Midpoint locally uniformly rotundity and a decomposition method for renorming.

by A. Moltó J. Orihuela S. Troyanski *and* M. Valdivia

1 Introduction

An excellent overview concerning results about midpoint locally uniformly rotund (**MLUR** for short) Banach spaces can be found in [5]. We would like to mention in addition the paper of R. Haydon [7] devoted to renormings of $C(T)$ where T is a tree. There he characterizes the trees T for which $C(T)$ is **MLUR** renormable and gives the first example of a Banach space which has an equivalent **MLUR** norm but no locally uniformly rotund (**LUR** for short) renorming. Actually the class of the trees T for which $C(T)$ is **MLUR**–renormable is the same as that of the trees T for which $C(T)$ has a rotund equivalent norm. In general this coincidence is not true (see [2], [1]). In this paper we characterize in terms of linear topological conditions the Banach spaces which admit an equivalent **MLUR** norm.

Definition 1 *[9] Let A be an arbitrary subset of a normed space X and ε, $\delta > 0$. The point $x \in A$ is said to be an (ε, δ)*–strongly extreme point of A *if*

$$\|u - v\| < \varepsilon \textit{ whenever } \|x - (u + v)/2\| < \delta \textit{ and } u, v \in A.$$

The point $x \in A$ is said to be ε–strongly extreme point of A *if there exists a $\delta > 0$ such that x is an (ε, δ)–strongly extreme point of A.*

Let us recall that a normed space (or the norm on) X is **MLUR** if all the points of its unit sphere are ε–strongly extreme points for B_X for all $\varepsilon > 0$. This assertion is equivalent to

$$\lim_k \|u_k - v_k\| = 0 \text{ whenever } \lim_k \|x - (u_k + v_k)/2\| = 0, \quad \|u_k\|, \|v_k\| \leq \|x\| = 1;$$

which in turn is equivalent to $\lim_k \|x_k\| = 0$ whenever $\lim_k \|x \pm x_k\| = \|x\| = 1$. A normed space (or the norm on) X is **LUR** if $\lim_k \|x - x_k\| = 0$ whenever $\lim_k \|(x + x_k)/2\| = \|x_k\| = \|x\| = 1$.

Theorem 1 *A normed space X is* **MLUR** *renormable if and only if for every positive number ε we can write*

$$X = \bigcup_{n=1}^{\infty} X_{n,\varepsilon} \tag{1}$$

in such a way that all points of $X_{n,\varepsilon}$ are ε–strongly extreme of conv$(X_{n,\varepsilon})$.

We have a similar result for dual **MLUR** renorming.

Theorem 2 *A normed space X has an equivalent norm $|\cdot|$ such that $(X, |\cdot|)^*$ is* **MLUR** *if and only if for any $\varepsilon > 0$ we can write $X^* = \bigcup_{n=1}^{\infty} X_{n,\varepsilon}$ in such a way that all points of $X_{n,\varepsilon}$ are ε–strongly extreme of $\overline{conv}^{w^*}(X_{n,\varepsilon})$.*

Remark 1 A similar characterization for the existence of **LUR** renormings has been obtained recently in [11] by means of probabilistic tools where, roughly speaking, ε–strongly extreme point has been replaced by ε–denting point. Let us recall that a point x in $A \subset X$ is said to be ε–denting for A if there exist $f \in X^*$ and a real number θ such that the open slice $S = \{u \in A : f(u) > \theta\}$ of A verifies $x \in S$ and diam $S < \varepsilon$.

The idea of splitting the space in countable pieces in such a way that every point of every piece is an ε–denting point has its origin in the paper [8] where the notion of countable cover by sets of small local diameter was introduced. In [12] the above result about **LUR** renorming was extended for dual norms in terms of ε–w^*–denting points. The method in [12] of construction of the norm is based on geometric convexity arguments mixed with the topological notion of network together with a reduction argument for the non–convex case based on the Bourgain–Namioka Superlemma [4, p. 157].

Deville's Master Lemma [3, Chapter VII, §1] is present in most of the constructions of the norms with different convex properties. R. Haydon [7] has extensively used it for some renormings of $C(T)$ where T is a tree. The roots of this approach can be traced back in [13] which in turn is based on some ideas of approximation theory. In § 3 of this paper we develop a linear topological method for **LUR** and **MLUR** renormings which plays the same role as Deville's Master Lemma when the renormings are obtained from the above covering characterizations. The geometrical part of this method is the following:

Proposition 1 *Let x be a point of a bounded subset A of a normed space X, let ε, η, θ be real numbers with ε, $\eta > 0$, let f be in X^*, and $T : X \to X$ be a bounded linear operator. Assume that the following hold:*

i) $\sup_A f = f(x) > \theta$ *and* $\|Tw - w\| < \eta$*, whenever w belongs to the open slice* $S = \{w \in A : \ f(w) > \theta\}$*;*

ii) Tx is an ε–strongly extreme (denting) point of conv (TS) (TS respectively).

Then x is a $2(\varepsilon + \eta)$–strongly extreme (denting) point of conv (A) (A respectively).

The condition of the existence of a bidual **MLUR** renorming in a Banach space is completely different from the **LUR** one. It is well known and easy to see that for every Banach space X and for every $z \in X^{**}$ such that $\|z\| = 1$, there exists a sequence

(x_k) in X, $\|x_k\| = 1$, such that $\lim_k \|x_k + z\| = 2$. Since we have $\|x_k - z\| \geq \operatorname{dist}(z, X)$ the unit sphere of $S_{X^{**}}$ has no **LUR** point in $S_{X^{**}} \setminus X$. In § 4 we prove that in James space J there exists an equivalent norm $|\cdot|$ such that $(J, |\cdot|)^{**}$ is **MLUR**. Let us mention that recently P. Hájek [6] has proved that J has an equivalent norm $|||\cdot|||$ such that $(J, |||\cdot|||)^{**}$ is rotund.

