathcal{S} = \{-1, 0, 1\}^{<\omega}$ be the set of all finite sequences of integers $s = (i_1, \ldots, i_n)$, where $i_k \in \{-1, 0, 1\}$ for $1 \leq k \leq n$; this n is called the *length* $|s|$ of s. We agree that the *empty* sequence $s = (\cdot)$ belongs to $\mathcal{S}$ and has length zero. If $s \in \mathcal{S}$ and $i \in \{-1, 0, 1\}$ we write (s, i) for the element of $\mathcal{S}$ which extends s and has i in its last place. $\mathcal{F}(K)$ will stand for the set of all closed subsets of K.

Proposition 4.7 ([3]). *Given $x \in C(K)$ and $\varepsilon > 0$ there exists a finite subset $\Upsilon \subset \mathcal{F}(K) \times \mathcal{S}$ and a tree order on Υ with the following properties*

(a) the unique minimal element of Υ is $(K, (\cdot))$;

(b) an element (L, s) is maximal in Υ if, and only if, $osc(x, L) < \varepsilon$;

(c) if (L, s) is not maximal in Υ and $|s| = n$ with $n \geq 0$ then there are two possibilities

- *(i) there exists a good choice $(\mathcal{M}, \mathcal{N})$ of x of type $\tau(n)$ on L. Then the immediate successors of (L, s) in Υ are $(L \cap X(\mathcal{M}, \mathcal{N}), (s, 0))$ and $(L \cap Y(\mathcal{M}, \mathcal{N}), (s, 1))$;*
- *(ii) no good choice of x of type $\tau(n)$ exists on L. In this case, the unique immediate successor of (L, s) in Υ is $(L, (s, -1))$.*

Moreover, the family

$$\mathcal{L} = \{L \in \mathcal{F}(K) : (L, s) \text{ is a maximal node of } \Upsilon \text{ for some } s \in \mathcal{S}\}$$

is a finite covering of K satisfying $osc(x, L) < \varepsilon$ for all $L \in \mathcal{L}$.

Proof. Conditions *(a)–(c)* define a tree Υ, we claim that it has no infinite branches. Indeed, otherwise there exists $\sigma \in \{-1, 0, 1\}^\omega$ and there is a sequence $\{(L_k, \sigma_{\restriction k})\}_{k \geq 0}$ in Υ such that $(L_{k+1}, \sigma_{\restriction k+1})$ is an immediate successor of $(L_k, \sigma_{\restriction k})$ for all $k \geq 0$. Since $(L_k)_k$ is a decreasing sequence of closed sets with the property that $osc(x, L_k) \geq \varepsilon$ for all $k \geq 0$, it follows that $L = \bigcap_{k \geq 0} L_k$ is a nonempty closed set satisfying $osc(x, L) \geq \varepsilon$. From Proposition 4.5 there exists $(\mathcal{M}, \mathcal{N})$ a good choice of x of type (ℓ, m, n, i, j) on L for some $\ell, m, n, i, j \in \mathbb{N}$. By compactness there must exist k_0 such that $(\mathcal{M}, \mathcal{N})$ is a good choice on L_k for all $k \geq k_0$. As $\tau^{-1}(\ell, m, n, i, j)$ is infinite there is some $k \geq k_0$ such that $\tau(k) = (\ell, m, n, i, j)$. By construction, either $L_{k+1} = L_k \cap X(\mathcal{M}, \mathcal{N})$ or $L_{k+1} = L_k \cap Y(\mathcal{M}, \mathcal{N})$. Therefore, either $\bigcup \mathcal{M} \cap L_{k+1} = \emptyset$ or $\bigcup \mathcal{N} \cap L_{k+1} = \emptyset$, contradicting that $\bigcup \mathcal{M} \cap L \neq \emptyset \neq \bigcup \mathcal{N} \cap L$.

According to König's Lemma, Υ is a finite tree. Consequently, if $\mathcal{L}$ is the family of all sets L for which (L, s) is maximal in Υ for some $s \in \mathcal{S}$, we have that $\mathcal{L}$ is finite. Moreover from the choice of Υ it follows that $\mathcal{L}$ is a finite covering of K such that for every $L \in \mathcal{L}$ we have $osc(x, L) < \varepsilon$. ■

Now we are ready to prove the main result about Namioka–Phelps compacta.

Theorem 4.8 ([3])**.** *Let K be a Namioka–Phelps compact space. Then $C(K)$ admits an equivalent pointwise lower semicontinuous **LUR** norm.*

Proof. We divide the proof of this theorem into three steps. Given $\varepsilon > 0$ we begin by decomposing the unit ball of $C(K)$ into countably many sets $\{C_n : n \in \mathbb{N}\}$ in such a way that for every $n \in \mathbb{N}$ the set C_n codifies the *countable* information relative to the tree Υ_x associated to each ε and $x \in C_n$ according to Proposition 4.7. In the second step, for every $n \in \mathbb{N}$ and every $x \in C_n$ we define a family of maps $\Phi(x)$ associated to x and we prove that $\Phi(x)$ fulfils the hypothesis of Lemma 3.1. Finally, we deduce that for every $n \in \mathbb{N}$ and every $x \in C_n$ there is a pointwise open half space H containing x such that $\Upsilon_y = \Upsilon_x$ for every $y \in H \cap C_n$. According to Proposition 4.7, the statement follows from Theorem 1.1.

Let $\varepsilon > 0$. We write $T(\mathcal{S})$ for the countable family of all finite trees $(\Upsilon, \sqsubseteq)$ in $\mathcal{S}$, where $\sqsubseteq$ is the end–extension order, with the property that every $\Upsilon \in T(\mathcal{S})$ has one minimal element and the set s^+ of the immediate successors of each s of Υ has at most two elements. Given $x \in B_{C(K)}$ let Υ_x be the tree associated to x and ε by Proposition 4.7. We denote by $P(\Upsilon_x)$ the tree made up by all $s \in \mathcal{S}$ for which there exists L such that $(L, s) \in \Upsilon_x$, with the order induced by $\mathcal{S}$. If $\Upsilon \in T(\mathcal{S})$ let $C_\Upsilon = \{x \in B_{C(K)} : P(\Upsilon_x) = \Upsilon\}$.

