WEAK BAIRE MEASURABILITY OF THE BALLS IN A BANACH SPACE

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Abstract. Let X be a Banach space. The property $(\star)$ *"the unit ball of X belongs to* Baire(X, weak)*"* holds whenever the unit ball of X^* is weak*-separable; on the other hand, it is also known that the validity of $(\star)$ ensures that X^* is weak*-separable. In this paper we use suitable renormings of $\ell^\infty(\mathbb{N})$ and the Johnson-Lindenstrauss spaces to show that $(\star)$ lies strictly between the weak*-separability of X^* and that of its unit ball. As an application, we provide a negative answer to a question raised by K. Musiał.

1. Introduction

There are several σ-algebras on a Banach space X, like the Borel σ-algebras associated to the weak (w) and norm topologies, as well as the Baire σ-algebra Baire(X, w) associated to w. G. A. Edgar [2] showed that Baire(X, w) is exactly the smallest σ-algebra on X for which each element of X^* (the topological dual of X) is measurable. In general, we have

$$\text{Baire}(X, w) \subset \text{Borel}(X, w) \subset \text{Borel}(X, \text{norm}).$$

Although this chain collapses for separable X, some of these inclusions may be strict beyond the separable case, see [2], [3] and [15].

Let $\|\cdot\|$ be an equivalent norm on X and denote by $\|\cdot\|^*$ its corresponding equivalent norm on X^*. Clearly, the unit ball $B(X, \|\cdot\|) = \{x \in X : \|x\| \leq 1\}$ belongs to Baire(X, w) provided that $B(X^*, \|\cdot\|^*)$ is separable for the weak* (w^*) topology (equivalently, $(X, \|\cdot\|)$ is isometric to a subspace of $\ell^\infty(\mathbb{N})$), because in this case

$$B(X, \|\cdot\|) = \bigcap_{x^* \in D} \{x \in X : |x^*(x)| \leq 1\}$$

for any countable w^*-dense set $D \subset B(X^*, \|\cdot\|^*)$. On the other hand, it is known (cf. [10, Theorem 1.5.3]) that the statement "$B(X, \|\cdot\|) \in \text{Baire}(X, w)$" (equivalent to saying that the mapping $\|\cdot\| : X \to \mathbb{R}$ is Baire(X, w)-measurable) implies that X^* is w^*-separable. In general, the w^*-separability of X^* is not sufficient to ensure the w^*-separability of $B(X^*, \|\cdot\|^*)$ (see the next paragraph), so it is natural to ask whether the statement "$B(X, \|\cdot\|) \in \text{Baire}(X, w)$" is equivalent to the w^*-separability of $B(X^*, \|\cdot\|^*)$ or that of X^*. We stress that the weak Baire measurability of the norm has important consequences in vector integration, see [6], [13] and [14].

The aim of this paper is to discuss the question above. We use some ideas of G. A. Edgar (from [3] and [16]) to construct suitable equivalent norms on the following Banach spaces with w^*-separable dual: $\ell^\infty(\mathbb{N})$ and the Johnson-Lindenstrauss spaces JL_0 and JL_2 [11] (see Section 2 for the definitions). In this way, *for each of*

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these spaces X *we find an equivalent norm* $\|\cdot\|$ *such that* $B(X,\|\cdot\|)$ *does not belong to* $\mathrm{Baire}(X,w)$ (Theorem 2.3). This improves the well known fact that each of these spaces admits an equivalent norm whose dual unit ball is not w^*-separable; see [5] (cf. [4, Theorem 12.58 and Exercise 12.40]) for $\ell^\infty(\mathbb{N})$ and JL_0, and [11] for JL_2 (in fact, no norm on this space has w^*-separable dual unit ball). Incidentally, the proof of Theorem 2.3 provides an example of an X-valued Pettis integrable function f for which there is no scalarly equivalent X-valued function g such that the mapping $\|g(\cdot)\|$ is measurable (Corollary 2.4). This answers negatively to a question posed by K. Musiał [12, Problem 4]. Finally, the previous constructions also allow us to show that *there is an equivalent norm* $\|\cdot\|$ *on* $\ell^\infty(\mathbb{N})$ *such that* $B(\ell^\infty(\mathbb{N}),\|\cdot\|)$ *belongs to* $\mathrm{Baire}(\ell^\infty(\mathbb{N}),w)$ *but* $B(\ell^\infty(\mathbb{N})^*,\|\cdot\|^*)$ *is not* w^**-separable* (Theorem 2.9).

