

ON THE MODULI OF CONVEXITY

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ABSTRACT. It is known that, given a Banach space $(X, \|\cdot\|)$, the modulus of convexity associated to this space δ_X is a non-negative function, non-decreasing, bounded above by the modulus of convexity of any Hilbert space and satisfies the equation $\frac{\delta_X(\varepsilon)}{\varepsilon^2} \leq 4L \frac{\delta_X(\mu)}{\mu^2}$ for every $0 < \varepsilon \leq \mu \leq 2$, where $L > 0$ is a constant. We show that, given a function f satisfying these properties then, there exists a Banach space in such a way its modulus of convexity is equivalent to f , in Figiel's sense. Moreover this Banach space can be taken two-dimensional.

1. INTRODUCTION

In this paper we are concerned with the basic properties of the moduli of convexity of Banach spaces. In relation with this properties it was shown by Figiel [?], that given a Banach space $(X, \|\cdot\|)$ and defining

Definition 1.1 (Modulus of convexity). The modulus of convexity of $(X, \|\cdot\|)$ is the function that for each $\varepsilon \in [0, 2]$ takes the value

$$\delta_X(\varepsilon) = \inf\{1 - \|x + y\|/2 : \|x\| = \|y\| = 1, \|x - y\| = \varepsilon\}.$$

Then:

Proposition 1.2. δ_X satisfy the following properties

- $\delta_X(\cdot)$ is non-decreasing
- If $0 < \varepsilon \leq \mu$ then, for a universal constant $0 < L < 3.18$,

$$\frac{\delta_X(\varepsilon)}{\varepsilon^2} \leq 4L \frac{\delta_X(\mu)}{\mu^2}.$$

Besides, due to some results by G. Nordlander [?], we can add another property to the previous proposition, which is the following result:

Proposition 1.3. For every $\varepsilon \in [0, 2]$ we have $\delta_X(\varepsilon) \leq \delta_E(\varepsilon)$.

Here, δ_E is the modulus of convexity of any Hilbert space. Obviously $\delta_E = \delta_{l_2}$, and from the parallelogram identity, it can be easily determined that

$$\delta_E(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}.$$

In the following section we will show that these conditions are enough for a function to be a modulus of convexity.

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2. CHARACTERIZATION OF MODULI OF CONVEXITY

Following the article [?], we will consider the concept of equivalence between functions; this is:

Definition 2.1. Given two non-negative functions f and g , each one defined on a segment $[0, a]$, let us write $f \prec g$ if there exist positive constants A, B, C such that $Af(Bt) \leq g(t)$ for $t \in [0, C]$; we shall consider f and g as equivalent, denoted by $f \sim g$, iff $f \prec g \prec f$.

Now we can consider all those functions that satisfy the properties described in the introduction. Indeed we state the following definition:

Definition 2.2. A “modulus function” is a non-negative real function f defined on a segment $[0, a]$ and satisfying the following conditions:

- (1) $f(\varepsilon) \leq \delta_E(\varepsilon)$.
- (2) $f(\cdot)$ is non-decreasing
- (3) If $0 < \varepsilon \leq \mu$ then, for a positive constant L ,

$$\frac{f(\varepsilon)}{\varepsilon^2} \leq L \frac{f(\mu)}{\mu^2}.$$

Our main result states that a function is a modulus function if and only if there exists a Banach space, in fact we may choose $X = l_2^2$, and a new norm in such a way the modulus of convexity of the new norm is equivalent to the function. In order to show that result we need to do previous work in relation with the general construction of the norm.

2.1. Construction. Let $(\mathbb{R}^2, \|\cdot\|)$ be the canonical two-dimensional real Banach space where $\|\cdot\|$ is the Euclidean norm and let $S = \{x \in \mathbb{R}^2 : \|x\| = 1\}$ be the unit sphere, that is, the circumference of radius one and whose center is the origin.

Let us take any point in S , say a_1 , its symmetric $-a_1$ and a positive value ε . Starting in a_1 and $-a_1$, we will take in the clockwise direction on S points $b_1, a_2, -b_1, -a_2$ such that the segments $[a_1, b_1]$ and $[b_1, a_2]$ have length ε . Now, we repeat the same process starting at a_2 but in such a way the length of the segments are ε^2 . Let us suppose that we have already chosen $\{(a_i, b_i)\}_{i=1}^n$ and a_{n+1} ; then we take points b_{n+1} and a_{n+2} such that the segments $[a_{n+1}, b_{n+1}]$ and $[b_{n+1}, a_{n+2}]$ have length ε^{n+1} .