Fix $\Upsilon \in T(\mathcal{S})$. We assume that $\#\Upsilon > 1$, otherwise $osc(x, K) < \varepsilon$ for every $x \in C_\Upsilon$. For every $s \in \Upsilon$ and every $x \in C_\Upsilon$ we write $L_{x,s}$ for the closed subset of K such that $(L_{x,s}, s) \in \Upsilon_x$. If $\Upsilon_2 = \{s \in \Upsilon : \#s^+ = 2\}$ then for every $s \in \Upsilon_2$ and every $x \in C_\Upsilon$ there exists a good choice $(\mathcal{M}_{x,s}, \mathcal{N}_{x,s})$ of x on $L_{x,s}$ of type $\tau(|s|)$ such that

$$L_{x,(s,0)} = L_{x,s} \cap X(\mathcal{M}_{x,s}, \mathcal{N}_{x,s}) \quad \text{and} \quad L_{x,(s,1)} = L_{x,s} \cap Y(\mathcal{M}_{x,s}, \mathcal{N}_{x,s}). \tag{11}$$

If $\tau(|s|) = (\ell_s, m_s, n_s, i_s, j_s)$ we let

$$\begin{aligned}\alpha_s(x) &= \sup\{\Phi(x, L_{x,s}, \mathcal{M}) : \mathcal{M} \in \mathcal{B}(L_{x,s}, \ell_s, m_s, i_s),\ \mathcal{M} \neq \mathcal{M}_{x,s}\}\,;\\ \beta_s(x) &= \inf\{\Psi(x, L_{x,s}, \mathcal{N}) : \mathcal{N} \in \mathcal{B}(L_{x,s}, \ell_s, n_s, j_s),\ \mathcal{N} \neq \mathcal{N}_{x,s}\}\,.\end{aligned}$$

From Lemma 4.6 it follows that $\Phi(x, L_{x,s}, \mathcal{M}_{x,s}) > \alpha_s(x)$ and $\Psi(x, L_{x,s}, \mathcal{N}_{x,s}) < \beta_s(x)$ for every $s \in \Upsilon_2$ and every $x \in C_\Upsilon$. Hence, for every $r \in \mathbb{N}$ and every family $\mathcal{U} = \{(U_s, V_s) : s \in \Upsilon_2\}$ of pairs (U_s, V_s) of open real intervals with rational end points and length equal to $(1/12r)^{2\#\Upsilon_2}$, let $C_{\Upsilon,r,\mathcal{U}}$ be the set of all $x \in C_\Upsilon$ such that $\Phi(x, L_{x,s}, \mathcal{M}_{x,s}) \in U_s$ and $\Psi(x, L_{x,s}, \mathcal{N}_{x,s}) \in V_s$ for every $s \in \Upsilon_2$ and

$$r^{-1} \leq \min_{s\in\Upsilon_2}\{\Phi(x, L_{x,s}, \mathcal{M}_{x,s}) - \alpha_s(x),\ \beta_s(x) - \Psi(x, L_{x,s}, \mathcal{N}_{x,s})\}.$$

It is clear that $B_{C(K)}$ is the (countable) union of the sets $C_{\Upsilon,r,\mathcal{U}}$.

Fix $\Upsilon, r, \mathcal{U}$ and $x \in C_{\Upsilon,r,\mathcal{U}}$. For every $s \in \Upsilon_2$ and every $i \in \{0,1\}$ we define the map $\varphi_s^i : B_{C(K)} \longrightarrow \mathbb{R}$ by

$$\varphi_s^i(y) = \begin{cases} \Phi(y, L_{x,s}, \mathcal{M}_{x,s})/2, & \text{if } i = 0;\\ -\Psi(y, L_{x,s}, \mathcal{N}_{x,s})/2, & \text{if } i = 1.\end{cases}$$

and fix the values $\theta = 1/3r$ and $\delta = (1/12r)^{2\#\Upsilon_2}$. If we write $\Phi(x)$ for the collection $\{\varphi_s^i : s \in \Upsilon_2,\ i = 0,1\}$ then $\Phi(x)$ is a family of convex and pointwise lower semicontinuous maps satisfying $osc(\varphi, B_{C(K)}) \leq 1$ for every $\varphi \in \Phi(x)$. The following result yields information about setting an order on $\Phi(x)$ to apply Lemma 3.1.

Lemma 4.9. *Let $y \in C_{\Upsilon,r,\mathcal{U}}$ be such that $L_{y,s} = L_{x,s}$ for some $s \in \Upsilon_2$. If $i \in \{0,1\}$ then*

$$\varphi_s^i(x) > \varphi_s^i(y) - \delta \text{ and } \varphi_s^i(x) - \varphi_s^i(y) \notin [\delta, \theta). \tag{12}$$

Moreover,

$$\text{if } \varphi_s^i(x) - \varphi_s^i(y) < \delta \text{ for } i = 0,1 \text{ then } L_{y,(s,0)} = L_{x,(s,0)} \text{ and } L_{y,(s,1)} = L_{x,(s,1)}. \tag{13}$$

Proof. Since $L_{y,s} = L_{x,s}$ we have $\mathcal{M}_{x,s} \in \mathcal{B}(L_{y,s}, \ell_s, m_s, i_s)$ and $\mathcal{N}_{x,s} \in \mathcal{B}(L_{y,s}, \ell_s, n_s, j_s)$. Moreover, $\Phi(x, L_{x,s}, \mathcal{M}_{x,s})$, $\Phi(y, L_{y,s}, \mathcal{M}_{y,s}) \in U_s$ and $\Psi(x, L_{x,s}, \mathcal{N}_{x,s})$, $\Psi(y, L_{y,s}, \mathcal{N}_{y,s}) \in V_s$, then

$$\begin{aligned}2\varphi_s^0(y) = \Phi(y, L_{y,s}, \mathcal{M}_{x,s}) &\leq \Phi(y, L_{y,s}, \mathcal{M}_{y,s})\\ &< \Phi(x, L_{x,s}, \mathcal{M}_{x,s}) + \text{length}(U_s) = 2\varphi_s^0(x) + \delta.\end{aligned}$$

Similarly we get $-2\varphi_s^1(y) > -2\varphi_s^1(x) - \delta$. Hence for each $i \in \{0,1\}$ we have $\varphi_s^i(x) > \varphi_s^i(y) - \delta$ and the first part of (12) follows.