For all unexplained terminology and notation we refer the reader to our standard references [4], [12] and [16]. The cardinality of the continuum is denoted by $\mathfrak{c}$ and the first uncountable ordinal by ω_1. Let X be a Banach space. Given $x^*\in X^*$ and $x\in X$, we write either $x^*(x)$ or $\langle x^*,x\rangle$ to denote the evaluation of x^* at x. We say that a function $f:\Omega\to X$ defined on a complete probability space (Ω,Σ,μ) is

(i) *scalarly measurable* if the composition $\langle x^*,f\rangle$ is Σ-measurable for every $x^*\in X^*$ (equivalently, f is Σ-$\mathrm{Baire}(X,w)$-measurable);
(ii) *Pettis integrable* if $\langle x^*,f\rangle$ is μ-integrable for every $x^*\in X^*$ and for each $E\in\Sigma$ there is $x_E\in X$ such that $\int_E\langle x^*,f\rangle\,d\mu=\langle x^*,x_E\rangle$ for every $x^*\in X^*$.

Two functions $f,g:\Omega\to X$ are said to be *scalarly equivalent* if for each $x^*\in X^*$ we have $\langle x^*,f\rangle=\langle x^*,g\rangle$ μ-a.e. (the null set depends on x^*). Recall that X has the *property (C)* of Corson if every family of convex closed subsets of X with empty intersection contains a countable subfamily with empty intersection.

2. The results

In order to recall the definition of the *Johnson-Lindenstrauss spaces* [11] we need to introduce the following notation:

- $T_n:=\{0,1\}^n$ for every $n\in\mathbb{N}$;
- $T:=\bigcup_{n=1}^\infty T_n$;
- $K:=\{0,1\}^{\mathbb{N}}$;
- $u|n:=(u_i)_{i=1}^n\in T_n$ for every $u=(u_i)_{i=1}^\infty\in K$ and every $n\in\mathbb{N}$;
- $B_u:=\{u|n:n\in\mathbb{N}\}\subset T$ for every $u\in K$.

Note that T is infinite countable and that (i) each B_u is infinite; (ii) $B_u\cap B_{u'}$ is finite whenever $u\neq u'$; and (iii) the family $\{B_u:u\in K\}$ has cardinality $\mathfrak{c}$. The existence of a family of subsets of $\mathbb{N}$ satisfying (i)–(ii)–(iii) was first proved by Sierpinski (cf. [4, Lemma 5.16]). We next isolate this fact for future reference.

Lemma 2.1. *Let A be an infinite countable set. Then there is a family $\{A_\gamma:\gamma<\mathfrak{c}\}$ of infinite subsets of A such that $A_\gamma\cap A_{\gamma'}$ is finite whenever $\gamma\neq\gamma'$.*

As usual, for each $A\subset T$ we write $\chi_A\in\ell^\infty(T)$ to denote the characteristic function of A. Let U_0 be the linear span of $c_0(T)\cup\{\chi_{B_u}:u\in K\}$ in $\ell^\infty(T)$. Any $x\in U_0$ can be written in a unique way as $x=y+\sum_{i=1}^p a_i\chi_{B_{u_i}}$, where $y\in c_0(T)$, $\{u_1,\dots,u_p\}\subset K$ and $a_i\in\mathbb{R}$ for all $1\le i\le p$. For such an x, set

$$\|x\|_{JL_2}:=\max\Big\{\|x\|_\infty,\Big(\sum_{i=1}^p|a_i|^2\Big)^{1/2}\Big\}.$$

$(U_0,\|\cdot\|_{JL_2})$ is a normed space whose completion will be denoted by $(JL_2,\|\cdot\|_{JL_2})$; this is the Banach space U studied in [11, Example 1]. On the other hand, the closure JL_0 of U_0 in $\ell^\infty(T)$ is a Banach space when equipped with $\|\cdot\|_\infty$; this space was discussed in [11, Example 2]. Our notation for the Johnson-Lindenstrauss spaces comes from [17], where the reader can find a lot of information on the role played by these spaces in Banach space theory.

