THE MCSHANE INTEGRAL IN WEAKLY COMPACTLY GENERATED SPACES

A. AVILÉS, G. PLEBANEK, AND J. RODRÍGUEZ

Abstract. Di Piazza and Preiss asked whether every Pettis integrable function defined on $[0,1]$ and taking values in a weakly compactly generated Banach space is McShane integrable. In this paper we answer this question in the negative.

1. Introduction

The classical Pettis' measurability theorem [15] ensures that scalar and strong measurability are equivalent for functions taking values in separable Banach spaces. This fact has many interesting consequences in vector integration. For instance, it is a basic tool to prove that Pettis and McShane integrability coincide in separable Banach spaces, [10, 12, 13]. However, for non-separable Banach spaces the notions of scalar and strong measurability are different in general. This leads to subtle problems when trying to compare different types of integrals.

In this paper we deal with the Pettis and McShane integrals. Di Piazza and Preiss [2] asked whether every Pettis integrable function $f : [0,1] \to X$ is McShane integrable if X is a weakly compactly generated (WCG) Banach space. Recently, Deville and the third author [1] have proved that the answer is affirmative when X is Hilbert generated, thus improving the previous results obtained in [2, 16]. Our main purpose here is to show that the question of Di Piazza and Preiss has negative answer in general.

The paper is organized as follows.

In Section 2 we introduce the MC-integral for Banach space-valued functions defined on probability spaces. This auxiliary tool is used as a substitute of the McShane integral at some stages. We prove that, for functions defined on quasi-Radon probability spaces, MC-integrability always implies McShane integrability (Proposition 2.2), while the converse holds if the topology on the domain has a countable basis (Proposition 2.3). This approach allows us to give a partial answer (Corollary 2.4) to a question posed by Fremlin in [10, 4G(a)].

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In Section 3 we show that the existence of scalarly null (hence Pettis integrable) WCG-valued functions which are not McShane integrable is strongly related to the existence of families of finite sets which are "measure filling" in the sense of the following definition. Throughout the paper (Ω, Σ, μ) is a probability space and we use the symbol $[S]^{<\omega}$ to denote the family of all finite subsets of a given set S.

Definition 1.1. *A hereditary family $\mathcal{F} \subset [\Omega]^{<\omega}$ is MC-filling on Ω if there exists $\varepsilon > 0$ such that for every countable partition (Ω_m) of Ω there is $F \in \mathcal{F}$ such that*

$$\mu^* \left(\bigcup \{ \Omega_m : F \cap \Omega_m \neq \emptyset \} \right) > \varepsilon,$$

where μ^* is the outer measure induced by μ.

This concept should be viewed as a measure-theoretic analogue of the notion of ε-filling families arising in Fremlin's problem DU [8]:

Definition 1.2. *Let $\varepsilon > 0$. A hereditary family $\mathcal{F} \subset [S]^{<\omega}$ is ε-filling on the set S if for every $H \in [S]^{<\omega}$ there is $F \in \mathcal{F}$ with $F \subset H$ and $|F| \geq \varepsilon |H|$.*

The existence of compact ε-filling families on uncountable sets is an open problem (the above mentioned problem DU). However, we show that compact MC-filling families on $[0, 1]$ can be constructed from some weaker versions of filling families that Fremlin proved to exist (Theorem 3.4). This leads to our main result:

Theorem 3.5. *There exist a WCG Banach space X and a scalarly null function $f : [0, 1] \to X$ which is not McShane integrable.*

In fact, the space X can be taken reflexive (Theorem 3.6). Observe that Theorem 3.5 also answers in the negative the question (attributed to Musial in [2]) whether every scalarly null Banach space-valued function on $[0, 1]$ is McShane integrable. Two counterexamples [2, 16] had been constructed under the Continuum Hypothesis (and having non WCG spaces in the range).

In Section 4 we prove that if a family $\mathcal{F} \subset [A]^{<\omega}$ is ε-filling on a set $A \subset \Omega$ of positive outer measure then it is MC-filling on Ω.

Finally, in Section 5 we provide an example of a McShane integrable function which is not MC-integrable (Theorem 5.5). Our example also makes clear that, in general, the results on the coincidence of Pettis and McShane integrability of [1, 2] do not hold when McShane integrability is replaced by MC-integrability.

Terminology. Our standard references are [4, 18] (vector integration) and [11] (topological measure theory). By a *partition* of a set S we mean a collection of pairwise disjoint (maybe empty) subsets whose union is S. A set is *countable* if it is either finite or countably infinite. As usual, the symbol $|S|$ stands for the cardinality of a set S. A family $\mathcal{F} \subset [S]^{<\omega}$ is *hereditary* if $G \in \mathcal{F}$ whenever $G \subset F \in \mathcal{F}$. A family $\mathcal{F} \subset [S]^{<\omega}$ is called *compact* if it is compact in 2^S equipped with the product topology. We say that a family $\mathcal{E} \subset \Sigma$ is *η-thick* (for some $\eta > 0$) if $\mu(\Omega \setminus \bigcup \mathcal{E}) \leq \eta$.

Throughout the paper X is a (real) Banach space. The norm of X is denoted by $\| \cdot \|$ if it is needed explicitly. We denote by X^* the topological dual of X and $B_X = \{x \in X : \|x\| \leq 1\}$. The space X is *WCG* if there is a weakly compact subset of X whose linear span is dense in X. Recall that a function $f : \Omega \to X$ is called *scalarly null* if, for each $x^* \in X^*$, the composition $x^* f : \Omega \to \mathbb{R}$ vanishes μ-a.e. (the exceptional set depending on x^*).

If $\mathfrak{T} \subset \Sigma$ is a topology on Ω, we say that $(\Omega, \mathfrak{T}, \Sigma, \mu)$ is a *quasi-Radon* probability space (following [11, Chapter 41]) if μ is complete, inner regular with respect to closed sets, and $\mu(\bigcup \mathcal{G}) = \sup\{\mu(G) : G \in \mathcal{G}\}$ for every upwards directed family $\mathcal{G} \subset \mathfrak{T}$. A *gauge* on Ω is a function $\delta : \Omega \to \mathfrak{T}$ such that $t \in \delta(t)$ for all $t \in \Omega$. Every Radon probability space is quasi-Radon [11, 416A].

The vector-valued McShane integral was first studied in [12, 13] for functions defined on $[0, 1]$ equipped with the Lebesgue measure. Fremlin [10] extended the theory to deal with functions defined on arbitrary quasi-Radon probability spaces. We next recall an alternative way of defining the McShane integral taken from [9, Proposition 3].