To show the second part of (12) suppose that $\varphi_s^i(x) - \varphi_s^i(y) \in [\delta, \theta)$. Since $\varphi_s^0(x) - \varphi_s^0(y) < \theta$ we get

$$\Phi(y, L_{y,s}, \mathcal{M}_{x,s}) > \Phi(x, L_{x,s}, \mathcal{M}_{x,s}) - 2\theta > \Phi(y, L_{y,s}, \mathcal{M}_{y,s}) - \text{length}(U_s) - 2\theta >$$
$$> \Phi(y, L_{y,s}, \mathcal{M}_{y,s}) - 3\theta \geq \alpha_s(y)$$

which implies $\mathcal{M}_{x,s} = \mathcal{M}_{y,s}$ so that $\Phi(y, L_{y,s}, \mathcal{M}_{x,s}) \in U_s$. Similarly we get

$$\Psi(y, L_{y,s}, \mathcal{N}_{x,s}) < \Psi(y, L_{y,s}, \mathcal{N}_{y,s}) + 3\theta \leq \beta_s(y) \text{ and } \mathcal{N}_{x,s} = \mathcal{N}_{y,s}$$

so that $\Psi(y, L_{y,s}, \mathcal{N}_{x,s}) \in V_s$. Nevertheless the inequalities $\varphi_s^i(x) - \varphi_s^i(y) \geq \delta$ for $i = 0, 1$ imply $\Phi(y, L_{y,s}, \mathcal{M}_{x,s}) \leq \Phi(x, L_{x,s}, \mathcal{M}_{x,s}) - 2\delta < \inf U_s$ and $\Psi(y, L_{y,s}, \mathcal{N}_{x,s}) \geq \Psi(x, L_{x,s}, \mathcal{N}_{x,s}) + 2\delta > \sup V_s$, a contradiction which shows the second part of (12).

Finally, suppose that $\varphi_s^i(x) - \varphi_s^i(y) < \delta$ for every $i \in \{0, 1\}$. Then $\varphi_s^i(x) - \varphi_s^i(y) < \theta$ and we get $(\mathcal{M}_{y,s}, \mathcal{N}_{y,s}) = (\mathcal{M}_{x,s}, \mathcal{N}_{x,s})$ as above. Hence $X(\mathcal{M}_{y,s}, \mathcal{N}_{y,s}) = X(\mathcal{M}_{x,s}, \mathcal{N}_{x,s})$ and $Y(\mathcal{M}_{y,s}, \mathcal{N}_{y,s}) = Y(\mathcal{M}_{x,s}, \mathcal{N}_{x,s})$ so (13) follows from (11). ■

Let us turn into the proof of Theorem 4.8. To enumerate the family $\Phi(x)$ we introduce an order $\prec$ as follows. Given two distinct maps $\varphi_s^i, \varphi_t^j \in \Phi(x)$ we write $\varphi_s^i \prec \varphi_t^j$ if, and only if, either $|s| < |t|$, or $|s| = |t|$ and $s <_{\text{lex}} t$, where $<_{\text{lex}}$ is the lexicographic order, or $s = t$ and $i < j$. Then we write $\Phi(x)$ as $\{\varphi_k : 1 \leq k \leq 2\#\Upsilon_2\}$ where $\varphi_k \prec \varphi_\ell$ if, and only if, $1 \leq k < \ell \leq 2\,\#\Upsilon_2$. Note that for every $y \in C_{\Upsilon,r,\mathcal{U}}$ we have $L_{y,(\cdot)} = L_{x,(\cdot)} = K$ and if $L_{y,s} = L_{x,s}$ for some $s \in \Upsilon$ with $\#s^+ = 1$ then it follows that $L_{y,(s,-1)} = L_{y,s} = L_{x,s} = L_{x,(s,-1)}$. If for some $n \leq 2\,\#\Upsilon_2$ there is $y \in C_{\Upsilon,r,\mathcal{U}}$ such that $\varphi_j(x) - \varphi_j(y) < \delta$ for all $j < n$, according to Lemma 4.9 we get $\varphi_n(x) > \varphi_n(y) - \delta$ and $\varphi_n(x) - \varphi_n(y) \notin [\delta, \theta)$. Since $0 < 4\delta^{\frac{1}{2\#\Upsilon_2}} \leq \theta < 1$, we can apply Lemma 3.1 to the family $\Phi(x)$ with $A_0 = C_{\Upsilon,r,\mathcal{U}}$ and $B = B_{C(K)}$. Then there exists a pointwise open half space H containing x such that $\Upsilon_y = \Upsilon_x$ for every $y \in H \cap C_{\Upsilon,r,\mathcal{U}}$. From Proposition 4.7 we have that the family $\mathcal{L}$ of the projections of all maximal elements of Υ_x into $\mathcal{F}(K)$ is a finite covering of K satisfying $osc(y, L) < \varepsilon$ for every $y \in H \cap C_{\Upsilon,r,\mathcal{U}}$ and every $L \in \mathcal{L}$. According to Theorem 1.1, $C(K)$ admits an equivalent pointwise lower semicontinuous **LUR** norm. ■

Remark. A compact Hausdorff space is said to be *descriptive* if its topology has a $\sigma-$isolated network. If $(K, \mathcal{T})$ is descriptive then there exists a metric ρ_K on K such that the metric topology induced by ρ_K has a $\mathcal{T} - \sigma-$isolated network. Thus, a proof similar to that of Theorem 4.8 shows that for any descriptive compact space K the linear subspace of all continuous functions on K which are ρ_K-uniformly continuous admits an equivalent pointwise lower semicontinuous **LUR** norm [3].

