NORMING SETS AND INTEGRATION WITH RESPECT TO VECTOR MEASURES

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Abstract. Let ν be a countably additive measure defined on a measurable space (Ω, Σ) and taking values in a Banach space X. Let $f : \Omega \to \mathbb{R}$ be a measurable function. In order to check the integrability (respectively, weak integrability) of f with respect to ν it is sometimes enough to test on a norming set $\Lambda \subset X^*$. In this paper we show that this is the case when Λ is a James boundary for B_{X^*} (respectively, Λ is weak*-thick). Some examples and applications are given as well.

1. Introduction

The modern theory of integration of scalar functions with respect to vector measures was introduced by Bartle, Dunford and Schwartz [2] in order to provide an analogue of Riesz's representation theorem for weakly compact operators defined on Banach spaces of continuous functions. After the seminal contributions by Lewis [10, 11] and Kluvanek and Knowles [9], many authors have worked on this topic and recently the spaces of scalar functions integrable with respect to vector measures have shown to play an important role within the theory of Banach lattices, see e.g. [3], [4] and [7]. Whereas the original definition of the integral given by Bartle, Dunford and Schwartz is based on approximation by simple functions, Lewis' equivalent approach uses scalar measures through a barycentric formula, as follows.

Let (Ω, Σ) be a measurable space, X a Banach space with dual X^* and $\nu : \Sigma \to X$ a countably additive measure. A Σ-measurable function $f : \Omega \to \mathbb{R}$ is called *integrable* (with respect to ν) if:

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- f is *weakly integrable*, i.e. $f \in \mathcal{L}^1(\langle\nu, x^*\rangle)$ for every $x^* \in X^*$, where $\langle\nu, x^*\rangle : \Sigma \to \mathbb{R}$ is the countably additive measure given by $\langle\nu, x^*\rangle(A) := \langle\nu(A), x^*\rangle$; and
- for each $A \in \Sigma$ there is a vector $\int_A f d\nu \in X$ such that

$$\left\langle \int_A f d\nu, x^* \right\rangle = \int_A f d\langle\nu, x^*\rangle \quad \text{for every } x^* \in X^*.$$

The set of all integrable (resp. weakly integrable) functions $f : \Omega \to \mathbb{R}$ is denoted by $\mathcal{L}^1(\nu)$ (resp. $\mathcal{L}^1_w(\nu)$).

A typical situation in Functional Analysis is the absence of a "good" description for the dual of certain Banach spaces. Sometimes we can deal only with a particular subset of the dual and it can be useful to obtain "global" information from that smaller set of functionals. This idea can be nicely applied to the setting of integration with respect to vector measures. Given a norming set $\Lambda \subset X^*$, we introduce the following terminology:

- $\mathcal{L}^1_\Lambda(\nu)$ is the set of all Σ-measurable functions $f : \Omega \to \mathbb{R}$ such that $f \in \mathcal{L}^1(\langle\nu, x^*\rangle)$ for every $x^* \in \Lambda$.
- $\mathcal{L}^{1,s}_\Lambda(\nu)$ is the set of all $f \in \mathcal{L}^1_\Lambda(\nu)$ with the following property: for each $A \in \Sigma$ there is $\xi_{f,\Lambda}(A) \in X$ such that

$$\langle \xi_{f,\Lambda}(A), x^* \rangle = \int_A f d\langle\nu, x^*\rangle \quad \text{for every } x^* \in \Lambda.$$

In this paper we discuss under which conditions the equalities $\mathcal{L}^1(\nu) = \mathcal{L}^{1,s}_\Lambda(\nu)$ and $\mathcal{L}^1_w(\nu) = \mathcal{L}^1_\Lambda(\nu)$ hold. The last one is satisfied whenever Λ is *weak*-thick* (in the sense of Fonf [8]), see Theorem 2.2. Nygaard [12, 13, 14], also together with Abrahamsen and Poldvere [1], has extended recently Fonf's ideas showing that *weak*-thick* sets can be used to test properties like boundedness, summability and integrability (for vector functions with respect to non-negative finite measures).

As regards the equality $\mathcal{L}^1(\nu) = \mathcal{L}^{1,s}_\Lambda(\nu)$, we prove that a function $f \in \mathcal{L}^{1,s}_\Lambda(\nu)$ belongs to $\mathcal{L}^1(\nu)$ if and only if the mapping $\xi_{f,\Lambda} : \Sigma \to X$ is countably additive, see Theorem 2.5. This fact paves the way to deduce that $\mathcal{L}^1(\nu) = \mathcal{L}^{1,s}_\Lambda(\nu)$ whenever X contains no isomorphic copy of ℓ_∞ or Λ is a James boundary for B_{X^*} (see the comments after Proposition 2.8). Some results in this fashion were obtained by Thomas [18] within his theory of integration of scalar functions with respect to a Radon vector measure.

In general, for any norming set $\Lambda \subset X^*$ we have

$$\mathcal{L}^1(\nu) \subset \mathcal{L}^{1,s}_\Lambda(\nu) \subset \mathcal{L}^1_w(\nu) \subset \mathcal{L}^1_\Lambda(\nu),$$

see Theorem 2.5. Our methods allow us to prove that $\mathcal{L}^1(\nu) = \mathcal{L}^1_\Lambda(\nu)$ provided that X contains no isomorphic copy of c_0 and Λ is a James boundary for B_{X^*} (Corollary 2.4). We also show that, if X is the dual of another Banach space Y, then $\mathcal{L}^{1,s}_Y(\nu) = \mathcal{L}^1_w(\nu) = \mathcal{L}^1_Y(\nu)$ (Theorem 2.10). We finish the paper with some examples making clear that in the previous chain of inclusions all combinations of "$\subsetneq$" and "$=$" are possible.