Definition 1.3. *Suppose $(\Omega, \mathfrak{T}, \Sigma, \mu)$ is quasi-Radon. A function $f : \Omega \to X$ is McShane integrable, with integral $x \in X$, if for every $\varepsilon > 0$ there exist $\eta > 0$ and a gauge δ on Ω such that: for every η-thick finite family (E_i) of pairwise disjoint measurable sets and every choice of points $t_i \in \Omega$ with $E_i \subset \delta(t_i)$, we have*

$$\Big\| \sum_i \mu(E_i) f(t_i) - x \Big\| \leq \varepsilon.$$

Every McShane integrable function is also Pettis integrable (and the corresponding integrals coincide), [10, 1Q]. The converse does not hold in general, see [1, 2, 12, 16] for examples.

2. Another look at the McShane integral

We next introduce a variant of the McShane integral that is defined in terms of the measure space only, without any reference to a topology.

Definition 2.1. *A function $f : \Omega \to X$ is MC-integrable, with integral $x \in X$, if for every $\varepsilon > 0$ there exist $\eta > 0$, a countable partition (Ω_m) of Ω and sets $A_m \in \Sigma$ with $\Omega_m \subset A_m$, such that: for every η-thick finite family (E_i) of pairwise disjoint elements of Σ with $E_i \subset A_{m(i)}$ and every choice of points $t_i \in \Omega_{m(i)}$, we have*

$$\Big\| \sum_i \mu(E_i) f(t_i) - x \Big\| \leq \varepsilon.$$

Clearly, given $\eta > 0$, a countable partition (Ω_m) of Ω and sets $\Omega_m \subset A_m \in \Sigma$, we can always find families (E_i) as in Definition 2.1. It is routine to check that the vector x in Definition 2.1 is unique.

The relationship between the MC-integral and the McShane integral is analyzed in the following two propositions.

Proposition 2.2. *Suppose $(\Omega, \mathfrak{T}, \Sigma, \mu)$ is quasi-Radon. If $f : \Omega \to X$ is MC-integrable, then it is McShane integrable (and the corresponding integrals coincide).*

Proof. Let $x \in X$ be the MC-integral of f and fix $\varepsilon > 0$. Since f is MC-integrable, there exist $\eta > 0$, a countable partition (Ω_m) of Ω and measurable sets $A_m \supset \Omega_m$ satisfying the condition of Definition 2.1.

For each $m, n \in \mathbb{N}$, set $\Omega_{m,n} := \{t \in \Omega_m : n - 1 \leq \|f(t)\| < n\}$ and choose $U_{m,n} \supset A_m$ open such that

$$\mu(U_{m,n} \setminus A_m) \leq \frac{1}{2^{m+n}} \min\Big\{\frac{\varepsilon}{n}, \frac{\eta}{2}\Big\}.$$

Clearly, $(\Omega_{m,n})$ is a partition of Ω. Define a gauge $\delta : \Omega \to \mathfrak{T}$ by $\delta(t) := U_{m,n}$ if $t \in \Omega_{m,n}$.

Let (E_i) be a $\frac{\eta}{2}$-thick finite family of pairwise disjoint measurable sets and let $t_i \in \Omega$ be points such that $E_i \subset \delta(t_i)$. We will check that

$$\Big\|\sum_i \mu(E_i) f(t_i) - x\Big\| \leq 2\varepsilon. \tag{1}$$

For each i, let $m(i), n(i) \in \mathbb{N}$ be such that $t_i \in \Omega_{m(i),n(i)}$. The set $F_i := E_i \cap A_{m(i)}$ is measurable, $F_i \subset A_{m(i)}$ and $t_i \in \Omega_{m(i)}$. The F_i's are pairwise disjoint. Since

$$E_i \setminus F_i = E_i \setminus A_{m(i)} \subset \delta(t_i) \setminus A_{m(i)} = U_{m(i),n(i)} \setminus A_{m(i)}$$

we have

$$\mu\Big(\Omega \setminus \bigcup_i F_i\Big) = \mu\Big(\Omega \setminus \bigcup_i E_i\Big) + \mu\Big(\bigcup_i E_i \setminus \bigcup_i F_i\Big) \leq$$
$$\leq \frac{\eta}{2} + \mu\Big(\bigcup_{m,n} U_{m,n} \setminus A_m\Big) \leq \frac{\eta}{2} + \sum_{m,n} \frac{\eta}{2^{m+n+1}} = \eta,$$

and so the family (F_i) is η-thick. From the MC-integrability condition it follows that

$$\Big\|\sum_i \mu(F_i) f(t_i) - x\Big\| \leq \varepsilon. \tag{2}$$

For each $m, n \in \mathbb{N}$, let $I(m,n)$ be the (maybe empty) set of all indexes i for which $m(i) = m$ and $n(i) = n$. Observe that

$$\sum_{i \in I(m,n)} \mu(E_i \setminus F_i)\|f(t_i)\| \leq \sum_{i \in I(m,n)} \mu(E_i \setminus F_i) n =$$
$$= \mu\Big(\bigcup_{i \in I(m,n)} E_i \setminus F_i\Big) n \leq \mu(U_{m,n} \setminus A_m) n \leq \frac{\varepsilon}{2^{m+n}}.$$

Therefore

$$\Big\|\sum_i \mu(E_i) f(t_i) - \sum_i \mu(F_i) f(t_i)\Big\| \leq \sum_i \mu(E_i \setminus F_i)\|f(t_i)\| = \tag{3}$$
$$= \sum_{m,n} \sum_{i \in I(m,n)} \mu(E_i \setminus F_i)\|f(t_i)\| \leq \sum_{m,n} \frac{\varepsilon}{2^{m+n}} = \varepsilon.$$

Inequality (1) now follows from (2) and (3). This shows that f is McShane integrable, with McShane integral x. □

The converse of Proposition 2.2 does not hold in general (see Theorem 5.5 below), although it is true for certain quasi-Radon spaces like $[0,1]$, as we next prove.