The final version of this paper was prepared during the stay of the third author in the University of Valencia and in the University of Murcia during the Academic Year 1998–1999. He acknowledges his gratitude for the hospitality and facilities provided by the University of Valencia and the University of Murcia.

2 Construction of an MLUR norm

Given a subset A of a normed space X and positive real numbers m, r, s, set

$$
\begin{aligned}
A^s &:= \{tw : \ 0 \leq t \leq 1, \ w \in A \cap (sB_X)\}, \\
A^{m,s} &:= A^s + m^{-1}B_X \quad \text{and} \\
A^{m,s}_r &:= A^{m,s} \cap (rB_X).
\end{aligned} \tag{2}
$$

Lemma 1 *Let A be a subset of a normed space X and let ε, δ, η be positive real numbers with $3\eta < \min(\varepsilon, \delta)$. Let x be a non–zero element of X which is an (ε, δ)–strongly extreme point of A. Then*

i) x is a $(2\varepsilon, 2\eta)$–strongly extreme point of $A + \eta B_X$;

ii) there exist rational numbers r, s with $0 < r < \|x\| < s$, such that x is a $(2\varepsilon, 2\eta)$–strongly extreme point of $A^s \setminus rB_X$;

iii) there exist $m \in \mathbb{N}$ and rational numbers r, s with $0 < r < \|x\| < s$, such that x belongs to the interior of $A^{m,s}$ and such that x is a $(3\varepsilon, \eta)$–strongly extreme point of $A^{m,s} \setminus A^{m,s}_r$.

Proof. i) Take u, $v \in A + \eta B_X$ with $\|x - (u+v)/2\| < 2\eta$. We can choose u_1, $v_1 \in A$ such that $\|u - u_1\| < \eta$ and $\|v - v_1\| < \eta$. Then

$$\left\| x - \frac{u_1 + v_1}{2} \right\| \leq \left\| x - \frac{u+v}{2} \right\| + \left\| \frac{u+v}{2} - \frac{u_1+v_1}{2} \right\| < 2\eta + \left\| \frac{u - u_1}{2} \right\| + \left\| \frac{v - v_1}{2} \right\| \leq \delta$$

so $\|u_1 - v_1\| < \varepsilon$ and

$$\|u - v\| \leq \|u - u_1\| + \|u_1 - v_1\| + \|v_1 - v\| < \eta + \varepsilon + \eta < 2\varepsilon.$$

ii) Choose rational numbers r, s such that $0 < r < \|x\| < s$ and $r - s < \eta/2$. Take u, $v \in A^s \setminus rB_X$,

$$\|(u+v)/2 - x\| < 2\eta. \tag{3}$$

There must exist u_1, $v_1 \in A \cap sB_X$ such that

$$u = t_1 u_1, \quad v = t_2 v_1, \quad 0 \leq t_1 \leq 1, \quad 0 \leq t_2 \leq 1.$$

We have that

$$r \leq \|u\| = t_1\|u_1\| \leq t_1 s, \quad r \leq \|v\| = t_2\|v_1\| \leq t_2 s,$$

so

$$\|u_1 - u\| = \|u_1\| - \|u\| \leq s - r < \eta,$$

and in a similar way we deduce $\|v_1 - v\| < \eta$. Consequently u, $v \in A + \eta B_X$. From i) and (3) we get $\|u - v\| < 2\varepsilon$.

iii) Since $x \in A^s$ and $A^{m,s} = A^s + m^{-1}B_X$ it follows that x is an internal point of $A^{m,s}$. On the other hand, according to ii) there are rational numbers r_1 and s, $0 < r_1 < \|x\| < s$, in such a way that x is a $(2\varepsilon, 2\eta)$–strongly extreme point of $A^s \setminus r_1 B_X$. Take a rational number r, $r_1 < r < \|x\|$ and a positive integer $m \in \mathbb{N}$ such that

$$m^{-1} < \min\left(r - r_1, \eta\right).$$

Let

$$u, v \in A^{m,s} \setminus A^{m,s}_r = A^{m,s} \setminus rB_X, \text{ with } \|x - (u+v)/2\| < \eta.$$

We can choose u_1, $v_1 \in A_s$ such that

$$\|u - u_1\| \leq m^{-1} \quad \text{and} \quad \|v - v_1\| \leq m^{-1}.$$

We have

$$\|u_1\| \geq \|u\| - \|u - u_1\| > r - m^{-1} > r_1,$$

so $u_1 \notin r_1 B_X$. The same argument shows that $v_1 \notin r_1 B_X$ hence u_1, $v_1 \in A^s \setminus r_1 B_X$. On the other hand

$$\left\|x - \frac{u_1 + v_1}{2}\right\| \leq \left\|x - \frac{u+v}{2}\right\| + \left\|\frac{u_1 - u}{2}\right\| + \left\|\frac{v_1 - v}{2}\right\| < \eta + m^{-1} < 2\eta.$$

Since x is a $(2\varepsilon, 2\eta)$–strongly extreme point of $A^s \setminus (r_1 B_X)$ we have $\|u_1 - v_1\| < 2\varepsilon$ and finally

$$\|u - v\| \leq \|u - u_1\| + \|u_1 - v_1\| + \|v_1 - v\| < m^{-1} + 2\varepsilon + m^{-1} < 3\varepsilon.$$

■

Lemma 2 *Let $(A_n)_1^\infty$ be a sequence of closed convex subsets of a normed space X such that, for every $x \in X$ and every $\varepsilon > 0$, there exists n such that x is an ε–strongly extreme point of A_n. Then X is* **MLUR** *renormable.*