4.2 A class of Rosenthal compacta

In what follows we denote by Γ a Polish space, i.e. a separable complete metric space. Let K be a separable and pointwise compact set of functions on Γ, assume further that each function $s \in K$ has only countably many discontinuities. It is clear that every

s in K is a Baire–1 function, so K is a Rosenthal compact [12]. For such subclass of Rosenthal compacta K it has been proved in [6] that $C(K)$ admits a pointwise lower semicontinuous **LUR** equivalent norm. Using some ideas of [6] we are going to deduce the existence of such renorming from Corollary 1.2 *describing* the controlling coordinates of the functions in $C(K)$, unlike Section 4.1 where Theorem 1.1 played the key role.

In what follows Q stands for a countable dense subset of Γ. As in Section 1 we assume that $K \subset [0,1]^\Gamma$. If $m \in \mathbb{N}$, R and S are subsets of Q and $\Gamma \setminus Q$ respectively, let

$$I(R,S,m) = \left\{(s,t) \in K \times K : \|(s-t)\restriction_R\|_\infty \leq (4m)^{-1},\ \|(s-t)\restriction_S\|_\infty \leq m^{-1}\right\}.$$

Fix $\varepsilon > 0$. The uniform continuity of every $x \in C(K)$ yields $m(x) \in \mathbb{N}$ and a finite subset F of Γ such that

$$|x(s) - x(t)| < \varepsilon \text{ whenever } s,t \in K \text{ and } \sup\{|s(\gamma) - t(\gamma)| : \gamma \in F\} \leq m(x)^{-1}.$$

If $S = F \cap (\Gamma \setminus Q)$ it is clear that $|x(s) - x(t)| < \varepsilon$ whenever $(s,t) \in I(Q,S,m(x))$. Furthermore we can associate to x a finite subset $S(x)$ of $\Gamma \setminus Q$ of minimal cardinality satisfying

$$(14) \qquad (s,t) \in I(Q,S(x),m(x)) \implies |x(s) - x(t)| < \varepsilon.$$

Since $I(Q,S(x),m(x))$ is pointwise compact, there must exist $p(x) \in \mathbb{N}$ such that $|x(s) - x(t)| \leq \varepsilon - p(x)^{-1}$ whenever $(s,t) \in I(Q,S(x),m(x))$. We claim that there exists a finite subset $R(x)$ of Q such that

$$(15) \qquad (s,t) \in I(R(x),S(x),m(x)) \implies |x(s) - x(t)| < \varepsilon.$$

Indeed, otherwise for every finite subset R of Q there is $(s_R,t_R) \in I(R,S(x),m(x))$ such that $|x(s_R) - x(t_R)| \geq \varepsilon$. By compactness, we can choose a cluster point (s,t) of the net $\{(s_R,t_R)\}_{R \in [Q]^{<\omega}}$ in $K \times K$. It is easy to check that $(s,t) \in I(Q,S(x),m(x))$ but applying continuity we get $|x(s) - x(t)| \geq \varepsilon$, a contradiction with (14) which proves our claim.

From (15) it follows that $R(x) \cup S(x)$ ε–controls x with $1/4m(x)$ so, in order to apply Corollary 1.2, we are going to split $C(K)$ up into countably many subsets and, fixed one of these subsets, to *describe* the set $S(x)$ for each x in it. Given $s \in K$ and $\delta > 0$ let

$$J(s,\delta) = \{\gamma \in \Gamma : osc(s,U) > \delta \text{ whenever } U \text{ is open and } \gamma \in U\}.$$

Each $J(s,\delta)$ is a countable closed subset of Γ, hence a scattered topological space. By means of arguments of Descriptive Set Theory, in [6, Theorem 3] it is proved that there exists a countable ordinal Ω such that for all $s \in K$ and all $\delta > 0$ the Ω^{th} derived set $J(s,\delta)^{(\Omega)}$ is empty; fix such Ω. Given $s,t \in K$ and $m \in \mathbb{N}$ we write $J(s,t,m) = J(s,1/4m) \cup J(t,1/4m)$. It is easily checked that $J(s,t,m)^{(\Omega)} = \emptyset$ for all $s,t \in K$ and all $m \in \mathbb{N}$. The proof of the lemma below can be found in [6], we include it here for the sake of completeness.

Lemma 4.10. ([6, Lemma 5]) *For every $x \in C(K)$ and every proper subset F of $S(x)$ the set $U(x,F) = \{(s,t) \in I(Q,F,m(x)) : |x(s) - x(t)| > \varepsilon - p(x)^{-1}\}$ is nonempty. Moreover, there exists an ordinal $\xi(x,F) < \Omega$ such that*

(i) $S(x) \cap J(s,t,m(x))^{(\xi(x,F))} \setminus F \neq \emptyset$ *for all* $(s,t) \in U(x,F)$*;*

(ii) *there is* $(s,t) \in U(x,F)$ *such that* $S(x) \cap J(s,t,m(x))^{(\xi(x,F)+1)} \setminus F = \emptyset$*.*

Proof. Since $S(x)$ is minimal the set $U(x,F)$ must be nonempty. For simplicity we write m instead of $m(x)$. By the choice of $S(x)$, given $(s,t) \in U(x,F)$ there is $\gamma \in S(x) \setminus F$ such that $|s(\gamma) - t(\gamma)| > m^{-1}$. We claim that any such γ belongs to $J(s,t,m)$. Indeed, if $\gamma \notin J(s,t,m)$ then there must exist an open set $U \subset \Gamma$ containing γ such that $|s(\alpha) - s(\gamma)| \leq (4m)^{-1}$ and $|t(\alpha) - t(\gamma)| \leq (4m)^{-1}$ for every $\alpha \in U$. By density of Q we can choose some $\alpha \in Q \cap U$. Since $(s,t) \in I(Q,F,m)$, it follows that $|s(\alpha) - t(\alpha)| \leq (4m)^{-1}$ and applying the triangle inequality we get $|s(\gamma) - t(\gamma)| \leq 3/4m$, a contradiction. Hence, for every $(s,t) \in U(x,F)$ the set $S(x) \cap J(s,t,m) \setminus F$ is nonempty. Since $S(x)$ is finite, for every $(s,t) \in U(x,F)$ there is an unique ordinal $\xi(s,t) < \Omega$ such that $S(x) \cap J(s,t,m)^{(\xi(s,t))} \setminus F \neq \emptyset$ and $S(x) \cap J(s,t,m)^{(\xi(s,t)+1)} \setminus F = \emptyset$. If $\xi(x,F) = \min\{\xi(s,t) : (s,t) \in U(x,F)\}$ then *(i)* and *(ii)* hold. ■