Proposition 2.3. *Suppose $(\Omega, \mathfrak{T}, \Sigma, \mu)$ is quasi-Radon and $\mathfrak{T}$ has a countable basis. Then $f : \Omega \to X$ is McShane integrable if and only if it is MC-integrable.*

Proof. It only remains to prove the "only if". Assume that f is McShane intgerable, with McShane integral $x \in X$. Let $\{U_m : m \in \mathbb{N}\}$ be a countable basis for $\mathfrak{T}$. Fix $\varepsilon > 0$. Since f is McShane integrable, there exist $\eta > 0$ and a gauge δ on Ω fulfilling the condition of Definition 1.3. We can suppose without loss of generality that $\delta(t) \in \{U_m : m \in \mathbb{N}\}$ for every $t \in \Omega$. Set

$$\Omega_m := \{t \in \Omega : \delta(t) = U_m\} \quad \text{and} \quad A_m := U_m \quad \text{for all } m \in \mathbb{N}.$$

Clearly, (Ω_m) is a partition of Ω and $\Omega_m \subset A_m \in \Sigma$. Now let (E_i) be an η-thick finite family of pairwise disjoint measurable sets with $E_i \subset A_{m(i)}$ and let $t_i \in \Omega_{m(i)}$. Then $\delta(t_i) = U_{m(i)} = A_{m(i)}$, hence $E_i \subset \delta(t_i)$ for all i. From the McShane integrability condition it follows that $\|\sum_i \mu(E_i) f(t_i) - x\| \leq \varepsilon$. This shows that f is MC-integrable. □

Fremlin raised in [10, 4G(a)] the following question: *Does any topology on Ω for which μ is quasi-Radon yield the same collection of McShane integrable X-valued functions?* In view of Propositions 2.2 and 2.3, we get a partial answer:

Corollary 2.4. *Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be two topologies on Ω for which μ is quasi-Radon. Suppose $\mathfrak{T}_1$ has a countable basis. If $f : \Omega \to X$ is McShane integrable with respect to $\mathfrak{T}_1$, then it is also McShane integrable with respect to $\mathfrak{T}_2$ (and the corresponding integrals coincide).*

3. MC-filling families vs the McShane integral

The connection between MC-filling families (Definition 1.1) and the MC-integral is explained in Proposition 3.2 below. First, it is convenient to characterize MC-filling families as follows:

Lemma 3.1. *A family $\mathcal{F} \subset [\Omega]^{<\omega}$ is MC-filling on Ω if and only if there exists $\varepsilon > 0$ such that for every countable partition (Ω_m) of Ω and sets $A_m \in \Sigma$ with $\Omega_m \subset A_m$, there is $F \in \mathcal{F}$ such that*

$$\mu\left(\bigcup\{A_m : F \cap \Omega_m \neq \emptyset\}\right) > \varepsilon.$$

Proof. The "only if" is obvious. For the converse, we will prove that the condition of Definition 1.1 holds for $0 < \eta < \varepsilon$. Suppose we are given a countable partition (Ω_m) of Ω. For every finite set $I \subset \mathbb{N}$, we choose $B_I \in \Sigma$ such that $B_I \supset \bigcup_{m \in I} \Omega_m$ and $\mu(B_I) - \mu^*(\bigcup_{m \in I} \Omega_m) < \varepsilon - \eta$. For each $m \in \mathbb{N}$, we define $A_m := \bigcap\{B_I : m \in I\}$. We have $\Omega_m \subset A_m \in \Sigma$, so we can apply the hypothesis to find $F \in \mathcal{F}$ such that

$$\mu\left(\bigcup\{A_m : F \cap \Omega_m \neq \emptyset\}\right) > \varepsilon.$$

Consider the finite set $I := \{m \in \mathbb{N} : F \cap \Omega_m \neq \emptyset\}$. Since $\bigcup_{m\in I} A_m \subset B_I$, we have

$$\mu^*\Big(\bigcup_{m\in I} \Omega_m\Big) > \mu(B_I) - (\varepsilon - \eta) \geq \mu\Big(\bigcup_{m\in I} A_m\Big) - (\varepsilon - \eta) > \eta.$$

This proves that $\mathcal{F}$ is MC-filling. □

A set $B \subset B_{X^*}$ is called *norming* if $\|x\| = \sup\{|x^*(x)| : x^* \in B\}$ for all $x \in X$. As usual, given a set $C \subset \Omega$, we write 1_C to denote the real-valued function on Ω defined by $1_C(t) = 1$ if $t \in C$ and $1_C(t) = 0$ if $t \notin C$.

Proposition 3.2. *Let $f : \Omega \to X$ be a function for which there exist a norming set $B \subset B_{X^*}$ and a family $(C_{x^*})_{x^*\in B}$ of subsets of Ω such that $x^* f = 1_{C_{x^*}}$ and $\mu^*(C_{x^*}) = 0$ for every $x^* \in B$. The following statements are equivalent:*

(i) *f is not MC-integrable;*
(ii) *$\bigcup_{x^*\in B}[C_{x^*}]^{<\omega}$ is MC-filling on Ω.*

Proof. Observe first that for every finite family (E_i) of pairwise disjoint elements of Σ and every choice of points $t_i \in \Omega$, we have

$$(4) \quad \Big\|\sum_i \mu(E_i) f(t_i)\Big\| = \sup_{x^*\in B} \Big|x^*\Big(\sum_i \mu(E_i) f(t_i)\Big)\Big| = \\ = \sup_{x^*\in B} \sum_i \mu(E_i) 1_{C_{x^*}}(t_i) = \sup_{x^*\in B} \mu\left(\bigcup\{E_i : t_i \in C_{x^*}\}\right).$$

Since B separates the points of X and $x^* f$ vanishes μ-a.e. for each $x^* \in B$, the MC-integral of f is $0 \in X$ whenever f is MC-integrable. Bearing in mind (4), statement (i) is equivalent to:

(iii) *There exists $\varepsilon > 0$ such that for every $\eta > 0$, every countable partition (Ω_m) of Ω and sets $A_m \in \Sigma$ with $\Omega_m \subset A_m$, there exist an η-thick finite family (E_i) of pairwise disjoint elements of Σ with $E_i \subset A_{m(i)}$, points $t_i \in \Omega_{m(i)}$ and a functional $x^* \in B$ such that $\mu\left(\bigcup\{E_i : t_i \in C_{x^*}\}\right) > \varepsilon$.*

Let us turn to the proof of (iii)⇔(ii). Assume first that (iii) holds and take a countable partition (Ω_m) of Ω and sets $\Omega_m \subset A_m \in \Sigma$. Choose $\eta > 0$ arbitrary and let (E_i), (t_i) and x^* be as in (iii). Observe that the set F made up of all t_i's belonging to C_{x^*} satisfies

$$\bigcup\{A_m : F \cap \Omega_m \neq \emptyset\} \supset \bigcup\{E_i : t_i \in C_{x^*}\}$$

and so $\mu(\bigcup\{A_m : F \cap \Omega_m \neq \emptyset\}) > \varepsilon$. According to Lemma 3.1, this proves that the family $\bigcup_{x^*\in B}[C_{x^*}]^{<\omega}$ is MC-filling on Ω.