Proof. Fix A_n, we set, as defined before (see (2))

$$A_n^s := (A_n)^s, \quad A_n^{m,s} := (A_n)^{m,s}, \quad A_{n,r}^{m,s} := (A_n)_r^{m,s}$$

where $m \in \mathbb{N}$, r and s are rational numbers such that $0 < r < s$. Every one of the sets $A_n^{m,s}$ and $A_{n,r}^{m,s}$ is convex and 0 belongs to its interior. If $\mathbb{Q}_+$ is the set of all positive rational numbers we write the sets

$$\{A_n^{m,s} : \ m,n \in \mathbb{N}, \ s \in \mathbb{Q}_+\} \cup \left\{A_{n,r}^{m,s} : \ m,n \in \mathbb{N}, \ r,s \in \mathbb{Q}_+, \ r < s\right\},$$

as a sequence $(C_j)_1^\infty$. Let $|\cdot|_j$ be the Minkowski functional of C_j. If

$$\|x\|_j := \left(|x|_j^2 + |-x|_j^2\right)^{1/2}, \quad x \in X,$$

we have that $\|\cdot\|_j$ is an equivalent norm so there exist constants $a_j > 0$ such that

$$|||x|||^2 := \sum_{j\geq 1} a_j \|x\|_j^2, \quad x \in X,$$

is an equivalent norm in X. We claim that $|||\cdot|||$ is **MLUR**. Indeed take x, u_k, v_k in X such that $|||u_k|||$,$|||v_k||| \leq |||x||| = 1$ and

$$\lim_k |||(u_k + v_k)/2 - x||| = 0.$$

Then

$$\lim_k |(u_k + v_k)/2|_j = |x|_j, \quad j \in \mathbb{N},$$

and

$$\lim_k |||(u_k + v_k)/2||| = |||x||| = |||u_k||| = |||v_k||| = 1.$$

By standard convexity arguments (see e.g. [3, Fact 2.3, p. 45]) from the above equalities we conclude that

$$\lim_k |u_k|_j = \lim_k |v_k|_j = |x|_j, \quad j \in \mathbb{N}.$$

Given $\varepsilon > 0$ there exists A_n such that x is an ε–strongly extreme point of A_n. From iii) of Lemma 1 it follows that there exist a positive integer m, positive rational numbers r, s, and $\eta > 0$ such that $0 < r < \|x\| < s$ and x is an internal $(3\varepsilon, \eta)$–strongly extreme point of $A_n^{m,s} \setminus A_{n,r}^{m,s}$. Set p, $q \in \mathbb{N}$ such that $C_p = A_n^{m,s}$ and $C_q = A_{n,r}^{m,s}$. Then x belongs to the interior of C_p. Since $\|x\| > r$, x does not belong to $\overline{C_q}^{\|\cdot\|}$. Hence $|x|_p < 1$ and $|x|_q > 1$. Choose a positive integer k_0 such that if $k \geq k_0$ we have

$$|u_k|_p < 1, \quad |v_k|_p < 1, \quad |u_k|_q > 1, \quad |v_k|_q > 1, \quad \|x - (u_k + v_k)/2\| < \eta.$$

Then

$$u_k, v_k \in A_n^{m,s} \setminus A_{n,r}^{m,s}, \quad k \geq k_0,$$

so

$$\|u_k - v_k\| < 3\varepsilon, \quad k \geq k_0.$$

Consequently

$$\lim_k \|u_k - v_k\| = 0.$$

■

Proof of Theorem 1. To show that the condition is necessary let us assume that the norm of X is **MLUR**. Fix $\varepsilon > 0$. For a non–negative rational number r we denote by $X_{r,\varepsilon}$ the set of all points that are ε–strongly extreme of rB_X. We claim that

$$X = \bigcup_r X_{r,\varepsilon}.$$

Indeed, let $x \in X$, $x \neq 0$. Since the norm of X is **MLUR** we can find $\delta > 0$ such that x is an $(\varepsilon/2, \delta)$–strongly extreme point of $\|x\|B_X$. According to i) of Lemma 1 there exists $\eta > 0$ such that x is an (ε, η)–strongly extreme point of $(\|x\| + \eta)B_X$, and $r = \|x\| + \eta$ is rational. So $x \in X_{r,\varepsilon}$.

To show that the condition is sufficient let $X = \bigcup \left\{X_{1/m,n} :\ n \in \mathbb{N}\right\}$ in such a way that all points of $X_{1/m,n}$ are $1/m$–strongly extreme points of $\overline{\mathrm{conv}}^{\|\cdot\|}\left(X_{1/m,n}\right) = A_{m,n}$. Since the sets $A_{m,n}$, m, $n \in \mathbb{N}$ satisfy the conditions of Lemma 2 we have that X admits an equivalent **MLUR** norm. ■

The proof of Theorem 2 is similar to that of Theorem 1. Since in this case the sets B_X and A_n are w^*–closed and so are $A_n^{m.s}$ and $A_{n,r}^{m.s}$ then the norm obtained following the above argument must be a dual norm.

Remark 2 Let us mention that if for every $\varepsilon > 0$ we can split a radial subset $R \subset X$ (i.e. $X = \bigcup_{\lambda \geq 0} \lambda R$) into countable pieces $R_{n,\varepsilon}$ in such a way that every $x \in R_{n,\varepsilon}$ is an ε–strongly extreme point of $\mathrm{conv}(R_{n,\varepsilon})$ then (1) is fulfilled. Indeed assume that for every $\varepsilon > 0$ we can write $R = \bigcup_n R_{n,\varepsilon}$ in such a way that for every $z \in R_{n,\varepsilon}$ there exists $\delta(z, \varepsilon) > 0$ such that z is an $(\varepsilon, \delta(z, \varepsilon))$–strongly extreme point of $\mathrm{conv}(R_{n,\varepsilon})$. Since R is radial for every $x \in X$, $x \neq 0$, there exists $\nu(x) > 0$ such that $z(x) = \nu(x)x \in R$.