Lemma 4.10 gives some information to *describe* the coordinates of $S(x)$, in fact, it shows that some of them belong to $J(s,t,m)^{(\xi)}$ for some $(s,t) \in K \times K$, $m \in \mathbb{N}$ and $\xi < \Omega$. From now on we will *codify* the *new countable information* about the controlling coordinates of the functions by taking countable decompositions of $C(K)$. Indeed, for $R \in [Q]^{<\omega}$, $m, p \in \mathbb{N}$ and $n \geq 0$ let $C^R_{m,n,p}$ be the set of all $x \in B_{C(K)}$ for which $p(x) = p$, $\#S(x) = n$ and (15) holds with $R(x) = R$ and $m(x) = m$. Given $n \geq 1$, $x \in C^R_{m,n,p}$ and a proper subset F of $S(x)$ let $\xi(x,F)$, or ξ for simplicity, as in Lemma 4.10. We will focus on the coordinates of $S(x) \setminus F$ which are in $J(s,t,m)^{(\xi)}$ for a pair (s,t) such that $J(s,t,m)^{(\xi+1)} \cap S(x) \setminus F = \emptyset$ so we introduce the set $I(x,F)$ below. To fix the minimum number of coordinates we may find there, consider $j(x,F)$ below. Therefore set

$$\begin{aligned} I(x,F) &= \{(s,t) \in U(x,F) :\ S(x) \cap J(s,t,m)^{(\xi+1)} \setminus F = \emptyset\}; \\ j(x,F) &= \min\{\#\left(S(x) \cap J(s,t,m)^{(\xi)} \setminus F\right) :\ (s,t) \in I(x,F)\}; \\ V(x,F) &= \{(s,t) \in I(x,F) :\ \#\left(S(x) \cap J(s,t,m)^{(\xi)} \setminus F\right) = j(x,F)\}; \\ \mathcal{H}(x,F) &= \{S(x) \cap J(s,t,m)^{(\xi)} \setminus F :\ (s,t) \in V(x,F)\}. \end{aligned}$$

According to Lemma 4.10 we have $I(x,F) \neq \emptyset$ and $j(x,F) \geq 1$ so $\mathcal{H}(x,F) \neq \emptyset$. For every $H \in \mathcal{H}(x,F)$ let

$$\alpha(x,F,H) = \sup\left\{|x(s) - x(t)| :\ (s,t) \in V(x,F),\ H = S(x) \cap J(s,t,m)^{(\xi)} \setminus F\right\}.$$

Since $|x(s) - x(t)| > \varepsilon - p^{-1}$ whenever $(s,t) \in V(x,F)$, it follows that

$$\alpha(x,F,H) > \varepsilon - p^{-1} \text{ for every } H \in \mathcal{H}(x,F). \tag{16}$$

Then if we let

$$D(x) = \{\alpha(x,F,H) : F \text{ is a proper subset of } S(x),\ H \in \mathcal{H}(x,F)\} \cup \{\varepsilon - p^{-1}\}$$

there exists $i(x) \in \mathbb{N}$ such that

$$\min\{|a-b| : a,b \in D(x),\ a \neq b\} > i(x)^{-1}. \tag{17}$$

For $i \in \mathbb{N}$ and $n \geq 1$ let $C^R_{m,n,p,i}$ be the set of all $x \in C^R_{m,n,p}$ such that $i(x) = i$. It is clear that $B_{C(K)}$ is the (countable) union of the sets $C^R_{m,0,p}$ and $C^R_{m,n,p,i}$.

In the proposition below, after some new countable decompositions, we will define some families of functions that will allow us to apply Lemma 3.1. Conditions (I) and (IId) will allow us to describe inductively $S(x)$, and (IIa)–(IIc) below to apply Lemma 3.1.

Proposition 4.11. *Given $m,n,p,i \in \mathbb{N}$ and $R \in [Q]^{<\omega}$ let $\delta = (40i)^{-2^n}$ and $\theta = (10i)^{-1}$. Then for every $k \geq 0$ there is a decomposition $C^R_{m,n,p,i} = \bigcup_{\ell=1}^{+\infty} B^k_\ell$ in such a way that*

- *(I) for every $\ell \geq 1$ and every $x \in B^k_\ell$ there is a subset $F_k(x)$ of $S(x)$ such that $\#F_k(x) \geq \min\{k, \#S(x)\}$;*
- *(II) if $k \geq 1$, for every $\ell \geq 1$ and every $x \in B^k_\ell$ there are $r \geq 1$ and a family $\{\varphi_j\}_{j=1}^r$ of convex and pointwise lower semicontinuous maps, $\varphi_j : B_{C(K)} \longrightarrow [0,+\infty)$, $1 \leq j \leq r$, such that*
  - *(a) $osc(\varphi_j, B_{C(K)}) \leq 1$ for all $1 \leq j \leq r$;*
  - *(b) if $y \in B^k_\ell$ and $F_{k-1}(y) = F_{k-1}(x)$ then $\varphi_j(x) > \varphi_j(y) - \delta$ for all $1 \leq j \leq r$;*
  - *(c) $\{y \in B^k_\ell : F_{k-1}(y) = F_{k-1}(x),\ \delta \leq \varphi_j(x) - \varphi_j(y) < \theta\} = \emptyset$ for all $1 \leq j \leq r$;*
  - *(d) if $y \in B^k_\ell$, $F_{k-1}(y) = F_{k-1}(x)$ and $\varphi_j(x) - \varphi_j(y) < \delta$ for all $1 \leq j \leq r$ then $F_k(y) = F_k(x)$.*