In the proofs of Theorems 2.3 and 2.9 we will use the following norm introduced by Edgar in his example [16, Example 3-3-5] (cf. [12, Example 3.4]) of a scalarly bounded function which is not scalarly equivalent to a bounded function.

Definition 2.2. *For each $u \in K$, consider the seminorm $\|\cdot\|_u$ on $\ell^\infty(T)$ given by*

$$\|x\|_u := \limsup_{n\to\infty} |x_{u|n}|, \quad x \in \ell^\infty(T).$$

Let $a : K \to [1, \infty)$ be a bounded function. Define

$$\|x\|_a := \max\Big\{\|x\|_\infty, \sup_{u\in K} a(u)\|x\|_u\Big\}, \quad x \in \ell^\infty(T).$$

Clearly, $\|\cdot\|_a$ is an equivalent norm on $\ell^\infty(T)$.

We write (K, Σ_K, μ_K) to denote the complete probability space obtained after completing the usual product probability measure on $\mathrm{Borel}(K)$. Recall that this measure space is isomorphic to $[0, 1]$ equipped with the Lebesgue measure on the σ-algebra of all Lebesgue measurable sets, cf. [8, 254K].

Theorem 2.3. *Let X be either $\ell^\infty(\mathbb{N})$, JL_0 or JL_2. Then X^* is w^*-separable and there is an equivalent norm $\|\cdot\|$ on X such that $B(X, \|\cdot\|)$ does not belong to* $\mathrm{Baire}(X, w)$*.*

Proof. The first assertion is obvious for $\ell^\infty(T)$ and its closed subspace JL_0. The w^*-separability of JL_2^* was proved in [11, Example 1].

Now fix a bounded *non Σ_K-measurable* function $a : K \to [1, \infty)$.

First case: JL_2. The identity mapping on U_0 can be extended to a linear continuous mapping $S : JL_2 \to JL_0$. Note that the formula

$$\|z\| := \|z\|_{JL_2} + \|S(z)\|_a, \quad z \in JL_2,$$

defines an equivalent norm on JL_2. On the other hand, Edgar showed in the proof of [3, Proposition 5.12(c)] that the function

$$\phi : K \to JL_2, \quad \phi(u) := \chi_{B_u},$$

is scalarly measurable, i.e. Σ_K-$\mathrm{Baire}(JL_2, w)$-measurable. For each $u \in K$ we have $\|\chi_{B_u}\|_a = a(u)$, hence $\|\phi(u)\| = 1 + a(u)$. Since a is not Σ_K-measurable, the mapping $\|\cdot\| : JL_2 \to \mathbb{R}$ cannot be $\mathrm{Baire}(JL_2, w)$-measurable.

Second case: JL_0 and $\ell^\infty(T)$. Clearly, the composition $S \circ \phi : K \to JL_0$ is also scalarly measurable, i.e. Σ_K-$\mathrm{Baire}(JL_0, w)$-measurable, and $\|(S \circ \phi)(u)\|_a = a(u)$ for every $u \in K$. It follows that the restriction of $\|\cdot\|_a$ to JL_0 is not $\mathrm{Baire}(JL_0, w)$-measurable. Finally, since

$$\mathrm{Baire}(JL_0, w) = \{C \cap JL_0 : \ C \in \mathrm{Baire}(\ell^\infty(T), w)\},$$

we infer that $\|\cdot\|_a$ is not $\mathrm{Baire}(\ell^\infty(T), w)$-measurable. The proof is over. □

Let (Ω, Σ, μ) be a complete probability space and $(X, \|\cdot\|)$ a Banach space. Musiał posed in [12, Problem 4] the following question. *Is it true that for each Pettis integrable function $f : \Omega \to X$ there is a function $g : \Omega \to X$ such that f and g are scalarly equivalent and the mapping $\|g(\cdot)\|$ is Σ-measurable?* Naturally, the answer is affirmative if (X, w) is measure compact (e.g. Lindelöf), since in this case every scalarly measurable X-valued function is scalarly equivalent to a strongly measurable one [2]. The following corollary provides a negative answer to Musiał's question even for spaces with property (C) (like JL_0 and JL_2, cf. [17, Section 2]).