For k, m, $n \in \mathbb{N}$, $q \in \mathbb{Q}_+$ by $X_{m,n}^{k,q}$ we denote the set of all $x \in X$ such that $z(x) \in R_{n,\varepsilon/m}$ and

$$\|x\| \leq m,\ \nu(x) \geq m^{-1},\ \delta\left(z(x), m^{-1}\varepsilon\right) \geq k^{-1},\ 4m\,|q - \nu(x)| \leq \min\left\{k^{-1}, m^{-1}\varepsilon\right\}.$$

Since $\nu(x) \geq m^{-1}$ and $|q - \nu(x)| \leq (4m)^{-1}$ for all $x \in X_{m,n}^{k,q}$ we have

$$q \geq \nu(x) - (4m)^{-1} \geq 3(4m)^{-1}, \tag{4}$$

if $X_{m,n}^{k,q} \neq \emptyset$.

We show that all points in $X_{m,n}^{k,q}$ are ε–strongly extreme of $\mathrm{conv}\left(X_{m,n}^{k,q}\right)$. Indeed, let $x \in X_{m,n}^{k,q}$ and u, $v \in \mathrm{conv}\left(X_{m,n}^{k,q}\right)$ be such that $\|x - (u+v)/2\| < (4kq)^{-1}$. Then there exist u_i, $v_i \in X_{m,n}^{k,q}$ and λ_i, $\mu_i \geq 0$, $\sum \lambda_i = \sum \mu_i = 1$ such that $u = \sum \lambda_i u_i$, $v = \sum \mu_i v_i$. We have

$$\|z(x) - \sum(\lambda_i z(u_i) + \mu_i z(v_i))/2\| = \|\nu(x)x - \sum(\lambda_i \nu(u_i)u_i + \mu_i \nu(v_i)v_i)/2\| \leq$$

$$\leq q\|x - (u+v)/2\| + |\nu(x) - q|\|x\| + \sum(\lambda_i|\nu(u_i) - q|\|u_i\| + \mu_i|\nu(v_i) - q|\|v_i\|)/2 <$$

$$< (4k)^{-1} + (4k)^{-1} + (4k)^{-1} + (4k)^{-1} = k^{-1} \leq \delta(z(x), m^{-1}\varepsilon).$$

Hence $\| \sum(\lambda_i z(u_i) - \mu_i z(v_i)\| < m^{-1}\varepsilon$. This implies

$$q\|u - v\| = \| \sum(\lambda_i q u_i - \mu_i q v_i)\| \leq$$

$$\leq \sum(\lambda_i |q - \nu(u_i)| \|u_i\| + \mu_i |q - \nu(v_i)| \|v_i\|) + \| \sum(\lambda_i z(u_i) - \mu_i z(v_i))\| <$$

$$< (4m)^{-1}\varepsilon + (4m)^{-1}\varepsilon + m^{-1}\varepsilon = 3\varepsilon(4m)^{-1}.$$

This and (4) imply $\|u - v\| < \varepsilon$.

3 Decomposition Method.

As we mention in the Introduction, Proposition 1 plays here the same role as the Decomposition Method does in [3, Chapter VII, §1]. We illustrate this in the following assertions which are the main tools of R. Haydon [7] for **LUR** and **MLUR** renormings of $C(T)$ where T is a tree.

If L is a locally compact scattered space by $C_0(L)$ we denote the set of all continuous real valued functions on L vanishing at infinity endowed with the supremum norm $\| \cdot \|_\infty$. For a clopen subset K of L and$x \in C_0(L)$ we write $P_K x = 1\!\!1_K x$. Clearly $P_K x \in C_0(L)$.Let $\varepsilon > 0$, we denote by $E_\varepsilon(K)$ the set of all $x \in \ell_\infty(K)$ such that $\|x - (a 1\!\!1_M + b 1\!\!1_N)\|_\infty < \varepsilon$ for some $a, b \in \mathbb{R}$ and $M, N \subset K$, $M \cup N = K$, $M \cap N = \emptyset$.

Proposition 2 [7, Proposition 5.3.] *Let L be a locally compact scattered space, let $\{K_\gamma\}_{\gamma\in\Gamma}$ be a family of clopen subsets of L and $U : C_0(L) \to c_0(\Gamma)$ a bounded linear operator. Assume that, for every $x \in C_0(L)$, every $t \in L$ with $x(t) \neq 0$, and every $\varepsilon > 0$, there exists $\gamma \in \Gamma$ such that $Ux(\gamma) \neq 0$, $t \in K_\gamma$ and either $C_0(K_\gamma)$ is* **MLUR** *renormable or $x \in E_\varepsilon(K_\gamma)$. Then $C_0(L)$ is* **MLUR** *renormable.*

The key point of our proof of the above proposition is the following assertion which is a consequence of Proposition 1.

Corollary 1 *Let ε, η be positive real numbers. Let Γ be a well ordered set, L a locally compact scattered space, $\{K_\gamma\}_{\gamma\in\Gamma}$ a family of clopen subsets of L and $U : C_0(L) \to c_0(\Gamma)$ a bounded linear operator. Let $\| \cdot \|_0$ be a* **LUR** *equivalent norm in $c_0(\Gamma)$, A a subset of $D = \{u \in C_0(L) : \|Uu\|_0 = 1\}$, and Δ a map from A into the set of all finite increasing sequences of elements of Γ such that for every $x \in A$ we have $\Delta(x) \subset \{\gamma \in \Gamma : Ux(\gamma) \neq 0\}$ and $\left\|P_{K(x)}x - x\right\|_\infty < \eta$, where $K(x) = \bigcup\{K_\beta : \beta \in \Delta(x)\}$ and $P_{K_\beta}x$ is an ε–strongly extreme (denting) point of conv $\left(P_{K_\beta}A(x)\right)$ (respectively $P_{K_\beta}A(x)$) for $\beta \in \Delta(x)$, where $A(x) = \{y \in A : \Delta(y) = \Delta(x)\}$.*

Then we can write $A = \bigcup_{n\in\mathbb{N}} A_n$ in such a way that all the points of A_n are $2(\varepsilon + \eta)$–strongly extreme (denting) points of conv (A_n) (respectively A_n).