All unexplained terminology can be found in our standard references [5] and [6]. The closed unit ball of X^* is denoted by B_{X^*} and the symbol w^* stands for the weak* topology on X^*. The evaluation of a functional $x^* \in X^*$ at $x \in X$ is denoted by $\langle x, x^*\rangle$. The semivariation of ν is the mapping $\|\nu\| : \Sigma \longrightarrow [0,\infty)$ defined by

$$\|\nu\|(A) := \sup\{|\langle \nu, x^*\rangle|(A) : \ x^* \in B_{X^*}\}, \quad A \in \Sigma,$$

where $|\langle \nu, x^*\rangle|$ denotes the total variation measure of $\langle \nu, x^*\rangle$. A set $\Lambda \subset X^*$ is called *norming* if there is $\lambda \geq 1$ such that

$$\|x\| \leq \lambda \cdot \sup\{\langle x, x^*\rangle : \ x^* \in \operatorname{span}(\Lambda) \cap B_{X^*}\} \quad \text{for every } x \in X$$

(we sometimes say that Λ is λ-*norming*). A set $B \subset B_{X^*}$ is called a *James boundary* for B_{X^*} if for every $x \in X$ there is $x^* \in B$ such that $\|x\| = \langle x, x^*\rangle$. The classical example of James boundary is given by the set $\operatorname{Ext}(B_{X^*})$ of extreme points of B_{X^*}, cf. [6, Fact 3.45]. A set $T \subset X^*$ is w^*-*thin* if we can write $T = \bigcup_{n=1}^\infty T_n$, where $T_n \subset T_{n+1}$ and

$$\inf_{\|x\|=1} \sup_{x^* \in T_n} |\langle x, x^*\rangle| = 0.$$

A subset of X^* is w^*-*thick* if it is not w^*-thin. Clearly, every w^*-thick set is norming. A simple example of a norming set (even James boundary) which is not w^*-thick is given by the set $\{e^*_n : n \in \mathbb{N}\} \subset c_0^* = \ell^1$ of all "coordinate projections" on c_0.

2. The results

We begin by discussing the relationship between the spaces $\mathcal{L}^1_w(\nu)$ and $\mathcal{L}^1_\Lambda(\nu)$ for a norming set $\Lambda \subset X^*$. Given $f \in \mathcal{L}^1_\Lambda(\nu)$, we define

$$\|f\|_{\mathcal{L}^1_\Lambda(\nu)} := \sup_{x^* \in \operatorname{span}(\Lambda) \cap B_{X^*}} \int_\Omega |f| d|\langle \nu, x^*\rangle| \in [0,\infty].$$

It is known that $\|f\|_{\mathcal{L}^1_w(\nu)} := \|f\|_{\mathcal{L}^1_{X^*}(\nu)} < \infty$ for every $f \in \mathcal{L}^1_w(\nu)$, see [10, p. 163] (cf. [17, Proposition 2]). Clearly, $\|\cdot\|_{\mathcal{L}^1_w(\nu)}$ is a seminorm on $\mathcal{L}^1_w(\nu)$.

Our starting point is the following characterization.

Theorem 2.1. *Let $f : \Omega \to \mathbb{R}$ be a function. The following conditions are equivalent:*

(i) $f \in \mathcal{L}^1_w(\nu)$.

(ii) *There exists a norming set $\Lambda \subset X^*$ such that $f \in \mathcal{L}^1_\Lambda(\nu)$ and $\|f\|_{\mathcal{L}^1_\Lambda(\nu)} < \infty$.*

In this case, for any λ-norming set $\Lambda \subset X^*$, we have

$$\|f\|_{\mathcal{L}^1_\Lambda(\nu)} \leq \|f\|_{\mathcal{L}^1_w(\nu)} \leq 2\lambda \|f\|_{\mathcal{L}^1_\Lambda(\nu)}.$$

Proof. For the implication (i)⇒(ii), just take $\Lambda = X^*$.

The proof of (ii)⇒(i) is as follows. Assume that Λ is λ-norming for some $\lambda \geq 1$. Fix $\eta > 0$. Since f is Σ-measurable, there is a function of the form $g = \sum_{n=1}^{\infty} a_n \chi_{A_n}$ (where $a_n \in \mathbb{R}$ for every $n \in \mathbb{N}$ and the A_n's belong to Σ and are pairwise disjoint) such that $|f - g| \leq \eta$ pointwise. Clearly, we have $f - g \in \mathcal{L}^1_w(\nu)$. *We claim that $g \in \mathcal{L}^1_w(\nu)$ as well.* Indeed, fix $x_0^* \in B_{X^*}$, $N \in \mathbb{N}$ and $\varepsilon > 0$. For each $1 \leq n \leq N$ we can find $D_n \subset A_n$, $D_n \in \Sigma$, such that

$$|a_n| \cdot |\langle \nu, x_0^* \rangle|(A_n) \leq 2|a_n| \cdot |\langle \nu, x_0^* \rangle(D_n)| + \frac{\varepsilon}{N}. \tag{1}$$

On the other hand, since Λ is λ-norming, the Hahn-Banach separation theorem ensures that

$$B_{X^*} \subset \lambda \cdot \overline{\operatorname{span}(\Lambda) \cap B_{X^*}}^{w^*}.$$

Therefore, we can find $x^* \in \operatorname{span}(\Lambda) \cap B_{X^*}$ such that

$$|a_n| \cdot |\langle \nu(D_n), \lambda x^* \rangle - \langle \nu(D_n), x_0^* \rangle| \leq \frac{\varepsilon}{N} \quad \text{for every } 1 \leq n \leq N. \tag{2}$$