Proof. We proceed by induction on $k \geq 0$. For $k = 0$ set $B^0_\ell = C^R_{m,n,p,i}$ for every $\ell \geq 1$, take $F_0(x) = \emptyset$ for every $x \in C^R_{m,n,p,i}$. Suppose that for some $k \geq 0$ there is a decomposition $C^R_{m,n,p,i} = \bigcup_{\ell=1}^{+\infty} B^k_\ell$ satisfying *(I)* and *(II)*. To complete the induction it suffices to split up $B^k_\ell = \bigcup_{s=1}^{+\infty} B^{k+1}_s$ for every $\ell \geq 1$ in such a way that *(I)* and *(II)* hold for every $s \geq 1$ and every $x \in B^{k+1}_s$. To this end fix $\ell \geq 1$, let $B_0 = \{x \in B^k_\ell : F_k(x) = S(x)\}$ and suppose $B^k_\ell \setminus B_0 \neq \emptyset$. Given $x \in B^k_\ell \setminus B_0$, $F_k(x)$ is a proper subset of $S(x)$ so choose $\xi\,(x, F_k(x))$ satisfying (i) and (ii) from Lemma 4.10. For every ordinal $\xi < \Omega$ and every $j \in \mathbb{N}$ let $B_{\xi,j}$ be the set of all $x \in B^k_\ell \setminus B_0$ such that $\xi(x, F_k(x)) = \xi$ and $j(x, F_k(x)) = j$. It is clear that $B^k_\ell \setminus B_0$ is the union of the sets $B_{\xi,j}$. We need the following lemma to codify the new countable information.

Lemma 4.12. *Given $\xi < \Omega$, $j \in \mathbb{N}$ and $x \in B_{\xi,j}$ let $\mathcal{I}(\xi, j, x)$ be the family of all $H \in \mathcal{H}(x, F_k(x))$ such that $\alpha(x, F_k(x), H) = \sup\{|x(s) - x(t)| : (s,t) \in V(x, F_k(x))\}$. Then there is a pair of open real intervals (L, M) with rational end points such that*

(18) $L \subset M,\ \sup L = \sup M,\ \inf M > \varepsilon - p^{-1},$ *length* $L = \delta,$ *length* $M \geq (2i)^{-1};$

(19) $\alpha(x, F_k(x), H) \in L$ *whenever* $H \in \mathcal{I}(\xi, j, x)$, *and*

(20) $\alpha(x, F_k(x), H) < \inf M$ *when* $H \in \mathcal{H}(x, F_k(x)) \setminus \mathcal{I}(\xi, j, x)$.

Moreover, if $|x(s) - x(t)| \in M$ for some $(s,t) \in V(x, F_k(x))$ then there exists $(s', t') \in V(x, F_k(x))$ such that $|x(s') - x(t')| \in L$ and

(21) $$S(x) \cap J(s', t', m)^{(\xi)} \setminus F_k(x) = S(x) \cap J(s, t, m)^{(\xi)} \setminus F_k(x).$$

Proof. Let $\{M_H : H \in \mathcal{H}(x, F_k(x))\}$ be a family of pairwise disjoint open real intervals with rational end points such that $\alpha(x, F_k(x), H) \in M_H$ for every $H \in \mathcal{H}(x, F_k(x))$ and with the property that $M_H = M_{H'}$ in case $H, H' \in \mathcal{H}(x, F_k(x))$ and $\alpha(x, F_k(x), H) = \alpha(x, F_k(x), H')$. From (16) and (17) it follows that for every $H \in \mathcal{H}(x, F_k(x))$ we can assume that $\inf M_H > \varepsilon - p^{-1}$ and that length $M_H \geq (2i)^{-1}$, moreover we can take L_H to be an open real interval with rational end points satisfying $L_H \subset M_H$, $\sup L_H = \sup M_H$, length $L_H = \delta$ and $\alpha(x, F_k(x), H) \in L_H$. If (L, M) is the pair (L_H, M_H) corresponding to any $H \in \mathcal{I}(\xi, j, x)$ then (18)–(20) are fulfilled.

Suppose that $|x(s) - x(t)| \in M$ for some $(s,t) \in V(x, F_k(x))$ and let $H = S(x) \cap J(s,t,m)^{(\xi)} \setminus F_k(x)$. Since $H \in \mathcal{H}(x, F_k(x))$, by the choice of M we have $\inf M_H \leq \inf M < |x(s) - x(t)| \leq \alpha(x, F_k(x), H) < \sup M_H$ so $M = M_H$. Then $\alpha(x, F_k(x), H) \in L$ and there must exist $(s', t') \in V(x, F_k(x))$ such that $|x(s') - x(t')| \in L$ and $H = S(x) \cap J(s', t', m)^{(\xi)} \setminus F_k(x)$. □

Let us turn into the proof of Proposition 4.11. Given $\xi < \Omega$, $j \in \mathbb{N}$, a real interval I and $x \in B_{\xi,j}$ let $\mathcal{H}(x, F_k(x), I) = \{H \in \mathcal{H}(x, F_k(x)) : \alpha(x, F_k(x), H) \in I\}$. For every pair of open real intervals (L, M) with rational end points satisfying (18) and for every $r \in \mathbb{N}$ we write $B_{\xi,j,L,M,r}$ for the set of all $x \in B_{\xi,j}$ with $\#\mathcal{H}(x, F_k(x), L) = r$ such that (18)–(21) hold for x, L and M. It is clear that each $B_{\xi,j}$ is the union of the sets $B_{\xi,j,L,M,r}$.

Given ξ, j, L, M and r as above let $\mathcal{B}$ be a countable basis for the topology of Γ. From the choice of $B_{\xi,j,L,M,r}$ and $\mathcal{H}(x, F_k(x), L)$ we have that for every $x \in B_{\xi,j,L,M,r}$ and every $H \in \mathcal{H}(x, F_k(x), L)$ there exists $(s,t) \in V(x, F_k(x))$ such that $|x(s) - x(t)| \in L$ and $H = S(x) \cap J(s,t,m)^{(\xi)} \setminus F_k(x)$. Since $J(s,t,m)^{(\xi+1)}$ is closed and $S(x) \cap J(s,t,m)^{(\xi+1)} \setminus F_k(x) = \emptyset$, for every $\gamma \in S(x) \setminus F_k(x)$ we can choose $U_\gamma \in \mathcal{B}$ with $\gamma \in U_\gamma$ such that if $\mathcal{U} = \{U_\gamma\}_{\gamma \in S(x) \setminus F_k(x)}$ then