Proof. We assume that $\|z\|_\infty \leq \|z\|_0$ for all $z \in c_0(\Gamma)$. Since U is bounded $|||x||| = \|Ux\|_0$ is a continuous seminorm on X. So for every $x \in D$ there exists $f \in C_0(L)^*$

supporting x with respect to $|||\cdot|||$, i.e.

$$f(x) = \sup_D f = 1 \tag{5}$$

Let us show that for every $x \in D$ and every $\xi > 0$ there exists $\delta = \delta(x, \xi)$ such that for all $y \in D$ with $f(y) > 1 - \delta$ we have

$$\|Ux - Uy\|_0 < \xi. \tag{6}$$

Indeed since $\|\cdot\|_0$ is a **LUR** norm in $c_0(\Gamma)$ we can find $\delta = \delta(x, \xi)$ such that $\|Ux - Uy\|_0 < \xi$ whenever $y \in D$ and $\|(Ux+Uy)/2\|_0 > 1 - \delta/2$. Take now $y \in D$ such that $f(y) > 1 - \delta$. From (5) we get

$$1 - \delta/2 < f(x+y)/2 \leq |||(x+y)/2||| = \|(Ux+Uy)/2\|_0.$$

Now we will split A into a countable number of pieces in such a way that in any of them we can apply Proposition 1. We say that the pair (ℓ, q), where $\ell \in \mathbb{N}$, and $q = (q_i)_{i=1}^m \in \mathbb{Q}^m$ is admissible if $|q_i| > \ell^{-1}$ and $q_{i_1} = q_{i_2}$ whenever $q_{i_1} \leq q_{i_2} < q_{i_1} + \ell^{-1}$. We denote by $A^\sigma_{j,\ell,q}$ the subset of A of all x such that $\|x\|_\infty \leq j$ and there exists an increasing sequence $\alpha_i = \alpha_i(x) \in \Gamma$, $i = 1, 2, \ldots, m$, such that, $\Delta(x) \subset (\alpha_i)_1^m$, $q_i \leq Ux(\alpha_i) < q_i + \ell^{-1}$, $i = 1, 2, \ldots, m$, $\min_i |q_i| > \max\ \{|Ux(\gamma)| :\ \gamma \notin (\alpha_i)_{i=1}^m\} + \ell^{-1}$ and $\sigma = (\sigma_i)_1^m$ with $\sigma_i = 0$ if $\alpha_i \notin \Delta(x)$, $\sigma_i = 1$ if $\alpha_i \in \Delta(x)$. Evidently

$$A = \bigcup \left\{ A^\sigma_{j,\ell,q} :\ \sigma \in \{0,1\}^m,\ j \in \mathbb{N},\ (\ell, q) \text{ is an admissible pair} \right\}.$$

Pick $x \in A^\sigma_{j,\ell,q}$. Take f satisfying (5) and

$$y \in S = \left\{ w \in A^\sigma_{j,\ell,q} :\ f(w) > 1 - \delta\left(x, \ell^{-1}\right) \right\}.$$

From (6) we get that $|Ux(\gamma) - Uy(\gamma)| \leq \|Ux - Uy\|_0 < \ell^{-1}$ for all $\gamma \in \Gamma$. Hence $\alpha_i(x) = \alpha_i(y)$ for $i = 1, 2, \ldots, m$. Taking into account that $(\alpha_i)_1^m$ is an increasing sequence we get $\Delta(x) = \Delta(y)$, so $K(x) = K(y)$ thus

$$\left\|P_{K(x)}y - y\right\|_\infty = \left\|P_{K(y)}y - y\right\|_\infty < \eta.$$

Since $\|P_{M\cup N}u\|_\infty = \max\{\|P_M u\|_\infty, \|P_N u\|_\infty\}$ for every M, $N \subset L$ and every $u \in C_0(L)$ we get that $P_{K(x)}x$ is an ε–strongly extreme (denting) point of conv $\left(P_{K(x)}S\right)$ (respectively $P_{K(x)}S$). Now we can apply Proposition 1 for $A = A^\sigma_{j,\ell,q}$, $T = P_{K(x)}$, f satisfying (5) and $\theta = 1 - \delta\left(x, 1/(2\ell)\right)$. ■

The next assertion is a reformulation of [7, Lemma 5.2.].

For $x \in \ell_\infty(K)$ we set $\omega(x) = \sup\{x(t) - x(s) : s, t \in K\}$.

Lemma 3 *Given $\varepsilon > 0$, let $x \in E_\varepsilon(K)$ and let y, $z \in \ell_\infty(K)$ with $\|x - (y+z)/2\|_\infty < \varepsilon$, $\|y\|_\infty$, $\|z\|_\infty \leq \|x\|_\infty + \varepsilon$, and $\omega(y)$, $\omega(z) \leq \omega(x) + \varepsilon$. Then $\|y - z\|_\infty < 15\varepsilon$.*

Corollary 2 *For any $\varepsilon > 0$ we can write $E_\varepsilon(K) = \bigcup_n E_{n,\varepsilon}$ in such a way that all the points of $E_{n,\varepsilon}$ are 15ε–strongly extreme points of conv$(E_{n,\varepsilon})$.*

Proof. Given q_ω, $q_\infty \in \mathbb{Q}_+$ we set

$$E_{q_\omega,q_\infty,\varepsilon} = \left\{ x \in E_\varepsilon : |\omega(x) - q_\omega| \le \varepsilon/2, \ \left| \|x\|_\infty - q_\infty \right| \le \varepsilon/2 \right\}.$$

Evidently for $x \in E_\varepsilon$ and $u \in \operatorname{conv}(E_{q_\omega,q_\infty,\varepsilon})$ we have $\omega(u) \le q_\omega + \varepsilon/2 \le \omega(x) + \varepsilon$, $\|u\|_\infty \le \|u\| + \varepsilon$. This and the former lemma complete the proof. ■

Proof of Proposition 2. Let Γ' be the set of all $\gamma \in \Gamma$ for which $C(K_\gamma)$ is **MLUR** renormable and let $\Gamma'' = \Gamma \setminus \Gamma'$. Fix $\varepsilon > 0$. From Theorem 1 and Corollary 2 it follows that

$$C(K_\gamma) = \bigcup_n X_n^\gamma, \quad \gamma \in \Gamma'; \qquad E_{\varepsilon/15}(K_\gamma) = \bigcup_{n\in\mathbb{N}} X_n^\gamma, \quad \gamma \in \Gamma'', \tag{7}$$

so that every $x \in X_n^\gamma$ is an ε–strongly extreme point of conv(X_n^γ), $\gamma \in \Gamma$, $n \in \mathbb{N}$.