Conversely, assume that (ii) holds. Let $\varepsilon > 0$ be as in Lemma 3.1 applied to the family $\bigcup_{x^*\in B}[C_{x^*}]^{<\omega}$. Fix $\eta > 0$, a countable partition (Ω_m) of Ω and sets $A_m \in \Sigma$ with $\Omega_m \subset A_m$. There exist $x^* \in B$ and $F \subset C_{x^*}$ finite such that $\mu(\bigcup_{m\in I} A_m) > \varepsilon$, where $I := \{m \in \mathbb{N} : F \cap \Omega_m \neq \emptyset\}$. Now take a finite set $J \subset \mathbb{N}$ disjoint from I such that $(A_m)_{m\in I\cup J}$ is η-thick. Enumerate $I = \{m(1), \dots, m(n)\}$ and $J = \{m(n+1), \dots, m(k)\}$. Set $E_1 := A_{m(1)}$ and $E_i := A_{m(i)} \setminus \bigcup_{j=1}^{i-1} A_{m(j)}$ for $i = 2, \dots, k$. Then (E_i) is an η-thick finite family of pairwise disjoint elements of Σ

with $E_i \subset A_{m(i)}$ and $\bigcup_{i=1}^{n} E_i = \bigcup_{m\in I} A_m$. Choose $t_i \in F \cap \Omega_{m(i)}$ for $i = 1,\dots,n$ and choose $t_i \in \Omega_{m(i)}$ arbitrary for $i = n+1,\dots,k$. Then

$$\mu\Big(\bigcup\{E_i : t_i \in C_{x^*}\}\Big) \geq \mu\Big(\bigcup_{i=1}^{n} E_i\Big) = \mu\Big(\bigcup_{m\in I} A_m\Big) > \varepsilon.$$

This shows that (iii) holds, that is, f is not MC-integrable. □

Given a compact Hausdorff topological space K, we write $C(K)$ to denote the Banach space of all real-valued continuous functions on K (with the sup norm).

Proposition 3.3. *Let $\mathcal{F} \subset [\Omega]^{<\omega}$ be a compact hereditary family made up of sets having outer measure* 0*. Let $f : \Omega \to C(\mathcal{F})$ be defined by $f(t)(F) := 1_F(t)$. Then:*

(i) *f is scalarly null;*
(ii) *f is not MC-integrable if and only if $\mathcal{F}$ is MC-filling on Ω.*

Proof. Part (i) follows from a standard argument which we include for the sake of completeness. Since $\mathcal{F}$ is an Eberlein compact (i.e., it is homeomorphic to a weakly compact subset of a Banach space), the space $C(\mathcal{F})$ is WCG and $B_{C(\mathcal{F})^*}$ is Eberlein compact when equipped with the w^*-topology, cf. [3, Theorem 4, p. 152]. Set $B := \{\delta_F : F \in \mathcal{F}\} \subset B_{C(\mathcal{F})^*}$, where δ_F denotes the "evaluation functional" at F. Since B is norming, its absolutely convex hull $\operatorname{aco}(B)$ is w^*-dense in $B_{C(\mathcal{F})^*}$. Bearing in mind that $(B_{C(\mathcal{F})^*}, w^*)$ is homeomorphic to a weakly compact subset of a Banach space, the Eberlein-Smulyan theorem (cf. [7, 3.10]) ensures that $\operatorname{aco}(B)$ is w^*-sequentially dense in $B_{C(\mathcal{F})^*}$. Since the composition $\delta_F f = 1_F$ vanishes μ-a.e. for every $F \in \mathcal{F}$, we conclude that f is scalarly null. Part (ii) follows from Proposition 3.2 applied to f and the norming set B defined above. □

On the other hand, it turns out that we can always find compact MC-filling families on $[0,1]$. As usual, we denote by $\mathfrak{c}$ the cardinality of the continuum.

Theorem 3.4. *There exists a compact MC-filling family on* $[0,1]$ *equipped with the Lebesgue measure.*

Proof. According to a result by Fremlin [8, 4B], there is a family $\mathcal{D} \subset [\mathfrak{c}]^{<\omega}$ which is hereditary and compact, and for every nonempty $A \in [\mathfrak{c}]^{<\omega}$ there is $D \in \mathcal{D}$ such that $D \subset A$ and $|D| \geq \log|A|$. In particular, $\mathcal{D}$ has the following property:

(*) If $P \subset \mathfrak{c}$ is infinite, then for every $n \in \mathbb{N}$ there is $D \in \mathcal{D}$ such that $D \subset P$ and $|D| = n$.

We denote by λ the Lebesgue measure on $[0,1]$. Fix a partition $\{Z_\alpha : \alpha < \mathfrak{c}\}$ of $[0,1]$ made up of sets of outer measure one (cf. [11, 419I]). Let $\varphi : [0,1] \to \mathfrak{c}$ be the function defined by $\varphi(t) = \alpha$ whenever $t \in Z_\alpha$. We define the family

$$\mathcal{F} := \big\{F \subset [0,1] \text{ finite} : \varphi \text{ is one-to-one on } F \text{ and } \varphi(F) \in \mathcal{D}\big\}.$$

It is clear that $\mathcal{F}$ is hereditary (because $\mathcal{D}$ is). We claim that $\mathcal{F}$ is compact or, equivalently, that every set $A \subset [0,1]$ with $[A]^{<\omega} \subset \mathcal{F}$ is finite. Indeed, observe that φ is one-to-one on A. Given any $C \in [\varphi(A)]^{<\omega}$, we have $C = \varphi(B)$ for some

$B \in [A]^{<\omega} \subset \mathcal{F}$ and so $C \in \mathcal{D}$. Hence $[\varphi(A)]^{<\omega} \subset \mathcal{D}$ and the compactness of $\mathcal{D}$ ensures that $\varphi(A)$ is finite. Since φ is one-to-one on A, we conclude that A is finite.