(22) $$J(s,t,m)^{(\xi+1)} \cap \bigcup \mathcal{U} = \emptyset \qquad \text{and} \qquad H = J(s,t,m)^{(\xi)} \cap \bigcup \mathcal{U}.$$

The second equality above shows that $\mathcal{U}$, a finite subset of the countable set $\mathcal{B}$, *codifies* which are the controlling coordinates of $S(x) \setminus F_k(x)$ lying in $J(s,t,m)^{(\xi)}$. Then for every $\mathcal{U} \in [\mathcal{B}]^{<\omega}$ we write $B^{\mathcal{U}}_{\xi,j,L,M,r}$ for the set of all $x \in B_{\xi,j,L,M,r}$ for which $S(x) \setminus$

$F_k(x) \subset \bigcup_{U\in\mathcal{U}} U$ and with the property that for each $H \in \mathcal{H}(x, F_k(x), L)$ there exists $(s,t) \in I(Q, F_k(x), m)$ such that the equalities of (22) hold and $|x(s) - x(t)| \in L$. It is clear that $B_{\xi,j,L,M,r}$ is the union of the sets $B^{\mathcal{U}}_{\xi,j,L,M,r}$.

Summarizing, we have written B^k_ℓ as the countable union of B_0 and the sets $B^{\mathcal{U}}_{\xi,j,L,M,r}$; the rest of the proof is devoted to show that this decomposition satisfies the requirements of Proposition 4.11. Indeed if $x \in B_0$ it is enough to take $F_{k+1}(x) = S(x)$ because (I) and (II) trivially hold associating to x the zero function on $B_{C(K)}$. On the other hand, fix ξ, j, L, M, r and $\mathcal{U}$ as above, if $x \in B^{\mathcal{U}}_{\xi,j,L,M,r}$ take $F_{k+1}(x)$ as the union of $F_k(x)$ and $\bigcup\{H : H \in \mathcal{H}(x, F_k(x), L)\}$, it is clear that (I) holds. To show (II), for every $H \in \mathcal{H}(x, F_k(x), L)$ we write $P(x, H)$ for the set of all $(s, t) \in I(Q, F_k(x), m)$ such that (22) hold. Take $\varphi_H : B_{C(K)} \longrightarrow [0, \infty)$ defined by $\varphi_H(y) = (1/4) \sup\{|y(s) - y(t)| : (s,t) \in P(x, H)\}$. Condition *(IIa)* is clearly fulfilled by any enumeration of the family $\{\varphi_H : H \in \mathcal{H}(x, F_k(x), L)\}$, to obtain one for which *(IIb)–(IId)* hold, we will prove the following

Claim Given $y \in B^{\mathcal{U}}_{\xi,j,L,M,r}$ and $H \in \mathcal{H}(x, F_k(x), L)$ if $F_k(y) = F_k(x)$ we have $4\,\varphi_H(y) < \sup L$.

Proof. Pick $(s,t) \in P(x, H)$, since $F_k(y) = F_k(x)$ we have $(s, t) \in I(Q, F_k(y), m)$, from this and (18) we get $|y(s) - y(t)| < \inf L$ whenever $|y(s) - y(t)| \le \varepsilon - p^{-1}$; so we can suppose that $|y(s) - y(t)| > \varepsilon - p^{-1}$. Then $(s,t) \in U(y, F_k(y))$ and

$$\emptyset \neq (S(y) \setminus F_k(y)) \cap J(s,t,m)^{(\xi)} \subset \bigcup_{U\in\mathcal{U}} U \cap J(s,t,m)^{(\xi)} = H.$$

Moreover from the choice of $P(x, H)$ we have $\bigcup_{U\in\mathcal{U}} U \cap J(s,t,m)^{(\xi+1)} = \emptyset$. Since $S(y) \setminus F_k(y) \subset \bigcup_{U\in\mathcal{U}} U$, it follows that $(S(y) \setminus F_k(y)) \cap J(s,t,m)^{(\xi+1)} = \emptyset$. Furthermore, since $j \le \#(S(y) \setminus F_k(y)) \cap J(s,t,m)^{(\xi)} \le \#H = j$, we have $(s,t) \in V(y, F_k(y))$ and $H = (S(y) \setminus F_k(y)) \cap J(s,t,m)^{(\xi)} \in \mathcal{H}(y, F_k(y))$. Hence, $|y(s) - y(t)| \le \alpha(y, F_k(y), H)$ and the claim follows from (19) and (20). □

According to the above claim and the choice of $B^{\mathcal{U}}_{\xi,j,L,M,r}$ we have $4\,\varphi_H(x) \in L$ for every $H \in \mathcal{H}(x, F_k(x), L)$. Given $y \in B^{\mathcal{U}}_{\xi,j,L,M,r}$ such that $F_k(y) = F_k(x)$ from the claim it follows that $4\varphi_H(y) < \sup L$, so $4\varphi_H(x) > \sup L - \text{length}\, L > 4\varphi_H(y) - \delta$ and *(IIb)* follows.

To show *(IIc)*, suppose there are $H \in \mathcal{H}(x, F_k(x), L)$ and $y \in B^{\mathcal{U}}_{\xi,j,L,M,r}$ with $F_k(y) = F_k(x)$ such that $\varphi_H(x) - \varphi_H(y) \in [\delta, \theta)$. Since $\varphi_H(x) - \varphi_H(y) < \theta$, we get

$$\begin{aligned} (23) \quad 4\varphi_H(y) &> 4\varphi_H(x) - 4\theta > \sup L - \text{length}\, L - 4\theta = \sup M - \delta - 4\theta > \\ &> \sup M - 5\theta = \sup M - (2i)^{-1} \ge \sup M - \text{length}\, M = \inf M. \end{aligned}$$

The inequality above, (18) and the claim imply $4\varphi_H(y) \in M$, then there is $(s,t) \in P(x, H)$ such that $|y(s) - y(t)| \in M$. From this and (18) we get $|y(s) - y(t)| > \varepsilon - p^{-1}$. Bearing in mind this inequality, the proof of the claim yields $(s,t) \in V(y, F_k(y))$ and $H \in \mathcal{H}(y, F_k(y))$ where $H = (S(y) \setminus F_k(y)) \cap J(s,t,m)^{(\xi)}$. According to Lemma 4.12 there exists $(s', t') \in V(y, F_k(y))$ such that $H = (S(y) \setminus F_k(y)) \cap J(s', t', m)^{(\xi)}$ and