It is a crucial fact that the family $\{(a_i, b_i)\}_{i \in \mathbb{N}}$ lies just in one hemisphere of the sphere S . Thus, in order to clarify the size that ε can take, we have the following result.

Lemma 2.3. *Given two points x and y of the unit sphere S , the angle α opposite to the segment $[x, y]$ of the triangle with extremes x, y , and the origin, takes the value*

$$\alpha = 2 \arcsin \left(\frac{\|x - y\|}{2} \right).$$

Then, in the previous construction, we have that the segments we have chosen cover an angle with arc length

$$4 \left(\sum_{n \in \mathbb{N}} \arcsin \left(\frac{\varepsilon^n}{2} \right) \right),$$

and if $\varepsilon \leq 1/2$, it is less than π .

Proof. All the results but the last are obvious. In order to show that for $\varepsilon \leq 1/2$ we have

$$4 \left(\sum_{n \in \mathbb{N}} \arcsin \left(\frac{\varepsilon^n}{2} \right) \right) \leq \pi,$$

let us note that the function $x/\sin(x)$ when $x \leq 0.25$ is bounded above by 1.02, and then when $\frac{\varepsilon^n}{2} \leq \frac{1}{4}1.02$,

$$\arcsin \left(\frac{\varepsilon^n}{2} \right) \leq \arcsin \left(\sin \left(\frac{\varepsilon^n}{2} 1.02 \right) \right) = \frac{\varepsilon^n}{2} 1.02,$$

since \arcsin is an increasing function. Then for $\varepsilon \leq 1/2$ we have the above estimation and

$$4 \left(\sum_{n \in \mathbb{N}} \arcsin \left(\frac{\varepsilon^n}{2} \right) \right) \leq 4 \left(\sum_{n \in \mathbb{N}} \frac{\varepsilon^n}{2} 1.02 \right) \leq 2.04 \sum_{n \in \mathbb{N}} \varepsilon^n = 2.04 \frac{\varepsilon}{1 - \varepsilon} \leq 2.04.$$

But, obviously this value is less than π . \square

Now we can assure that our election of points is good in the sense that they do not overlap each other and with their symmetric points.

We are going to define a new unit sphere for \mathbb{R}^2 . In order to do it, we consider a fixed $i \in \mathbb{N}$, take the segment $[a_i, b_i]$ and delete from S the points of the arc lying between a_i and b_i . Instead of these points, we add a new set of points which are defined using a new affine coordinate system.

Indeed, given a non-negative value c_i , let us consider the system of coordinates defined by the following equations:

$$x^i = \frac{-1}{\sqrt{1 - \frac{\varepsilon^{2i}}{4}}} \left(x \left(\frac{a_i'' + b_i''}{2} \right) + y \left(\frac{a_i' + b_i'}{2} \right) \right) + \left(1 + \frac{c_i}{\sqrt{1 - \frac{\varepsilon^{2i}}{4}}} \right) \left(\frac{a_i' + b_i'}{2} \right),$$

$$y^i = \frac{1}{\sqrt{1 - \frac{\varepsilon^{2i}}{4}}} \left(x \left(\frac{a_i' + b_i'}{2} \right) - y \left(\frac{a_i'' + b_i''}{2} \right) \right) + \left(1 + \frac{c_i}{\sqrt{1 - \frac{\varepsilon^{2i}}{4}}} \right) \left(\frac{a_i'' + b_i''}{2} \right),$$

where $a_i = (a_i', a_i'')$ and $b_i = (b_i', b_i'')$ are expressed in the canonical system of coordinates. It happens to be that the y^i -axis contains both the initial origin of coordinates and the midpoint of the segment $[a_i, b_i]$. Therefore, the origin of the new system will lie on the y^i -axis, at distance c_i from the segment $[a_i, b_i]$ (see figure 1).

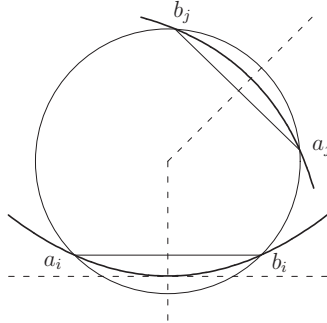


FIGURE 1

Thus the old arc is replaced by the set $\{(t, y_i(t)) : |2t| \leq \varepsilon^i\}$ (expressed in terms of the system (x^i, y^i)), where y_i is the function defined by $y_i(t) = (4c_i/\varepsilon^i)t^2$.