We shall check that $\mathcal{F}$ is MC-filling on $[0,1]$ with arbitrary constant $0 < \varepsilon < 1$. Fix a countable partition (Ω_m) of $[0,1]$. For each $\alpha < \mathfrak{c}$ we have

$$1 = \lambda^*(Z_\alpha) = \lim_{n\to\infty} \lambda^*\Big(Z_\alpha \cap \bigcup_{m=1}^{n} \Omega_m\Big),$$

so we can pick $n(\alpha) \in \mathbb{N}$ such that

$$\lambda^*\Big(Z_\alpha \cap \bigcup_{m=1}^{n(\alpha)} \Omega_m\Big) > \varepsilon. \tag{5}$$

Fix $n \in \mathbb{N}$ such that $P_n := \{\alpha < \mathfrak{c} : n(\alpha) = n\}$ is infinite. By property (*), there is $D \in \mathcal{D}$ such that $D \subset P_n$ and $|D| = n$. Write $D = \{\alpha_1, \dots, \alpha_n\}$.

We next define $t_j \in Z_{\alpha_j}$ and $m_j \in \{1, \dots, n\}$ inductively as follows. By (5) the set $Z_{\alpha_1} \cap \bigcup_{m=1}^n \Omega_m$ is nonempty and we pick any $t_1 \in Z_{\alpha_1} \cap \bigcup_{m=1}^n \Omega_m$. Choose $m_1 \in \{1, \dots, n\}$ so that $t_1 \in \Omega_{m_1}$. Now suppose we have already constructed a set $\{m_1, \dots, m_l\} \subset \{1, \dots, n\}$ and points $t_j \in Z_{\alpha_j} \cap \Omega_{m_j}$ for $j = 1, \dots, l$. If $\lambda^*(\bigcup_{j=1}^l \Omega_{m_j}) > \varepsilon$, the construction stops. Otherwise $\lambda^*(\bigcup_{j=1}^l \Omega_{m_j}) \le \varepsilon$ and therefore $l < n$ (bear in mind that $\lambda^*(\bigcup_{m=1}^n \Omega_m) > \varepsilon$ by (5)). Writing $N := \{1, \dots, n\} \setminus \{m_1, \dots, m_l\}$, another appeal to (5) yields

$$\lambda^*\Big(Z_{\alpha_{l+1}} \cap \bigcup_{m\in N} \Omega_m\Big) \ge \lambda^*\Big(Z_{\alpha_{l+1}} \cap \bigcup_{m=1}^{n} \Omega_m\Big) - \lambda^*\Big(Z_{\alpha_{l+1}} \cap \bigcup_{j=1}^{l} \Omega_{m_j}\Big) > 0,$$

so we can find $t_{l+1} \in Z_{\alpha_{l+1}} \cap \Omega_{m_{l+1}}$ for some $m_{l+1} \in N$. Repeating the process, the construction stops for some $l \in \{1, \dots, n\}$.

After that, we obtain a set $\{m_1, \dots, m_l\} \subset \{1, \dots, n\}$ with $\lambda^*(\bigcup_{j=1}^l \Omega_{m_j}) > \varepsilon$ and points $t_j \in Z_{\alpha_j} \cap \Omega_{m_j}$ for all $j = 1, \dots, l$. Putting $F := \{t_1, \dots, t_l\}$ we have

$$\lambda^*\Big(\bigcup\{\Omega_m : F \cap \Omega_m \neq \emptyset\}\Big) = \lambda^*\Big(\bigcup_{j=1}^{l} \Omega_{m_j}\Big) > \varepsilon.$$

Since $\varphi(t_j) = \alpha_j$ for all j, it follows that φ is one-to-one on F and $\varphi(F) \subset D$, thus $\varphi(F) \in \mathcal{D}$ and so $F \in \mathcal{F}$. The proof is complete. □

We now arrive at our main result:

Theorem 3.5. *There exist a WCG Banach space X and a scalarly null function $f : [0,1] \to X$ which is not McShane integrable.*

Proof. By Theorem 3.4, there is a compact MC-filling family $\mathcal{F}$ on $[0,1]$. As we observed in the proof of Proposition 3.3, the space $X := C(\mathcal{F})$ is WCG. The function

$$f : [0,1] \to C(\mathcal{F}), \quad f(t)(F) := 1_F(t),$$

from Proposition 3.3 is scalarly null and fails to be MC-integrable. According to Proposition 2.3, f is not McShane integrable. □

Moreover, the Banach space in the range can be taken reflexive:

Theorem 3.6. *There exist a reflexive Banach space Y and a scalarly null function $g : [0,1] \to Y$ which is not McShane integrable.*

Proof. Let $\mathcal{F}$ and f be as in the proof of Theorem 3.5. Observe first that $f([0,1])$ is relatively weakly compact in $C(\mathcal{F})$. Indeed, by the Eberlein-Smulyan theorem (cf. [7, 3.10]), it is enough to check that $(f(t_n))$ converges weakly to 0 whenever (t_n) is a sequence of distinct points of $[0,1]$. But this follows directly from Grothendieck's theorem (cf. [7, 4.2]) just bearing in mind that for each $F \in \mathcal{F}$ (finite!) we have $f(t_n)(F) = 1_F(t_n) = 0$ for n large enough.

Then, by the Davis-Figiel-Johnson-Pelczynski theorem (cf. [3, Chapter 5, §4]), there exist a reflexive Banach space Y and a one-to-one linear continuous mapping $T : Y \to C(\mathcal{F})$ such that $f([0,1]) \subset T(B_Y)$. The set of compositions

$$V := \{\phi \circ T : \phi \in C(\mathcal{F})^*\}$$

is a linear subspace of Y^* which separates the points of Y (because T is one-to-one). Since Y is reflexive, V is norm dense in Y^*. Let $g : [0,1] \to Y$ be the function satisfying $T \circ g = f$. For each $y^* \in V$ the composition y^*g vanishes a.e. (f is scalarly null). This fact and the norm density of V imply that g is scalarly null. Moreover, since f is not McShane integrable and T is linear and continuous, g is not McShane integrable either. □

Remark 3.7. A glance at the proof of Proposition 3.3 reveals that the function f from Theorem 3.5 satisfies that, for each $x^* \in X^*$, the composition x^*f vanishes up to a countable set. This property and the boundedness of f ensure that f is *universally Pettis integrable*, that is, Pettis integrable with respect to any Radon probability on $[0,1]$. The same holds true for the function g from Theorem 3.6. Thus, we answer Question 2.2 in [17]: there exist ZFC examples of universally Pettis integrable functions which are not universally McShane integrable.