Observe that $g \in \mathcal{L}^1(\langle \nu, \lambda x^* \rangle)$ and

$$\begin{aligned} \int_\Omega |g| d|\langle \nu, \lambda x^* \rangle| &\leq \int_\Omega |f - g| d|\lambda \langle \nu, x^* \rangle| + \int_\Omega |f| d|\lambda \langle \nu, x^* \rangle| \\ &\leq \eta \lambda \|\nu\|(\Omega) + \lambda \|f\|_{\mathcal{L}^1_\Lambda(\nu)}. \end{aligned} \tag{3}$$

By putting together (1), (2) and (3) we obtain

$$\begin{aligned} \sum_{n=1}^{N} |a_n| \cdot |\langle \nu, x_0^* \rangle|(A_n) &\leq \sum_{n=1}^{N} 2|a_n| \cdot |\langle \nu, x_0^* \rangle(D_n)| + \varepsilon \\ &\leq \sum_{n=1}^{N} 2|a_n| \cdot |\lambda \langle \nu, x^* \rangle(D_n)| + 3\varepsilon \leq \sum_{n=1}^{N} 2|a_n| \cdot |\lambda \langle \nu, x^* \rangle|(A_n) + 3\varepsilon \\ &\leq 2 \int_\Omega |g| d|\langle \nu, \lambda x^* \rangle| + 3\varepsilon \leq 2\eta\lambda \|\nu\|(\Omega) + 2\lambda \|f\|_{\mathcal{L}^1_\Lambda(\nu)} + 3\varepsilon. \end{aligned}$$

As $N \in \mathbb{N}$ and $\varepsilon > 0$ are arbitrary, it follows that

$$\int_\Omega |g| d|\langle \nu, x_0^* \rangle| \leq 2\eta\lambda \|\nu\|(\Omega) + 2\lambda \|f\|_{\mathcal{L}^1_\Lambda(\nu)} < \infty.$$

Therefore, $g \in \mathcal{L}^1_w(\nu)$ and $\|g\|_{\mathcal{L}^1_w(\nu)} \leq 2\eta\lambda\|\nu\|(\Omega) + 2\lambda\|f\|_{\mathcal{L}^1_\Lambda(\nu)}$. Finally, notice that $(f-g)+g = f \in \mathcal{L}^1_w(\nu)$ and

$$\|f\|_{\mathcal{L}^1_w(\nu)} \leq \|f-g\|_{\mathcal{L}^1_w(\nu)} + \|g\|_{\mathcal{L}^1_w(\nu)} \leq \eta\|\nu\|(\Omega) + 2\eta\lambda\|\nu\|(\Omega) + 2\lambda\|f\|_{\mathcal{L}^1_\Lambda(\nu)}.$$

As $\eta > 0$ was arbitrary, the proof is over. □

The norming sets $\Lambda \subset X^*$ for which the equality $\mathcal{L}^1_w(\vartheta) = \mathcal{L}^1_\Lambda(\vartheta)$ holds for *any* countably additive X-valued measure ϑ can be characterized as those which are w^*-thick. To this end, we will apply a result of Abrahamsen, Nygaard and Poldvere (see [1, Corollary 2.4]) saying that *a set $\Lambda \subset X^*$ is w^*-thick if and only if every series $\sum_n x_n$ in X satisfying $\sum_{n=1}^\infty |\langle x_n, x^*\rangle| < \infty$ for every $x^* \in \Lambda$ is weakly unconditionally Cauchy (i.e. $\sum_{n=1}^\infty |\langle x_n, x^*\rangle| < \infty$ for every $x^* \in X^*$).*

Theorem 2.2. *Let $\Lambda \subset X^*$ be a w^*-thick set. Then $\mathcal{L}^1_w(\nu) = \mathcal{L}^1_\Lambda(\nu)$.*

Proof. Fix $f \in \mathcal{L}^1_\Lambda(\nu)$. Since f is Σ-measurable, there exists a function of the form $g = \sum_{n=1}^\infty a_n \chi_{A_n}$ (where $a_n \in \mathbb{R}$ for every $n \in \mathbb{N}$ and the A_n's belong to Σ and are pairwise disjoint) such that $|f-g| \leq 1$ pointwise. Of course, we have $f-g \in \mathcal{L}^1_w(\nu)$ and we only have to check that $g \in \mathcal{L}^1_w(\nu)$. To this end, fix $x_0^* \in B_{X^*}$ and for each $n \in \mathbb{N}$ choose $D_n \subset A_n$, $D_n \in \Sigma$, such that

$$|a_n| \cdot |\langle \nu, x_0^*\rangle|(A_n) \leq 2|a_n| \cdot |\langle \nu, x_0^*\rangle(D_n)| + \frac{1}{2^n}.$$

Since $f - (f-g) = g \in \mathcal{L}^1_\Lambda(\nu)$, we have

$$\sum_{n=1}^\infty |\langle a_n \nu(D_n), x^*\rangle| \leq \sum_{n=1}^\infty |a_n| \cdot |\langle \nu, x^*\rangle|(A_n) = \int_\Omega |g| d|\langle \nu, x^*\rangle| < \infty$$

for every $x^* \in \Lambda$. Bearing in mind that Λ is w^*-thick, an appeal to the result of Abrahamsen, Nygaard and Poldvere ensures us that

$$\int_\Omega |g| d|\langle \nu, x_0^*\rangle| = \sum_{n=1}^\infty |a_n| \cdot |\langle \nu, x_0^*\rangle|(A_n) \leq 2 \cdot \left(\sum_{n=1}^\infty |\langle a_n \nu(D_n), x_0^*\rangle| \right) + 1 < \infty,$$

and the proof is over. □

Proposition 2.3. *Let $\Lambda \subset X^*$ be a norming set such that the equality $\mathcal{L}^1_w(\vartheta) = \mathcal{L}^1_\Lambda(\vartheta)$ holds for every countably additive X-valued measure ϑ. Then Λ is w^*-thick.*