Assume now that Γ is well ordered . From the assumption of the proposition it follows that for every $x \in C_0(L)$ there exists $\Delta(x) = \{\gamma_i(x)\}_1^m \subset \Gamma$, $\gamma_1(x) < \gamma_2(x) < \ldots < \gamma_m(x)$ such that for every $t \in L$ with $|x(t)| \ge \varepsilon$ we can find k, $1 \le k \le m$ in such a way that $t \in K_{\gamma_i(x)}$ and either $\gamma_i(x) \in \Gamma'$ or $P_{K_{\gamma_i(x)}} x \in E_{\varepsilon/15}\left(K_{\gamma_i(x)}\right)$. Let D be from Corollary 1 and $m \in \mathbb{N}$, $n = (n_i)_1^m$. By $A_{m,n}$ we denote the set of all $x \in D$ such that $\Delta(x) = \{\gamma_i(x)\}_1^m$ and $P_{K_{\gamma_i(x)}} x \in X_{n_i}^{\gamma_i(x)}$, $i = 1, 2, \ldots, m$. Set $K(x) = \bigcup_1^m K_{\gamma_i(x)}$ and $A_{m,n}(x) = \{y \in A_{m,n} : \Delta(x) = \Delta(y)\}$. Then $\left\| P_{K(x)} x - x \right\|_\infty < \varepsilon$. Since $\Delta(x)$ is an increasing sequence we get $\gamma_i(y) = \gamma_i(x)$ for all $y \in A_{m,n}(x)$. Hence according to (7) and Corollary 1 we can write $A_{m,n} = \bigcup A_{m,n}^\ell$ in such a way that all the points of $A_{m,n}^\ell$ are 4ε–strongly extreme points of conv$\left(A_{m,n}^\ell\right)$. Since D is a radial set for $C_0(L)$ and $D = \bigcup \left\{ A_{m,n}^\ell : \ell, m, n \in \mathbb{N} \right\}$ from Remark 2 we get that $C_0(L)$ is **MLUR** renormable. ■

In a similar way from Corollary 1 and [11, Main Theorem] we can deduce Proposition 4.2 of [7].

In order to prove Proposition 1 we need the following

Lemma 4 *Let A, x, f, θ, and S be as in Proposition 1. Then for every convex combination*

$$y = \sum \lambda_i y_i, \ y_i \in A, \ \lambda_i > 0, \ \sum \lambda_i = 1$$

we have

$$\sum \{\lambda_i : \ y_i \notin S\} \le \|f\| \, \|x - y\| / (f(x) - \theta). \tag{8}$$

Proof. Set $I = \{i : \ y_i \in S\}$ then

$$\sum_{i\notin I} \lambda_i f(y_i) \le \theta \sum_{i\notin I} \lambda_i, \quad \sum_{i\in I} \lambda_i f(y_i) \le \left(\sup_A f \right) \sum_{i\in I} \lambda_i = f(x) \sum_{i\in I} \lambda_i.$$

Hence

$$\begin{aligned}\|f\|\,\|x-y\| &\geq f(x-y)=f(x)-f(y)=f(x)-\sum_{i\notin I}\lambda_i f\,(y_i)-\sum_{i\in I}\lambda_i f\,(y_i)\geq\\ &\geq f(x)-\theta\sum_{i\notin I}\lambda_i-f(x)\sum_{i\in I}=f(x)-\theta\sum_{i\notin I}\lambda_i-f(x)\left(1-\sum_{i\notin I}\lambda_i\right)=\\ &=(f(x)-\theta)\sum_{i\notin I}\lambda_i,\end{aligned}$$

which implies (8). ■

Proof of Proposition 1. We can find a $\delta>0$ such that Tx is an (ε,δ)–strongly extreme point of conv (TS). Take

$$a=\sup_A\|w\| \text{ and } \tau=\min\{\varepsilon/8a,\ \delta/(1+4a)\|T\|\}.$$

Let

$y_i, z_i\in A,\ \mu_i,\nu_i>0,\ \sum\mu_i=\sum\nu_i=1,\ \|x-(y+z)/2\|<\tau\min\{1,\ (f(x)-\theta)/\|f\|\},$

where $y=\sum\mu_i y_i$, $z=\sum\nu_i z_i$.

Set $I_y=\{i: y_i\in S\}$, $I_z=\{i: z_i\in S\}$ it follows from Lemma 4 that

$$\frac{1}{2}\left(\sum_{i\notin I_y}\mu_i+\sum_{i\notin I_z}\nu_i\right)<\tau. \tag{9}$$

Set

$$u=\left(\sum_{i\notin I_y}\mu_i\right)x+\sum_{i\in I_y}\mu_i y_i,\quad v=\left(\sum_{i\notin I_z}\nu_i\right)x+\sum_{i\in I_z}\nu_i z_i. \tag{10}$$

Since $\|x\|$, $\|y_i\|$, $\|z_i\|\leq a$ from (9) we get

$$\|u-y\|\leq 4a\tau<\varepsilon/2,\quad \|v-z\|\leq 4a\tau<\varepsilon/2 \tag{11}$$

$$\left\|x-\frac{u+v}{2}\right\|\leq\left\|x-\frac{y+z}{2}\right\|+\left\|\frac{u-y}{2}\right\|+\left\|\frac{v-z}{2}\right\|<\tau+4a\tau\leq\frac{\delta}{\|T\|}. \tag{12}$$