4. Filling vs MC-filling families

In this section we prove that ε-filling families (Definition 1.2) on sets of positive outer measure are MC-filling. This result is less powerful than Theorem 3.4, in the sense that the existence of ε-filling families on uncountable sets is unknown while Theorem 3.4 is a ZFC result. Yet, we have decided to include it as it may have some interest in relation with problem DU.

Theorem 4.1. *Suppose μ is atomless. Let $A \subset \Omega$ with $\mu^*(A) > 0$ and $\mathcal{F} \subset [A]^{<\omega}$ be a family which is ε-filling on A for some $\varepsilon > 0$. Then $\mathcal{F}$ is MC-filling on Ω.*

Proof. Write $\eta := \mu^*(A)$ and fix $\eta > \eta_1 > \eta_2 > 0$. We take a countable partition (Ω_m) of Ω and sets $A_m \supset \Omega_m$ with $A_m \in \Sigma$. We will prove that there is $F \in \mathcal{F}$ such that

$$\mu\left(\bigcup\{A_m : F \cap \Omega_m \neq \emptyset\}\right) > \varepsilon(\eta - \eta_1).$$

According to Lemma 3.1, this means that $\mathcal{F}$ is MC-filling on Ω.

To this end, take $m_0 \in \mathbb{N}$ large enough such that

$$\mu^*(A) - \mu^*\Big(A \cap \bigcup_{m=1}^{m_0} \Omega_m\Big) < \eta_2. \tag{6}$$

Since μ is atomless, every finite subset of Ω has outer measure 0, so we can assume without loss of generality that $A \cap \Omega_m$ is infinite for all $m = 1, \dots, m_0$. Take $0 < \eta_3 < (\eta_1 - \eta_2)/m_0$.

We can find pairwise disjoint $B_1, \dots, B_{m_0} \in \Sigma$ such that $\bigcup_{m=1}^{m_0} B_m = \bigcup_{m=1}^{m_0} A_m$ and $B_m \subset A_m$. Let M be the set of all $m \in \{1, \dots, m_0\}$ for which $\mu(B_m) > 0$. For each $m \in M$, choose a positive rational α_m such that

$$\mu(B_m) > \alpha_m > \mu(B_m) - \eta_3.$$

We can write $\alpha_m = p_m/q$ for some $p_m \in \mathbb{N}$ and $q \in \mathbb{N}$, for $m = 1, \dots, m_0$. Set $\theta := 1/q$. Since μ is atomless, for each $m \in M$ we can find pairwise disjoint $E_1^m, \dots, E_{p_m}^m \in \Sigma$ contained in B_m with $\mu(E_i^m) = \theta$. Then

$$\mu\Big(B_m \setminus \bigcup_{i=1}^{p_m} E_i^m\Big) < \eta_3$$

and we have

$$\begin{aligned}
\mu^*\Big(A \cap \bigcup_{m=1}^{m_0} \Omega_m\Big) &\leq \mu^*\Big(A \cap \bigcup_{m=1}^{m_0} A_m\Big) = \mu^*\Big(A \cap \bigcup_{m \in M} B_m\Big) \leq \\
&\leq \sum_{m \in M} \mu\Big(B_m \setminus \bigcup_{i=1}^{p_m} E_i^m\Big) + \sum_{m \in M} \sum_{i=1}^{p_m} \mu(E_i^m) \leq \\
&\leq |M|\eta_3 + \Big(\sum_{m \in M} p_m\Big)\theta \leq m_0\eta_3 + \Big(\sum_{m \in M} p_m\Big)\theta < (\eta_1 - \eta_2) + \Big(\sum_{m \in M} p_m\Big)\theta.
\end{aligned}$$

From these inequalities and (6) we obtain

$$\eta = \mu^*(A) < \eta_1 + \Big(\sum_{m \in M} p_m\Big)\theta. \tag{7}$$

For each $m \in M$ and $i = 1, \dots, p_m$ we pick a point $t_{(m,i)} \in A \cap \Omega_m$. This can be done in such a way that the points $t_{(m,i)}$'s are different, since $A \cap \Omega_m$ is infinite for all $m \in M$. Now $H := \{t_{(m,i)} : m \in M, i = 1, \dots, p_m\}$ is a subset of A with cardinality $\sum_{m \in M} p_m$. Since $\mathcal{F}$ is ε-filling on A, there exists $F \subset H$ with $F \in \mathcal{F}$ such that $|F| \geq \varepsilon(\sum_{m \in M} p_m)$. By (7), we get

$$\begin{aligned}
&\mu\Big(\bigcup\{A_m : F \cap \Omega_m \neq \emptyset\}\Big) \geq \\
&\qquad \geq \mu\Big(\bigcup\{E_i^m : t_{(m,i)} \in F\}\Big) = |F|\theta \geq \varepsilon\Big(\sum_{m \in M} p_m\Big)\theta > \varepsilon(\eta - \eta_1).
\end{aligned}$$

The proof is over. □

5. McShane integrability vs MC-integrability

This section is devoted to ensure the existence of McShane integrable functions which are not MC-integrable (Theorem 5.5). The proof is divided into several auxiliary lemmas. The first one translates the problem into the language of MC-filling families.

Lemma 5.1. *Let Γ be a set. The following statements are equivalent:*

(i) *there exists a scalarly null function $f : \Omega \to c_0(\Gamma)$ which is not MC-integrable and satisfies $f(\Omega) \subset \{e_\gamma : \gamma \in \Gamma\}$, where $e_\gamma(\gamma') = \delta_{\gamma,\gamma'}$ (the Kronecker symbol) for all $\gamma, \gamma' \in \Gamma$;*

(ii) *there exists a partition $(C_\gamma)_{\gamma\in\Gamma}$ of Ω into sets having outer measure 0 such that the family $\bigcup_{\gamma\in\Gamma}[C_\gamma]^{<\omega}$ is MC-filling on Ω.*

Proof. The set $B := \{e^*_\gamma : \gamma \in \Gamma\} \subset B_{c_0(\Gamma)^*}$ is norming, where $e^*_\gamma(x) = x(\gamma)$ for all $x \in c_0(\Gamma)$ and $\gamma \in \Gamma$.

(i)⇒(ii) For each $\gamma \in \Gamma$ we have $e^*_\gamma f = 1_{C_\gamma}$, where $C_\gamma := \{t \in \Omega : f(t) = e_\gamma\}$ has outer measure 0 (because f is scalarly null). Clearly, $(C_\gamma)_{\gamma\in\Gamma}$ is a partition of Ω. Since f is not MC-integrable, we can apply Proposition 3.2 to conclude that the family $\bigcup_{\gamma\in\Gamma}[C_\gamma]^{<\omega}$ is MC-filling on Ω.