Proof. Suppose if possible otherwise. By the aforementioned result of Abrahamsen, Nygaard and Poldvere, there is a sequence (x_n) in X such that $\sum_{n=1}^\infty |\langle x_n, x^*\rangle| < \infty$ for every $x^* \in \Lambda$, but $\sum_{n=1}^\infty |\langle x_n, x_0^*\rangle| = \infty$ for some $x_0^* \in X^*$. Define

$$y_n := \frac{1}{2^n(\|x_n\|+1)} \cdot x_n \in X \quad \text{for every } n \in \mathbb{N}.$$

Since the series $\sum_n y_n$ is unconditionally convergent, the mapping

$$\vartheta : \mathcal{P}(\mathbb{N}) \to X, \quad \vartheta(A) := \sum_{n\in A} y_n,$$

is a countably additive measure. It is now clear that the function $f : \mathbb{N} \to \mathbb{R}$ given by $f(n) := 2^n(\|x_n\| + 1)$ satisfies

$$\int_{\mathbb{N}} |f|d|\langle \nu, x^*\rangle| = \sum_{n=1}^{\infty} |\langle x_n, x^*\rangle|$$

for every $x^* \in X^*$. It follows that $f \in \mathcal{L}^1_\Lambda(\vartheta)$ but $f \notin \mathcal{L}^1_w(\vartheta)$. □

The proof of the previous proposition, together with Theorems 2.1 and 2.2, make clear that *a norming set $\Lambda \subset X^*$ is w^*-thick if and only if every series $\sum_n x_n$ in X satisfying $\sum_{n=1}^{\infty} |\langle x_n, x^*\rangle| < \infty$ for every $x^* \in \Lambda$ has the property that*

$$\sup_{x^*\in \mathrm{span}(\Lambda)\cap B_{X^*}} \sum_{n=1}^{\infty} |\langle x_n, x^*\rangle| < \infty.$$

A deep result of Fonf [8] (cf. [13, Theorem 6.2] or [14, Theorem 2.3]) states that *if X does not contain subspaces isomorphic to c_0, then every James boundary $B \subset B_{X^*}$ is w^*-thick.* On the other hand, Lewis [11] (cf. [9, Theorem 1, p. 31]) showed that the absence of isomorphic copies of c_0 implies the equality $\mathcal{L}^1(\nu) = \mathcal{L}^1_w(\nu)$. As a consequence we get:

Corollary 2.4. *Suppose X does not contain subspaces isomorphic to c_0 and let $B \subset B_{X^*}$ be a James boundary. Then $\mathcal{L}^1(\nu) = \mathcal{L}^1_B(\nu)$.*

We now focus on $\mathcal{L}^{1,s}_\Lambda(\nu)$ for a norming set $\Lambda \subset X^*$. We start by showing that this space lies between $\mathcal{L}^1(\nu)$ and $\mathcal{L}^1_w(\nu)$. Recall that, given $f \in \mathcal{L}^1_w(\nu)$, for each $A \in \Sigma$ there is $\nu_f(A) \in X^{**}$ such that

$$\langle x^*, \nu_f(A)\rangle = \int_A f d\langle \nu, x^*\rangle \quad \text{for every } x^* \in X^*,$$

see [10, p. 163] (cf. [17, Corollary 3]).

Theorem 2.5. *Let $\Lambda \subset X^*$ be a norming set and $f \in \mathcal{L}^{1,s}_\Lambda(\nu)$. Then:*

(i) *The mapping $\xi_{f,\Lambda} : \Sigma \to X$ is a bounded finitely additive measure.*
(ii) *$f \in \mathcal{L}^1_w(\nu)$.*
(iii) *The following conditions are equivalent:*
  (a) *$f \in \mathcal{L}^1(\nu)$;*
  (b) *$\xi_{f,\Lambda}(A) = \nu_f(A)$ for every $A \in \Sigma$;*
  (c) *$\xi_{f,\Lambda}$ is countably additive.*

Proof. (i) follows directly from a result of Dieudonné and Grothendieck (cf. [5, Corollary 3, p. 16]), because the composition $\langle \xi_{f,\Lambda}, x^* \rangle$ is bounded and finitely additive (in fact, it is countably additive) for every $x^* \in \Lambda$, and Λ separates the points of X. Moreover, we have

$$\|f\|_{\mathcal{L}^1_\Lambda(\nu)} = \sup_{x^* \in \operatorname{span}(\Lambda) \cap B_{X^*}} |\langle \xi_{f,\Lambda}, x^* \rangle|(\Omega) \leq \|\xi_{f,\Lambda}\|(\Omega) < \infty,$$

hence Theorem 2.1 can be applied to conclude that $f \in \mathcal{L}^1_w(\nu)$.

Let us turn to the proof of (iii). Assume that Λ is λ-norming for some $\lambda \geq 1$.

(a)⇒(b) If f is integrable then ν_f takes its values in X and, since Λ separates the points of X, it follows from the very definitions that $\nu_f = \xi_{f,\Lambda}$.

(b)⇒(c) Since $\nu_f = \xi_{f,\Lambda}$ takes its values in X, the Orlicz-Pettis theorem (cf. [5, Corollary 4, p. 22]) ensures that it is countably additive.