Taking into account (10) we can write

$$u=\sum\lambda_i u_i,\ v=\sum\lambda_i v_i,\ \lambda_i\geq 0,\ \sum\lambda_i=1,\ u_i,v_i\in S.$$

From (12) we get

$$\|Tx-(Tu+Tv)/2\|<\delta.$$

Since Tu, $Tv\in\text{conv}(TS)$ and Tx is an (ε,δ)–strongly extreme point of conv(TS) from the above inequality we get

$$\|Tu-Tv\|<\varepsilon. \tag{13}$$

Since u_i, $v_i \in S$ we have $\|Tu_i - u_i\| < \eta$, $\|Tv_i - v_i\| < \eta$. So

$$\|Tu - u\| \le \sum \lambda_i \|Tu_i - u_i\| < \eta, \quad \|Tv - v\| < \eta.$$

Then from (13) we deduce

$$\|u - v\| \le \|u - Tu\| + \|Tu - Tv\| + \|Tv - v\| \le \varepsilon + 2\eta.$$

This and (11) imply $\|y - z\| < 2(\varepsilon + \eta)$.

The proof of Proposition 1 in the case when Tx is an ε–denting point of TS can be done in a similar way. ■

4 A bidual renorming of the James space.

We start with the following

PROPOSITION 3 *Let X be a Banach space with a monotone shrinking basis (e_i) and let u be an element of X^{**}. Assume that, for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\left\|R_j^{**}z\right\| < \varepsilon$ whenever the element z of X^{**} and the natural number j satisfy*

$$\left\|R_j^{**}(u \pm z)\right\| - \left\|R_j^{**}u\right\| < \delta(\varepsilon) \tag{14}$$

where $R_j x = \sum_{i>j} f_i(x)e_i$ and (f_i) is the conjugate system to the basis (e_i).

Then there exists an equivalent norm $|\cdot|$ in X such that all the points of $S_{(X,|\cdot|)^{**}} \cap Y$ are strongly extreme of $B_{(X,|\cdot|)^{**}}$, where $Y = span\{u, X\}$.

Proof. For $x \in X$ set

$$|x| = \left(\|x\|^2 + \sum_{j\ge 1} 2^{-j}\left(\|R_j x\|^2 + (f_j(x)/\|f_j\|)^2\right)\right)^{1/2}.$$

Since the basis (e_i) is monotone and shrinking we have for all $z \in X^{**}$ (see e.g. [10, p. 8]) that

$$\lim_{\ell} \|P_\ell^{**}z\| = \|z\|, \tag{15}$$

where $P_j x = x - R_j x$ for $x \in X$.

Since (e_i) is a monotone basis with respect to $|\cdot|$ replacing in (15) z by $R_j^{**}z$ we get

$$|z| = \lim_{\ell} |P_\ell^{**}z| = \left(\|z\|^2 + \sum_{j\ge 1} 2^{-j}\left(\|R_j z\|^2 + \left(z\left(f_j\right)/\|f_j\|\right)^2\right)\right)^{1/2}$$

for all $z \in X^{**}$.

Pick $y \in Y$. Then $y = x + bu$ for some $x \in X$ and $b \in \mathbb{R}$. let $z_k \in X^{**}$ and

$$\lim_k |y \pm z_k| = |y|.$$

By convexity arguments we have

$$\lim_k \left\| R_j^{**} (y \pm z_k) \right\| = \left\| R_j^{**} y \right\|, \quad j = 1, 2, \dots \tag{16}$$

and

$$\lim_k f_j (z_k) = 0, \quad j = 1, 2, \dots$$

This implies

$$\lim_k \left\| P_j^{**} z_k \right\| = 0, \quad j = 1, 2, \dots \tag{17}$$

If $b = 0$ then $y \in X$ and $\lim_j \left\| R_j^{**} y \right\| = 0$. Since for all j and k we have

$$\|z_k\| \leq \left\| P_j^{**} z_k \right\| + \left\| R_j^{**} (y + z_k) \right\| + \|R_j y\|$$

from (16) and (17) we get

$$\limsup_k \|z_k\| \leq 2 \|R_j y\|,$$

so $\lim_k \|z_k\| = 0$.

Assume now that $b \neq 0$. By homogeneity we may assume $b = 1$. Suppose that for all k

$$\|z_k\| \geq 2\varepsilon > 0. \tag{18}$$

Since $x \in X$ we can find m such that

$$\|R_m x\| < \delta(\varepsilon)/4. \tag{19}$$

From (17) it follows that there exists n such that for $k > n$ we have $\|P_m^{**} z_k\| < \varepsilon$. Then from (18) we have for $k > n$

$$\|R_m^{**} z_k\| \geq \|z_k\| - \|P_m^{**} z_k\| \geq \varepsilon.$$

From (14) we deduce that for $k > n$

$$\max_{\alpha = \pm 1} \|R_m^{**} (u + \alpha z_k)\| \geq \|R_m^{**} u\| + \delta(\varepsilon). \tag{20}$$

From (19) we have for all k

$$\|R_m^{**} (y + \alpha z_k)\| \geq \|R_m^{**} (u + \alpha z_k)\| - \|R_m^{**} x\| \geq \|R_m^{**} (u + \alpha z_k)\| - \delta(\varepsilon)/4,$$

$$\|R_m^{**} u\| \geq \|R_m^{**} y\| - \|R_m^{**} x\| \geq \|R_m^{**} y\| - \delta(\varepsilon)/4.$$

The last two inequalities and (20) imply that for $k > n$

$$\max_{\alpha = \pm 1} \|R_m^{**} (y + \alpha z_k)\| \geq \|R_m^{**} y\| + \delta(\varepsilon)/2,$$

which contradicts (16). ■

COROLLARY 3 *The James space J has an equivalent norm $|\cdot|$ such that $(J,|\cdot|)^{**}$ is* **MLUR**.