(ii)⇒(i) Define $f : \Omega \to c_0(\Gamma)$ by $f(t) := e_\gamma$ whenever $t \in C_\gamma$, $\gamma \in \Gamma$. Then $e^*_\gamma f = 1_{C_\gamma}$ for all $\gamma \in \Gamma$ and f is scalarly null, because $\mu^*(C_\gamma) = 0$ for all $\gamma \in \Gamma$ and the linear span of $\{e^*_\gamma : \gamma \in \Gamma\}$ is norm dense in $c_0(\Gamma)^* = \ell^1(\Gamma)$. By Proposition 3.2, since $\bigcup_{\gamma\in\Gamma}[C_\gamma]^{<\omega}$ is MC-filling on Ω, the function f is not MC-integrable. □

Thus, bearing in mind that Pettis and McShane integrability are equivalent for $c_0(\Gamma)$-valued functions [2], in order to find McShane integrable functions which are not MC-integrable we will look for MC-filling families like in condition (ii) of Lemma 5.1. The following sufficient condition will be helpful.

Lemma 5.2. *Let $(C_\gamma)_{\gamma\in\Gamma}$ be a partition of Ω and $\varepsilon > 0$ such that, whenever $(\Gamma_A)_{A\subset\mathbb{N}}$ is a partition of Γ, there is some $A \subset \mathbb{N}$ such that $\mu^*(\bigcup_{\gamma\in\Gamma_A} C_\gamma) > \varepsilon$. Then the family $\bigcup_{\gamma\in\Gamma}[C_\gamma]^{<\omega}$ is MC-filling on Ω.*

Proof. Fix a countable partition (Ω_m) of Ω. For each $A \subset \mathbb{N}$, set

$$\Gamma_A := \{\gamma \in \Gamma : \ C_\gamma \cap \Omega_m \neq \emptyset \Leftrightarrow m \in A\}.$$

Then $(\Gamma_A)_{A\subset\mathbb{N}}$ is a partition of Γ and so there is $A \subset \mathbb{N}$ such that $\mu^*(\bigcup_{\gamma\in\Gamma_A} C_\gamma) > \varepsilon$. Observe that

$$\bigcup_{m\in A} \Omega_m \supset \bigcup_{\gamma\in\Gamma_A} C_\gamma,$$

hence $\mu^*(\bigcup_{m\in A} \Omega_m) > \varepsilon$. Choose $B \subset A$ finite with $\mu^*(\bigcup_{m\in B} \Omega_m) > \varepsilon$. Take $\gamma \in \Gamma_A$. We can find a finite set $F \subset C_\gamma$ such that $F \cap \Omega_m \neq \emptyset$ for every $m \in B$, hence

$$\mu^*\Big(\bigcup\{\Omega_m : \ F \cap \Omega_m \neq \emptyset\}\Big) \geq \mu^*\Big(\bigcup_{m\in B} \Omega_m\Big) > \varepsilon.$$

This shows that $\bigcup_{\gamma\in\Gamma}[C_\gamma]^{<\omega}$ is MC-filling on Ω. □

We now focus on 2^κ (for a cardinal κ), which is a Radon probability space when equipped with (the completion of) the usual product probability, cf. [11, 416U].

Lemma 5.3. *Let κ be an uncountable cardinal, $(A_\alpha)_{\alpha<\kappa}$ a partition of κ into infinite sets and consider, for each $\alpha < \kappa$, the sets*

$$D_\alpha := \{x \in 2^\kappa : \ x(\gamma) = 0 \text{ for all } \gamma \in A_\alpha\} \quad \text{and} \quad E_\alpha := D_\alpha \setminus \bigcup_{\beta<\alpha} D_\beta.$$

Then $\bigcup_{\alpha\in I} E_\alpha$ has outer measure 1 for every uncountable set $I \subset \kappa$.

Proof. It suffices to check that $Z \cap (\bigcup_{\alpha\in I} E_\alpha) \neq \emptyset$ whenever Z belongs to the product σ-algebra of 2^κ and has positive measure. Fix a countable set $A \subset \kappa$ such that, for any $z \in Z$, we have

$$\{x \in 2^\kappa : \ x(\gamma) = z(\gamma) \text{ for all } \gamma \in A\} \subset Z. \tag{8}$$

Since the A_α's are disjoint, the set $J := \{\alpha < \kappa : A \cap A_\alpha \neq \emptyset\}$ is countable. Clearly, the D_α's have measure zero (because A_α is infinite) and so $Z \setminus \bigcup_{\alpha\in J} D_\alpha$ has positive measure. In particular, we can choose $z \in Z \setminus \bigcup_{\alpha\in J} D_\alpha$. Since J is countable and I is not, there is $\beta \in I \setminus J$. We now define an element $x \in 2^\kappa$ by declaring

$$x(\gamma) := \begin{cases} z(\gamma) & \text{if } \gamma \in \bigcup_{\alpha\in J} A_\alpha, \\ 0 & \text{if } \gamma \in A_\beta, \\ 1 & \text{otherwise.} \end{cases}$$

We claim that $x \in Z \cap E_\beta$. Indeed, we have $x \in Z$ by (8) (bear in mind that $A \subset \bigcup_{\alpha\in J} A_\alpha$). On the other hand, take any $\alpha < \kappa$ with $\alpha \neq \beta$. If $\alpha \in J$ then $z \notin D_\alpha$ and so $x \notin D_\alpha$ as well. If $\alpha \notin J$, then $x(\gamma) = 1$ for all $\gamma \in A_\alpha$ and so $x \notin D_\alpha$. It follows that $x \in Z \cap E_\beta$, as claimed. Therefore $Z \cap (\bigcup_{\alpha\in I} E_\alpha) \neq \emptyset$. □

Lemma 5.4. *Let κ be a cardinal with $\kappa > \mathfrak{c}$. Then there is a partition $(C_\gamma)_{\gamma\in\Gamma}$ of 2^κ into sets of measure zero such that, whenever $(\Gamma_A)_{A\subset\mathbb{N}}$ is a partition of Γ, there is some $A \subset \mathbb{N}$ such that $\bigcup_{\gamma\in\Gamma_A} C_\gamma$ has outer measure 1.*