(c)⇒(a) Given $A \in \Sigma$, we can apply Theorem 2.1 to the restriction $f|_A$ obtaining

$$\begin{aligned} (4) \quad \|\nu_f(A)\| &= \sup_{x^* \in B_{X^*}} \langle x^*, \nu_f(A) \rangle = \sup_{x^* \in B_{X^*}} \int_A f d\langle \nu, x^* \rangle \\ &\leq 2\lambda \cdot \left(\sup_{x^* \in \operatorname{span}(\Lambda) \cap B_{X^*}} \int_A |f| d|\langle \nu, x^* \rangle| \right) \leq 2\lambda \cdot \|\xi_{f,\Lambda}\|(A). \end{aligned}$$

Let μ be a non-negative finite measure on Σ such that $\nu \ll \mu$ (i.e. $\lim_{\mu(A)\to 0} \nu(A) = 0$), cf. [5, Corollary 6, p. 14]. Since $\xi_{f,\Lambda}$ is countably additive and vanishes on all μ-null sets, we have $\xi_{f,\Lambda} \ll \mu$ (cf. [5, Theorem 1, p. 10]) and we can use (4) to deduce that $\lim_{\mu(A)\to 0} \nu_f(A) = 0$. It follows that $f \in \mathcal{L}^1(\nu)$, see [10, Theorem 2.6]. The proof is finished. □

Given a norming set $\Lambda \subset X^*$, the linear space $L^{1,s}_\Lambda(\nu)$ (obtained from $\mathcal{L}^{1,s}_\Lambda(\nu)$ by identifying functions which coincide $\|\nu\|$-a.e.) is a normed space when endowed with any of the two equivalent norms $\|\cdot\|_{L^1_\Lambda(\nu)}$ and $\|\cdot\|_{L^1_w(\nu)}$ (Theorem 2.1). In fact, we have:

Proposition 2.6. *Let $\Lambda \subset X^*$ be a norming set. Then $L^{1,s}_\Lambda(\nu)$ is a Banach space.*

Proof. Since $(L^1_w(\nu), \|\cdot\|_{L^1_w(\nu)})$ is complete (see [17, Theorem 9]), it suffices to check that $L^{1,s}_\Lambda(\nu)$ is a closed subspace of $L^1_w(\nu)$. To this end, take a sequence (f_n) in $\mathcal{L}^{1,s}_\Lambda(\nu)$ that $\|\cdot\|_{\mathcal{L}^1_w(\nu)}$-converges to some $f \in \mathcal{L}^1_w(\nu)$. In order to check that $f \in \mathcal{L}^{1,s}_\Lambda(\nu)$, fix $A \in \Sigma$ and observe that for every $n, m \in \mathbb{N}$ we have

$$\|\xi_{f_n,\Lambda}(A) - \xi_{f_m,\Lambda}(A)\| \leq \lambda \cdot \|f_n - f_m\|_{\mathcal{L}^1_\Lambda(\nu)} \leq \lambda \cdot \|f_n - f_m\|_{\mathcal{L}^1_w(\nu)},$$

where $\lambda \geq 1$ is a constant such that Λ is λ-norming. Therefore, there exists $\lim_n \xi_{f_n,\Lambda}(A) = x_A \in X$ for the norm topology. Finally, given $x^* \in \operatorname{span}(\Lambda) \cap B_{X^*}$ we have

$$\left|\langle \xi_{f_n,\Lambda}(A), x^*\rangle - \int_A f d\langle \nu, x^*\rangle\right| \leq \|f_n - f\|_{\mathcal{L}^1_w(\nu)} \to 0,$$

hence $\langle x_A, x^*\rangle = \int_A f d\langle \nu, x^*\rangle$. It follows that $f \in \mathcal{L}^{1,s}_\Lambda(\nu)$. □

The following proposition shows that $\mathcal{L}^{1,s}_\Lambda(\nu)$ is an *ideal* of the lattice of all Σ-measurable real-valued functions on Ω (with the $\|\nu\|$-a.e. order).

Proposition 2.7. *Let $\Lambda \subset X^*$ be a norming set. Let $f \in \mathcal{L}^{1,s}_\Lambda(\nu)$ and $g : \Omega \to \mathbb{R}$ a Σ-measurable function such that $|g| \leq |f|$ $\|\nu\|$-a.e. Then $g \in \mathcal{L}^{1,s}_\Lambda(\nu)$.*

Proof. We can assume without loss of generality that $|g(t)| \leq |f(t)|$ for every $t \in \Omega$. Define $h : \Omega \to \mathbb{R}$ by $h(t) := \frac{g(t)}{f(t)}$ if $f(t) \neq 0$, $h(t) := 0$ otherwise. Since h is bounded and Σ-measurable, there is a sequence of simple functions $s_n : \Omega \to \mathbb{R}$ such that $\lim_n \|s_n - h\|_\infty = 0$. Clearly, $s_n f \in \mathcal{L}^{1,s}_\Lambda(\nu)$ for every $n \in \mathbb{N}$. Since

$$\begin{aligned}\|s_n f - g\|_{\mathcal{L}^1_w(\nu)} &= \|s_n f - hf\|_{\mathcal{L}^1_w(\nu)} \\ &= \sup_{x^* \in B_{X^*}} \int_\Omega |s_n f - hf| d|\langle \nu, x^*\rangle| \leq \|s_n - h\|_\infty \cdot \|f\|_{\mathcal{L}^1_w(\nu)} \to 0,\end{aligned}$$

an appeal to Proposition 2.6 establishes that $g \in \mathcal{L}^{1,s}_\Lambda(\nu)$. □

We next provide some conditions ensuring that $\mathcal{L}^1(\nu) = \mathcal{L}^{1,s}_\Lambda(\nu)$ for a norming set $\Lambda \subset X^*$. As usual, we write $\sigma(X,\Lambda)$ to denote the topology on X of pointwise convergence on Λ. Following [18, Appendice II], we say that a norming set $\Lambda \subset X^*$ has the *Orlicz property* if, for every sequence (x_n) in X, the series $\sum_n x_n$ is unconditionally convergent whenever all subseries are $\sigma(X,\Lambda)$-unconditionally convergent. Clearly, Λ has the Orlicz property if and only if every X-valued mapping ϑ defined on a σ-algebra such that the composition $\langle \vartheta, x^*\rangle$ is countably additive for all $x^* \in \Lambda$ is a countably additive vector measure.