Proof. Given $x=(x_i)_1^\infty \in J$ let us consider the norm

$$\|x\| = \sup\left\{\left(x_{i_m}^2 + \sum_{j=1}^m \left(x_{i_{j-1}} - x_{i_j}\right)^2\right)^{1/2} : 1 \le i_0 < i_1 < \ldots < i_m\right\}.$$

Taking into account that $x_i \to 0$ it is easy to see that $\|\cdot\|$ is an equivalent norm in J.

For $x=(x_i)_1^\infty \in J$, set $P_j x = (x_1, x_2, \ldots, x_j, 0, 0, \ldots)$ and $R_j x = x - P_j x$. Since the unit vector basis in $(J, \|\cdot\|)$ is monotone and shrinking we have (see e.g. [10, p. 8]) for $z = (z_i)_1^\infty \in J^{**}$ that

$$\|z\| =$$

$$(21) \qquad = \lim_\ell \|P_\ell^{**} z\| = \sup\left\{\left(z_{i_m}^2 + \sum_{j=1}^m \left(z_{i_{j-1}} - z_{i_j}\right)^2\right)^{1/2} : 1 \le i_0 < i_1 < \ldots < i_m\right\}.$$

It is known that $J^{**} = \text{span}\{u, J\}$ where $u = (1,1,\ldots)$. From (21) it follows that for every $z \in J^{**}$ and $j \in \mathbb{N}$ we have

$$(22) \qquad \left\|R_j^{**}(u+z)\right\|^2 + \left\|R_j^{**}(u-z)\right\|^2 \ge 2\left(\left\|R_j^{**}u\right\|^2 + \left\|R_j^{**}z\right\|^2\right).$$

Now we show that u satisfies (14). Given $\varepsilon > 0$ we set $\delta(\varepsilon) = \min\{\varepsilon^2/2, 1\}$ and assume that for $z \in J^{**}$ and $j \in \mathbb{N}$ we have

$$(23) \qquad \max_{\alpha=\pm 1}\left(\left\|R_j^{**}(u+\alpha z)\right\| - \left\|R_j^{**}u\right\|\right) < \delta(\varepsilon).$$

Taking into account that $\left\|R_j^{**}u\right\| = 1$ for all j we deduce from (22) and (23)

$$2\left\|R_j^{**}z\right\|^2 \le \max_{\alpha=\pm 1}\left\{\left\|R_j^{**}(u+\alpha z)\right\|^2 - \left\|R_j^{**}u\right\|^2\right\} < \delta(\varepsilon)\left(\left\|R_j^{**}z\right\| + 2\right).$$

It is easy to see that this implies $\left\|R_j^{**}z\right\| < \varepsilon$. Now we can apply the previous Proposition. ■

5 Acknowledgments

The first and the fourth authors have been partially supported by DGES, project PB96-0758, the second author has been partially supported by DGES, project PB96-0758 and DGICYT, project PB95–125, the third author has been partially supported by NFSR of Bulgaria, Grant MM–808/98, by a grant of the University of Valencia and by Fundación Séneca Grant 00 404/CV/99.

References

[1] G. A. Alexandrov, V. D. Babev, 'Banach spaces not isomorphic to weakly midpoint locally uniformly rotund spaces', *Comptes rendus de l'Academie bulgare des Sciences.* **41**, (1988), 29–32.

[2] G. A. Alexandrov, I. P. Dimitrov, 'On equivalent weakly midpoint locally uniformly rotund renormings of the space ℓ_∞', *Math. and math. education, proceedings of the 14$^{\text{th}}$ Spring Conference of the Union of Bulg. Mathematicians.* Sunny Beach, 1985, 189–191. (Russian)

[3] R. Deville, G. Godefroy, and V. Zizler, *Smoothness and renorming in Banach spaces.* Pitman Monographs and Surveys in Pure and Appl. Math. 64, Longman Scientific & Technical, Longman House, Burnt Mill, Harlow. 1993.

[4] J. Diestel *Sequences and series in Banach spaces,* Springer–Verlag, Berlin 1984.

[5] P. N. Dowling, Z. Hu, and M. A. Smith, 'MLUR renormings of Banach spaces', *Pacific J. Math.*, **170**,(1995), 473–482.

[6] P. Hájek, 'Dual renormings of Banach spaces', *Commentationes Mathematicae Universitatis Carolinae*, **37**, (1996), 241–253.

[7] R. Haydon, 'Trees in renorming theory', *Proc. London Math. Soc.* **78**, (1999), 541–585.

[8] J. E. Jayne, I. Namioka, and C. A. Rogers, 'σ–fragmentable Banach spaces', *Mathematika*, **39**, (1992), 161–188, 197–215.

[9] K. Kunen, H. Rosenthal, 'Martingale proofs of some geometrical results in Banach space theory', *Pacific J. Math.*, **100**, (1982), 153–175.

[10] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces I, Sequence spaces.* Springer–Verlag, Berlin 1977.

[11] A. Moltó, J. Orihuela, and S. Troyanski, 'Locally uniformly rotund renorming and fragmentability', *Proc. London Math. Soc.*, **75**, (1997), 619–640.

[12] M. Raja, 'On locally uniformly rotund norms', *Mathematika.* (To appear)

[13] S. Troyanski, 'On locally uniformly convex and differentiable norms in certain non separable Banach spaces', *Studia Math.*, **37**, (1971), 173–180.

A. Moltó & M. Valdivia
Departamento de
Análisis Matemático
Facultad de Matemáticas
Universidad de Valencia
Dr. Moliner 50
46100 Burjasot (Valencia)
Spain

J. Orihuela
Departamento de
Matemáticas
Universidad de Murcia
Campus de Espinardo
30100 Espinardo
Murcia
Spain

S. Troyanski
Department of
Mathematics
and Informatics
Sofia University
5, James Bourchier Blvd.
1126 Sofia
Bulgaria