Corollary 2.4. *Let X be either $\ell^\infty(\mathbb{N})$, JL_0 or JL_2. Then there exist an equivalent norm $\|\cdot\|$ on X and a Pettis integrable function $f : K \to X$ for which there is no function $g : K \to X$ such that f and g are scalarly equivalent and the mapping $\|g(\cdot)\|$ is Σ_K-measurable.*

Proof. We first deal with JL_2. Let $f := \phi$ be the function considered in the proof of Theorem 2.3. Since JL_2 has property (C) and f is bounded and scalarly measurable, we can apply [16, Theorem 5-2-2] to conclude that f is Pettis integrable. As we have shown in the proof of Theorem 2.3, there is an equivalent norm $\|\cdot\|$ on JL_2 such that the mapping $\|f(\cdot)\|$ is not Σ_K-measurable. Now, if a function $g : K \to JL_2$ is scalarly equivalent to f, the w^*-separability of JL_2^* ensures that $f = g$ μ_K-a.e. and, therefore, $\|g(\cdot)\|$ is not Σ_K-measurable as well.

The proof for JL_0 and $\ell^\infty(T)$ is similar, bearing in mind the Pettis integrable function $f := S \circ \phi$. □

Remark 2.5. A. S. Granero et al. [9] have shown that any Banach space X without property (C) admits an equivalent norm $\|\cdot\|$ such that $B(X^*, \|\cdot\|^*)$ is not w^*-separable (see [1] for related results). In general, the failure of property (C) does not ensure the existence of an equivalent norm $\|\cdot\|$ such that $B(X, \|\cdot\|) \notin \mathrm{Baire}(X, w)$. For instance, $\ell^1(\omega_1)$ fails property (C) [3] and

$$\mathrm{Baire}(\ell^1(\omega_1), w) = \mathrm{Borel}(\ell^1(\omega_1), \mathrm{norm}),$$

according to a theorem of D. H. Fremlin [7].

In Proposition 2.8 below we study the w^*-separability of $B(\ell^\infty(T)^*, \|\cdot\|_a^*)$ in terms of a. To this end we need a couple of lemmas. The first one follows easily from the "lifting property" of $\ell^1(\mathbb{N})$, cf. [4, Proposition 5.10]. As usual, we write

$$c_0(T)^\perp := \{x^* \in \ell^\infty(T)^* : \ \langle x^*, x\rangle = 0 \ \text{ for every } x \in c_0(T)\}.$$

Lemma 2.6. *There is a decomposition $\ell^\infty(T)^* = Y \oplus c_0(T)^\perp$, where Y is isomorphic to $\ell^1(T)$. The isomorphism $\Theta : \ell^1(T) \to Y$ is given by*

$$\langle \Theta(z), x\rangle = \sum_{u|n \in T} z_{u|n} x_{u|n}, \quad z \in \ell^1(T), \ x \in \ell^\infty(T),$$

and the projection $P : \ell^\infty(T)^* \to Y$ is given by

$$P(x^*) = \Theta\big((x^*(\chi_{\{u|n\}}))_{u|n \in T}\big), \quad x^* \in \ell^\infty(T)^*.$$

Let $a : K \to [1, \infty)$ be a bounded function. If $\ell^\infty(T)^*$ is equipped with $\|\cdot\|_a^*$ and $\ell^1(T)$ is equipped with its canonical norm $\|\cdot\|_{\ell^1(T)}$, then Θ, Θ^{-1} and P have norm 1.

The second lemma isolates a property used by J. Hagler in his example (cf. [16, Example 3-2-4] or [12, Example 3.3]) of a scalarly measurable function which is not scalarly equivalent to a strongly measurable one. The original proof for the family $\{B_u : u \in K\}$ can be extended straightforwardly to this more general case.