Proposition 2.8. *Let $\Lambda \subset X^*$ be a norming set having the Orlicz property. Then $\mathcal{L}^1(\nu) = \mathcal{L}^{1,s}_\Lambda(\nu)$.*

Proof. Fix $f \in \mathcal{L}^{1,s}_\Lambda(\nu)$ and consider the finitely additive measure $\xi_{f,\Lambda}$. Since $\langle \xi_{f,\Lambda}, x^*\rangle$ is countably additive for every $x^* \in \Lambda$ and Λ has the Orlicz property, we deduce that $\xi_{f,\Lambda}$ is countably additive. In view of Theorem 2.5, this means that $f \in \mathcal{L}^1(\nu)$. □

Some examples of norming sets having the Orlicz property are:

- Norming sets $\Lambda \subset X^*$ when X does not contain subspaces isomorphic to ℓ_∞, see [5, Corollary 7, p. 23].
- James boundaries, as we show in the following proposition.

Proposition 2.9. *Let $B \subset B_{X^*}$ be a James boundary. Then B has the Orlicz property.*

Proof. Let ξ be an X-valued mapping defined on a σ-algebra Σ such that $\langle \xi, x^* \rangle$ is a countably additive measure for every $x^* \in B$.

Note first that ξ is finitely additive and bounded, by the Dieudonné-Grothendieck criterion already mentioned in the proof of Theorem 2.5.

Let (A_n) be a disjoint sequence in Σ. We claim that $\sum_n \xi(A_n)$ converges unconditionally to $\xi(\bigcup_n A_n)$. Indeed, fix an increasing sequence $n_1 < n_2 < \dots$ in $\mathbb{N}$ and define $x_k = \sum_{i=1}^k \xi(A_{n_i}) = \xi(\bigcup_{i=1}^k A_{n_i}) \in X$ for every $k \in \mathbb{N}$. Then (x_k) is bounded and $\sigma(X, B)$-converges to $\xi(\bigcup_i A_{n_i})$. Since B is a James boundary, Simons' [16] extension of Rainwater's theorem [15] ensures that (x_k) converges weakly to $\xi(\bigcup_i A_{n_i})$. An appeal to the Orlicz-Pettis theorem (cf. [5, Corollary 4, p. 22]) now establishes that $\sum_n \xi(A_n)$ is unconditionally convergent (with sum $\xi(\bigcup_n A_n)$), as claimed. It follows that ξ is a countably additive measure. □

We now deal with a particular case of special interest.

Theorem 2.10. *Suppose $X = Y^*$ for another Banach space Y. Then $\mathcal{L}_Y^{1,s}(\nu) = \mathcal{L}_w^1(\nu) = \mathcal{L}_Y^1(\nu)$.*

Proof. Fix $f \in \mathcal{L}_Y^1(\nu)$. Since f is Σ-measurable, there is a sequence of simple functions $s_n : \Omega \to \mathbb{R}$ such that $|s_n| \leq |f|$ for every $n \in \mathbb{N}$ and $\lim_n s_n = f$ pointwise. For each $n \in \mathbb{N}$ the vector measure $\nu_{s_n} : \Sigma \to X$ is countably additive and satisfies

$$\langle \nu_{s_n}(A), y \rangle = \int_A s_n d\langle \nu, y \rangle \quad \text{for every } A \in \Sigma \text{ and every } y \in Y.$$

Fix $A \in \Sigma$. The dominated convergence theorem yields

$$\Big| \int_A f d\langle \nu, y \rangle - \langle \nu_{s_n}(A), y \rangle \Big| \leq \int_\Omega |f - s_n| d|\langle \nu, y \rangle| \to 0$$

for every $y \in Y$. From the Banach-Steinhaus theorem (applied to the sequence $(\nu_{s_n}(A))$ in $X = Y^*$) it follows that the linear mapping

$$x_A : Y \to \mathbb{R}, \quad x_A(y) := \int_A f d\langle \nu, y \rangle,$$

is continuous, that is, $x_A \in X$. As $A \in \Sigma$ is arbitrary, $f \in \mathcal{L}_Y^{1,s}(\nu)$. □

We finish the paper with some examples. We write $\mathcal{M}$ to denote the σ-algebra of all Lebesgue measurable subsets of $[0,1]$ and μ stands for the Lebesgue measure on $\mathcal{M}$. We fix a countable partition (A_n) of $[0,1]$ in $\mathcal{M}$ such that $\mu(A_n) > 0$ for every $n \in \mathbb{N}$. Define

$$\vartheta : \mathcal{M} \to \mathbb{R}^{\mathbb{N}}, \quad \vartheta(A) := (\mu(A \cap A_n)),$$

and consider the function $f : [0,1] \to \mathbb{R}$ given by $f := \sum_{n=1}^{\infty} \frac{n}{\mu(A_n)} \chi_{A_n}$.

Example 2.11. Consider ϑ as a countably additive ℓ_1-valued measure. Then there is a norming set $\Lambda \subset \ell_1^* = \ell_\infty$ such that

$$\mathcal{L}^1(\vartheta) = \mathcal{L}^{1,s}_\Lambda(\vartheta) = \mathcal{L}^1_w(\vartheta) \subsetneq \mathcal{L}^1_\Lambda(\vartheta).$$

Thus, *the conclusion of Corollary 2.4 is not valid for arbitrary norming sets.*

Proof. Take the norming set

$$\Lambda := \{(c_n) \in \ell_\infty : \text{ there is } N \in \mathbb{N} \text{ such that } c_n = 0 \text{ for every } n \geq N\}.$$