$|y(s') - y(t')| \in L$. Therefore, we have $\inf L < |y(s') - y(t')| \leq \alpha(y, F_k(y), H)$ that together with (19) and (20) imply $H \in \mathcal{H}(y, F_k(y), L)$. From the choice of $B^{\mathcal{U}}_{\xi,j,L,M,r}$ we deduce that there is $(s'', t'') \in P(y, H)$ such that $|y(s'') - y(t'')| \in L$ so, taking in mind that $P(x, H) = P(y, H)$, we get $4\varphi_H(y) \in L$. However, the inequality $\varphi_H(x) - \varphi_H(y) \geq \delta$ implies $4\varphi_H(y) \leq 4\varphi_H(x) - 4\delta < \inf L$, a contradiction which proves *(IIc)*.

To show *(IId)* let $y \in B^{\mathcal{U}}_{\xi,j,L,M,r}$ with $F_k(y) = F_k(x)$ such that $\varphi_H(x) - \varphi_H(y) < \delta$ for every $H \in \mathcal{H}(x, F_k(x), L)$. Arguing as in (23) we get that $\mathcal{H}(x, F_k(x), L)$ is included in $\mathcal{H}(y, F_k(y), L)$. Since both sets have the same cardinality r it follows that $\mathcal{H}(x, F_k(x), L) = \mathcal{H}(y, F_k(y), L)$. Hence, $F_{k+1}(y) = F_{k+1}(x)$ and *(IId)* follows. The proof of Proposition 4.11 is now complete. ■

Now we are ready to prove the following

Theorem 4.13. ([6, Theorem 1]) *Let Γ be a Polish space and let K be a separable and pointwise compact subset of functions on Γ with the property that each $s \in K$ has at most countably many discontinuities. Then $C(K)$ admits an equivalent pointwise lower semicontinuous **LUR** norm.*

Proof. The ball $B_{C(K)}$ has already been decomposed as the union of the sets $C^R_{m,0,p}$ and $C^R_{m,n,p,i}$ for $R \in [Q]^{<\omega}$ and $m, n, p, i \in \mathbb{N}$. From the choice of $C^R_{m,0,p}$ it follows that R $\varepsilon-$controls every $y \in C^R_{m,0,p}$ with $(4m)^{-1}$. On the other hand, given $R \in [Q]^{<\omega}$ and $m, n, p, i \in \mathbb{N}$ let B^k_ℓ and $\{\varphi_j\}_{j=1}^r$ as in Proposition 4.11. For every $\ell \geq 1$ and $k \in \mathbb{N}$ let $\mathcal{F}_{k,\ell}(x) = \{\varphi_j\}_{j=1}^r$.
For every $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{N}^n$ set

$$C^{R,\ell}_{m,n,p,i} = \left\{ x \in \bigcap_{k=1}^n B^k_{\ell_k} : B^k_{\ell_k} \subset C^R_{m,n,p,i} \text{ for all } 1 \leq k \leq n \right\}.$$

Then $C^R_{m,n,p,i} = \bigcup_{\ell \in \mathbb{N}^n} C^{R,\ell}_{m,n,p,i}$. We are going to see that each $C^{R,\ell}_{m,n,p,i}$ satisfies the requirement of Corollary 1.2.

Indeed, given $\ell \in \mathbb{N}^n$ and $x \in C^{R,\ell}_{m,n,p,i}$ let $\mathcal{F}(x) = \bigcup_{k=1}^n \mathcal{F}_{k,\ell_k}(x)$. To enumerate $\mathcal{F}(x)$ we introduce an order $\prec$ as follows. Given $\varphi_j \in \mathcal{F}_{k,\ell_k}(x)$, $\varphi_{j'} \in \mathcal{F}_{k',\ell_{k'}}(x)$ with $\varphi_j \neq \varphi_{j'}$ we write $\varphi_j \prec \varphi_{j'}$ if, and only if, $(k, j) <_{\text{lex}} (k', j')$, where $<_{\text{lex}}$ is the lexicographic order. If $N = \#\mathcal{F}(x)$ then we can write $\mathcal{F}(x)$ as $\{\varphi_k\}_{k=1}^N$ where $\varphi_k \prec \varphi_{k'}$ if, and only if, $k < k'$. If for some $k < N$ there is $y \in C^{R,\ell}_{m,n,p,i}$ such that $\varphi_j(x) - \varphi_j(y) < \delta$ for all $j < k$, Proposition 4.11 shows that $\varphi_k(x) > \varphi_k(y) - \delta$ and $\varphi_k(x) - \varphi_k(y) \notin [\delta, \theta)$. By the choice of δ and θ we have $0 < 4\delta^{1/N} \leq \theta < 1$, so applying Lemma 3.1 to $\mathcal{F}(x)$, with $A_0 = C^{R,\ell}_{m,n,p,i}$ and $B = B_{C(K)}$ we get a pointwise open half space H, containing x, such that

$$H \cap C^{R,\ell}_{m,n,p,i} \subset \left\{ y \in C^R_{m,n,p,i} : S(y) = S(x) \right\}.$$

From this and (15) we deduce that $R \cup S(x)$ $\varepsilon-$controls every $y \in H \cap C^{R,\ell}_{m,n,p,i}$ with $(4m)^{-1}$. According to Corollary 1.2 we conclude that $C(K)$ admits an equivalent pointwise lower semicontinuous **LUR** norm. ■

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J. F. Martínez, Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Valencia, Dr. Moliner 50, 46100 Burjassot, Valencia, Spain

E-mail address: `j.francisco.martinez@uv.es`

A. Moltó, Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Valencia, Dr. Moliner 50, 46100 Burjassot, Valencia, Spain

E-mail address: `anibal.molto@uv.es`

J. Orihuela, Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100 Espinardo, Murcia, Spain
E-mail address: joseori@um.es

S. Troyanski, Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100 Espinardo, Murcia, Spain
E-mail address: stroya@um.es