Lemma 2.7. *Let $\{C_i : i \in I\}$ be a family of subsets of T such that $C_i \cap C_{i'}$ is finite whenever $i \neq i'$. Let $x^* \in c_0(T)^\perp$. Then the set $\{i \in I : x^*(\chi_{C_i}) \neq 0\}$ is countable.*

Proposition 2.8. *Let $a : K \to [1, \infty)$ be a bounded function. The following statements are equivalent:*

(i) *$B(\ell^\infty(T)^*, \|\cdot\|_a^*)$ is w^*-separable.*
(ii) *$a(u) = 1$ for every $u \in K$.*

Proof. Clearly, (ii) implies that $\|\cdot\|_a = \|\cdot\|_\infty$, so it only remains to prove (i)⇒(ii). Suppose that (ii) fails, that is, there is $u \in K$ such that $a(u) > 1$. Take any countable set $D \subset B(\ell^\infty(T)^*, \|\cdot\|_a^*)$. Since B_u is infinite countable, we can find a family $\{A_\gamma : \gamma < \mathfrak{c}\}$ of infinite subsets of B_u such that $A_\gamma \cap A_{\gamma'}$ is finite whenever

$\gamma \neq \gamma'$ (Lemma 2.1). With the notations of Lemma 2.6, for each $x^* \in D$ we can write $x^* = P(x^*) + (x^* - P(x^*))$, where $x^* - P(x^*) \in c_0(T)^\perp$, $P(x^*) \in Y$ and $\|P(x^*)\|_a^* \leq 1$. Since D is countable, we can apply Lemma 2.7 to find $\gamma < \mathfrak{c}$ such that $x^*(\chi_{A_\gamma}) = P(x^*)(\chi_{A_\gamma})$ for every $x^* \in D$. By the definition of P, we have

$$P(x^*)(\chi_{A_\gamma}) = \sum_{u|n \in A_\gamma} x^*(\chi_{\{u|n\}}).$$

On the other hand, since Θ^{-1} has norm 1, we get

$$\sum_{v|n \in T} |x^*(\chi_{\{v|n\}})| = \|\Theta^{-1}(P(x^*))\|_{\ell^1(T)} \leq 1.$$

It follows that $|x^*(\chi_{A_\gamma})| \leq 1$ for every $x^* \in D$. Since $A_\gamma \subset B_u$ is infinite, we conclude that

$$\|\chi_{A_\gamma}\|_a = a(u) > 1 \geq \sup_{x^* \in D} |x^*(\chi_{A_\gamma})|.$$

Therefore, D is not w^*-dense in $B(\ell^\infty(T)^*, \|\cdot\|_a^*)$. This shows that $B(\ell^\infty(T)^*, \|\cdot\|_a^*)$ is not w^*-separable. □

Theorem 2.9. *There is an equivalent norm $\|\cdot\|$ on $\ell^\infty(\mathbb{N})$ such that:*

(i) *$B(\ell^\infty(\mathbb{N}), \|\cdot\|)$ belongs to* $\mathrm{Baire}(\ell^\infty(\mathbb{N}), w)$.

(ii) *$B(\ell^\infty(\mathbb{N})^*, \|\cdot\|^*)$ is not w^*-separable.*

Proof. Fix a bounded function $a : K \to [1, \infty)$ such that $a^{-1}((1, \infty))$ is countable and non empty. Then $B(\ell^\infty(T)^*, \|\cdot\|_a^*)$ is not w^*-separable, by Proposition 2.8. On the other hand, note that for each $u \in K$ the mapping

$$\|\cdot\|_u : \ell^\infty(T) \to \mathbb{R}, \quad \|x\|_u = \limsup_{n\to\infty} |x_{u|n}| = \inf_{k\in\mathbb{N}} \sup_{n\geq k} |x_{u|n}|,$$

is $\mathrm{Baire}(\ell^\infty(T), w)$-measurable. Since

$$\|x\|_a = \max\Big\{\|x\|_\infty, \sup_{u \in a^{-1}((1,\infty))} a(u)\|x\|_u\Big\}, \quad x \in \ell^\infty(T),$$

it follows that $\|\cdot\|_a$ is $\mathrm{Baire}(\ell^\infty(T), w)$-measurable, as required. □

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