As we mentioned just before Corollary 2.4, the fact that ℓ_1 contains no isomorphic copy of c_0 ensures that $\mathcal{L}^1(\vartheta) = \mathcal{L}^1_w(\vartheta)$. On the other hand, the function f belongs to $\mathcal{L}^1_\Lambda(\vartheta)$ and fulfills $\|f\|_{\mathcal{L}^1_\Lambda(\vartheta)} = \infty$, so $f \notin \mathcal{L}^1_w(\vartheta)$ (by Theorem 2.1). □

Example 2.12. Consider ϑ as a countably additive c_0-valued measure. Then:

(i) There is a James boundary $B \subset B_{c_0^*} = B_{\ell_1}$ such that

$$\mathcal{L}^1(\vartheta) = \mathcal{L}^{1,s}_B(\vartheta) \subsetneq \mathcal{L}^1_w(\vartheta) \subsetneq \mathcal{L}^1_B(\vartheta).$$

(ii) Taking $\Lambda = c_0^* = \ell_1$, we have

$$\mathcal{L}^1(\vartheta) = \mathcal{L}^{1,s}_\Lambda(\vartheta) \subsetneq \mathcal{L}^1_w(\vartheta) = \mathcal{L}^1_\Lambda(\vartheta).$$

Proof. Consider the James boundary $B := \{e_n^* : n \in \mathbb{N}\} \subset B_{\ell_1}$, where $e_n^*(m) = \delta_{n,m}$ (the Kronecker symbol) for every $n, m \in \mathbb{N}$. In view of the comments after Proposition 2.8, we have the equality $\mathcal{L}^1(\vartheta) = \mathcal{L}^{1,s}_B(\vartheta)$. On the other hand, it is clear that f belongs to $\mathcal{L}^1_B(\vartheta)$ but not to $\mathcal{L}^1_w(\vartheta)$ (bear in mind that $\|f\|_{\mathcal{L}^1_B(\vartheta)} = \infty$). Finally, the function $g : [0,1] \to \mathbb{R}$ given by

$$g := \sum_{n=1}^{\infty} \frac{1}{\mu(A_n)} \chi_{A_n}$$

belongs to $\mathcal{L}^1_B(\vartheta)$ and $\|g\|_{\mathcal{L}^1_B(\vartheta)} < \infty$, so we have $g \in \mathcal{L}^1_w(\vartheta)$ (by Theorem 2.1). An easy computation shows that $\vartheta_g : \mathcal{M} \to \ell_\infty = c_0^{**}$ is given by $\vartheta_g(A) = \left(\frac{\mu(A \cap A_n)}{\mu(A_n)}\right)$, hence $\vartheta_g(\mathcal{M}) \not\subset c_0$ and, therefore, $g \notin \mathcal{L}^1(\vartheta)$. □

Example 2.13. Consider ϑ as a countably additive ℓ_∞-valued measure.

(i) There is a norming set $\Lambda \subset \ell_\infty^*$ such that

$$\mathcal{L}^1(\vartheta) \subsetneq \mathcal{L}^{1,s}_\Lambda(\vartheta) = \mathcal{L}^1_w(\vartheta) \subsetneq \mathcal{L}^1_\Lambda(\vartheta).$$

(ii) There is a norming set $\Lambda' \subset \ell_\infty^*$ such that

$$\mathcal{L}^1(\vartheta) \subsetneq \mathcal{L}^{1,s}_{\Lambda'}(\vartheta) = \mathcal{L}^1_w(\vartheta) = \mathcal{L}^1_{\Lambda'}(\vartheta).$$

Proof. Take $\Lambda := \{e_n^* : n \in \mathbb{N}\} \subset \ell_\infty^*$ and $\Lambda' = \ell_1$. By Theorem 2.10 we have $\mathcal{L}^{1,s}_{\Lambda'}(\vartheta) = \mathcal{L}^1_w(\vartheta) = \mathcal{L}^1_{\Lambda'}(\vartheta)$. Since $\mathcal{L}^{1,s}_\Lambda(\vartheta) \subset \mathcal{L}^1_w(\vartheta)$ and $\Lambda \subset \Lambda'$, it follows that

$$\mathcal{L}^{1,s}_{\Lambda'}(\vartheta) = \mathcal{L}^{1,s}_\Lambda(\vartheta) = \mathcal{L}^1_w(\vartheta) = \mathcal{L}^1_{\Lambda'}(\vartheta).$$

As in the previous example, $f \in \mathcal{L}^1_\Lambda(\vartheta)$ but $f \notin \mathcal{L}^1_w(\vartheta)$. We claim that the function g defined in the previous example belongs to $\mathcal{L}^{1,s}_\Lambda(\vartheta)$ but not to $\mathcal{L}^1(\vartheta)$. Indeed, it is not difficult to check that g belongs to $\mathcal{L}^1_{\Lambda'}(\vartheta) = \mathcal{L}^{1,s}_{\Lambda'}(\vartheta)$ and that

$$\xi_{g,\Lambda}(A) = \left(\frac{\mu(A \cap A_n)}{\mu(A_n)}\right) \quad \text{for every } A \in \mathcal{M}.$$

Since the series $\sum_k \xi_{g,\Lambda}(A_k)$ is not convergent in ℓ_∞, $\xi_{g,\Lambda}$ is not countably additive and so $g \notin \mathcal{L}^1(\vartheta)$ (by Theorem 2.5). □

Our last examples are based on the following standard construction (cf. [9, II.7] for the case of spaces of integrable functions).