Proof. Let $(A_\alpha)_{\alpha<\kappa}$ be a partition of κ into infinite sets. Clearly, the E_α's of Lemma 5.3 are pairwise disjoint and have measure zero (since A_α is infinite). We claim that the following partition of 2^κ satisfies the desired property:

$$\mathcal{C} := \Big\{E_\alpha : \ \alpha < \kappa\Big\} \cup \Big\{\{x\} : \ x \in 2^\kappa \setminus \bigcup_{\alpha<\kappa} E_\alpha\Big\}.$$

Indeed, let $(\mathcal{C}_A)_{A\subset\mathbb{N}}$ be any partition of $\mathcal{C}$. Since $\kappa > \mathfrak{c}$, there is some $A \subset \mathbb{N}$ such that $\mathcal{C}_A$ contains uncountably many E_α's. By Lemma 5.3, the outer measure of $\bigcup \mathcal{C}_A$ is 1, as required. □

We can now state the aforementioned result:

Theorem 5.5. *Let κ be a cardinal with $\kappa > \mathfrak{c}$. Then there is a McShane integrable function $f : 2^\kappa \to c_0(\Gamma)$ (for some set Γ) which is not MC-integrable.*

Proof. By Lemmas 5.1, 5.2 and 5.4, there is a scalarly null function $f : 2^\kappa \to c_0(\Gamma)$ (for some set Γ) which is not MC-integrable. Since f is Pettis integrable, it is also McShane integrable [2]. □

In [1] it is proved that Pettis and McShane integrability are equivalent for X-valued functions defined on quasi-Radon probability spaces whenever X is *Hilbert generated* (i.e., there exist a Hilbert space Y and a linear continuous map $T : Y \to X$ such that $T(Y)$ is dense in X). Clearly, every Hilbert generated space is WCG. Typical examples of Hilbert generated spaces are the separable ones, $c_0(\Gamma)$ (for any set Γ) and $L^1(\nu)$ (for any probability measure ν). Moreover, any super-reflexive space embeds into a Hilbert generated space. For more information on this class of spaces, we refer the reader to [5, 6] and [14, Chapter 6].

In view of our Theorem 5.5, we cannot replace McShane integrability by MC-integrability in the results of [1]. However, something can be said for a particular class of functions. The following proposition is inspired in [1, Lemma 3.3]. Recall that a *Markushevich basis* of X is a family $\{(x_i, x_i^*) : i \in I\} \subset X \times X^*$ such that $x_i^*(x_j) = \delta_{i,j}$ for every $i, j \in I$, the linear span of $\{x_i : i \in I\}$ is dense in X and $\{x_i^* : i \in I\}$ separates the points of X.

Proposition 5.6. *Suppose μ is atomless and X is a closed subspace of a Hilbert generated Banach space. Let $\{(x_i, x_i^*) : i \in I\}$ be a Markushevich basis of X with $x_i \in B_X$ for all $i \in I$. Let $\varphi : \Omega \to I$ be a one-to-one function and define $f : \Omega \to X$ by $f(t) := x_{\varphi(t)}$. Then f is scalarly null and MC-integrable.*

Proof. Fix $\varepsilon > 0$. Since X embeds into a Hilbert generated space, there is a partition $I = \bigcup_{m\in\mathbb{N}} I_m$ such that

$$\text{for all } x^* \in B_{X^*} \text{ and all } m \in \mathbb{N}, \quad |\{i \in I_m : |x^*(x_i)| > \varepsilon\}| \le m, \tag{9}$$

see [6, Theorem 6] (cf. [14, Theorem 6.30]). In particular, for each $x^* \in B_{X^*}$, the set $\{t \in \Omega : |x^* f(t)| > \varepsilon\}$ is countable (φ is one-to-one) and so it has outer measure 0 (because μ is atomless). As $\varepsilon > 0$ is arbitrary, f is scalarly null.

For each $m \in \mathbb{N}$, define $\Omega_m := \{t \in \Omega : \varphi(t) \in I_m\}$ and choose finitely many disjoint sets $A_{1,m}, \dots, A_{N(m),m} \in \Sigma$ with

$$\mu(A_{n,m}) \le \frac{\varepsilon}{2^m m}, \quad n = 1, \dots, N(m),$$

and $\Omega_m \subset \bigcup_{n=1}^{N(m)} A_{n,m}$. Set $\Omega_{n,m} := \Omega_m \cap A_{n,m}$ for all $m \in \mathbb{N}$ and $n = 1, \dots, N(m)$, so that $(\Omega_{n,m})$ is a countable partition of Ω.

Fix a finite family (E_j) of pairwise disjoint elements of Σ with $E_j \subset A_{n(j),m(j)}$ and choose points $t_j \in \Omega_{n(j),m(j)}$. Fix $x^* \in B_{X^*}$ and set

$$C := \{i \in I : |x^*(x_i)| \le \varepsilon\} \quad \text{and} \quad B_m := \{i \in I_m : |x^*(x_i)| > \varepsilon\} \text{ for all } m \in \mathbb{N}.$$

We can write

$$\sum_j \mu(E_j) f(t_j) = \sum_{i\in C} \sum_{\varphi(t_j)=i} \mu(E_j) x_i + \sum_{m\in\mathbb{N}} \sum_{i\in B_m} \sum_{\varphi(t_j)=i} \mu(E_j) x_i. \tag{10}$$

On the one hand

$$\Big|x^*\Big(\sum_{i\in C}\sum_{\varphi(t_j)=i}\mu(E_j)x_i\Big)\Big| \le \mu\Big(\bigcup_{i\in C}\bigcup_{\varphi(t_j)=i}E_j\Big)\varepsilon \le \varepsilon. \tag{11}$$

On the other hand, for each $m\in\mathbb{N}$ and $i\in B_m$, we have $\bigcup_{\varphi(t_j)=i}E_j\subset A_{n,m}$ for some n (here we use again that φ is one-to-one) and therefore

$$\Big|x^*\Big(\sum_{\varphi(t_j)=i}\mu(E_j)x_i\Big)\Big| \le \mu\Big(\bigcup_{\varphi(t_j)=i}E_j\Big) \le \frac{\varepsilon}{2^m m}. \tag{12}$$

From (9), (10), (11) and (12) it follows that

$$\Big|x^*\Big(\sum_j \mu(E_j)f(t_j)\Big)\Big| \le 2\varepsilon.$$

As $x^*\in B_{X^*}$ is arbitrary, we have $\|\sum_j\mu(E_j)f(t_j)\|\le 2\varepsilon$. Hence f is MC-integrable, with MC-integral $0\in X$. □

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