Lemma 2.14. *For $i = 1, 2$, let (Ω_i, Σ_i) be a measurable space, X_i a Banach space, $\Lambda_i \subset X_i^*$ a norming set and $\nu_i : \Sigma_i \to X_i$ a countably additive measure. Suppose $\Omega_1 \cap \Omega_2 = \varnothing$. Let us consider $\Omega := \Omega_1 \cup \Omega_2$, the σ-algebra $\Sigma := \{A \subset \Omega : \ A \cap \Omega_i \in \Sigma_i$ for $i = 1, 2\}$ and the countably additive measure*

$$\nu : \Sigma \to X_1 \oplus X_2, \quad \nu(A) := \nu_1(A \cap \Omega_1) \oplus \nu_2(A \cap \Omega_2).$$

The set $\Lambda := \Lambda_1 \oplus \Lambda_2 \subset (X_1 \oplus X_2)^*$ is norming. Let $f : \Omega \to \mathbb{R}$ be a function. The following conditions are equivalent:

(i) *f belongs to $\mathcal{L}^{1,s}_\Lambda(\nu)$ (resp. $\mathcal{L}^1_\Lambda(\nu)$).*
(ii) *$f|_{\Omega_i}$ belongs to $\mathcal{L}^{1,s}_{\Lambda_i}(\nu_i)$ (resp. $\mathcal{L}^1_{\Lambda_i}(\nu_i)$) for $i = 1, 2$.*

Applying the previous lemma to Examples 2.12 (ii) and 2.13 (ii) (resp. Examples 2.12 (ii) and 2.13 (i)) we get:

Example 2.15. There exist a countably additive $c_0 \oplus \ell_\infty$-valued measure ϑ and a norming set $\Lambda \subset (c_0 \oplus \ell_\infty)^*$ such that

$$\mathcal{L}^1(\vartheta) \subsetneq \mathcal{L}^{1,s}_\Lambda(\vartheta) \subsetneq \mathcal{L}^1_w(\vartheta) = \mathcal{L}^1_\Lambda(\vartheta).$$

Example 2.16. There exist a countably additive $c_0 \oplus \ell_\infty$-valued measure ϑ and a norming set $\Lambda \subset (c_0 \oplus \ell_\infty)^*$ such that

$$\mathcal{L}^1(\vartheta) \subsetneq \mathcal{L}^{1,s}_\Lambda(\vartheta) \subsetneq \mathcal{L}^1_w(\vartheta) \subsetneq \mathcal{L}^1_\Lambda(\vartheta).$$

References

[1] T. A. Abrahamsen, O. Nygaard, and M. Põldvere, *On weak integrability and boundedness in Banach spaces*, J. Math. Anal. Appl. **314** (2006), no. 1, 67–74. MR **2183537 (2006g:**28028)

[2] R. G. Bartle, N. Dunford, and J. Schwartz, *Weak compactness and vector measures*, Canad. J. Math. **7** (1955), 289–305. MR 0070050 (16,1123c)

[3] G. P. Curbera, *Operators into L^1 of a vector measure and applications to Banach lattices*, Math. Ann. **293** (1992), no. 2, 317–330. MR **1166123 (93b:**46083)

[4] G. P. Curbera and W. J. Ricker, *Banach lattices with the Fatou property and optimal domains of kernel operators*, Indag. Math. (N.S.) **17** (2006), no. 2, 187–204. MR **2321380 (2008e:**46036)

[5] J. Diestel and J. J. Uhl, Jr., *Vector measures*, American Mathematical Society, Providence, R.I., 1977, With a foreword by B. J. Pettis, Mathematical Surveys, No. 15. MR 0453964 (56 #12216)

[6] M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucía, J. Pelant, and V. Zizler, *Functional analysis and infinite-dimensional geometry*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, vol. 8, Springer-Verlag, New York, 2001. MR **1831176 (2002f:**46001)

[7] A. Fernández, F. Mayoral, F. Naranjo, C. Sáez, and E. A. Sánchez-Pérez, *Spaces of p-integrable functions with respect to a vector measure*, Positivity **10** (2006), no. 1, 1–16. MR **2223581 (2006m:**46053)

[8] V. P. Fonf, *Weakly extremal properties of Banach spaces*, Mat. Zametki **45** (1989), no. 6, 83–92, 112, English translation in Math. Notes 45 (1989), no. 5-6, 488–494. MR **1019040 (90k:**46032)

[9] I. Kluvánek and G. Knowles, *Vector measures and control systems*, North-Holland Publishing Co., Amsterdam, 1976, North-Holland Mathematics Studies, Vol. 20, Notas de Matemática, No. 58. [Notes on Mathematics, No. 58]. MR 0499068 (58 #17033)

[10] D. R. Lewis, *Integration with respect to vector measures*, Pacific J. Math. **33** (1970), 157–165. MR 0259064 (41 #3706)

[11] ———, *On integrability and summability in vector spaces*, Illinois J. Math. **16** (1972), 294–307. MR 0291409 (45 #502)

[12] O. Nygaard, *A strong uniform boundedness principle in Banach spaces*, Proc. Amer. Math. Soc. **129** (2001), no. 3, 861–863. MR **1626466 (2001f:**46026)

[13] ———, *Boundedness and surjectivity in normed spaces*, Int. J. Math. Math. Sci. **32** (2002), no. 3, 149–165. MR **1938513 (2003h:**46024)

[14] ———, *Thick sets in Banach spaces and their properties*, Quaest. Math. **29** (2006), no. 1, 59–72. MR **2209791 (2007a:**46016)

[15] J. Rainwater, *Weak convergence of bounded sequences*, Proc. Amer. Math. Soc. **14** (1963), 999. MR 0155171 (27 #5111)

[16] S. Simons, *A convergence theorem with boundary*, Pacific J. Math. **40** (1972), 703–708. MR 0312193 (47 #755)

[17] G. F. Stefánsson, L_1 *of a vector measure*, Matematiche (Catania) **48** (1993), no. 2, 219–234 (1994). MR **1320665 (96a:**46061)

[18] G. E. F. Thomas, *L'intégration par rapport à une mesure de Radon vectorielle*, Ann. Inst. Fourier (Grenoble) **20** (1970), no. 2, 55–191 (1971). MR 57 #3348

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