For the symmetric segment $[-a_i, -b_i]$ we repeat this process in order to obtain a symmetric curve.

Let us suppose that we have done the same construction for each one of the segments $[a_i, b_i]$. Thus we have defined a new sphere S' , which defines a new norm in \mathbb{R}^2 ; we will denote this norm by $\|\cdot\|_{\{\varepsilon, \{c_i\}\}}$.

Remark 2.4. Let us note that, in order to assure that the sphere S' defines a new norm, the values c_i must verify

$$c_i \leq \delta_E(\varepsilon^i) = 1 - \sqrt{1 - \frac{\varepsilon^{2i}}{4}},$$

where, let us recall, δ_E is the modulus of convexity of the Euclidean norm. Hence, in particular, for every modulus function f and for every $0 < \varepsilon \leq 1/2$, we can take in the previous construction $c_i = f(\varepsilon^i)$. In this case, we will denote the resulting norm $\|\cdot\|_{\{\varepsilon, f\}}$.

The next step is to show the main property of this new norm. In order to do it, we need the following definition.

Definition 2.5. Given an arbitrary norm $\|\cdot\|$ in \mathbb{R}^2 , we can define the function

$$\eta(t) = \inf\{\varphi(x, y) : \|x\| = \|y\| = 1, \|x - y\| \geq t\},$$

where $\varphi(x, y) = \left\| \frac{x+y}{2} \right\| \left(\frac{1}{\left\| \frac{x+y}{2} \right\|} - 1 \right)$.

Lemma 2.6. *It holds $\eta \sim \delta_{(\mathbb{R}^2, \|\cdot\|)}$.*

Proof. Trivial. □

Proposition 2.7. *For every $0 < \varepsilon \leq 1/2$, and for every modulus function δ , there exists a positive constant $L'(\varepsilon)$ such that*

$$\delta(\varepsilon^i) \geq \eta(\varepsilon^i) \geq L'(\varepsilon)\delta(\varepsilon^i),$$

for every $i \in \mathbb{N}$, where η is defined with the norm $\|\cdot\|_{\{\varepsilon, \delta\}}$.

Proof. Let us denote just by $\|\cdot\|$, the norm $\|\cdot\|_{\{\varepsilon, \delta\}}$. Let $i \in \mathbb{N}$ fixed, and let us take two arbitrary points x and y such that $\|x\| = \|y\|$ and $\|x - y\| = \varepsilon^i$. We have to distinguish two cases:

- (1) Let us suppose that $\|x + y\| = \|x + y\|$. Obviously $\|x\|, \|y\| \leq 1$. Then, the parallelogram identity assures that

$$\|x + y\|^2 = 2\|x\|^2 + 2\|y\|^2 - \|x - y\|^2 \leq 4 - \varepsilon^{2i},$$

and thus,

$$\varphi(x, y) = 1 - \left\| \frac{x + y}{2} \right\| \geq 1 - \sqrt{1 - \frac{\varepsilon^{2i}}{4}} = \delta_E(\varepsilon^i) \geq \delta(\varepsilon^i).$$

- (2) Let us suppose that $\|x + y\| \neq \|x + y\|$. In this case there exists $j \in \mathbb{N}$ such that the point $x + y$ lies inside the cone with vertex the origin, and determined by a_j and b_j . If $j > i$ then $\varepsilon^j < \varepsilon^i$ and then

$$\begin{aligned} \varphi(x, y) &\geq \sqrt{1 - \frac{\varepsilon^{2j}}{4}} - \sqrt{1 - \frac{\varepsilon^{2i}}{4}} = \delta_E(\varepsilon^i) - \delta_E(\varepsilon^j) \\ &\geq \delta_E(\varepsilon^i) - \delta_E(\varepsilon^{i+1}) \geq (1 - \varepsilon^2)\delta_E(\varepsilon^i) \geq (1 - \varepsilon^2)\delta(\varepsilon^i) \end{aligned}$$

But if $j \leq i$ we have to be more careful. Let us consider the system of coordinates corresponding to the interval $[a_j, b_j]$, in order to define the norm $\|\cdot\|$. And let us take the first coordinates, say t and s , in this system, of the points x and y , respectively. Now, we denote by D the midpoint of the segment $[x, y]$, by C the intersection point of the segment $[0, x + y]$ with the sphere S' , and by C' the orthogonal projection of D on the graph of the function y_j (see figure ??).

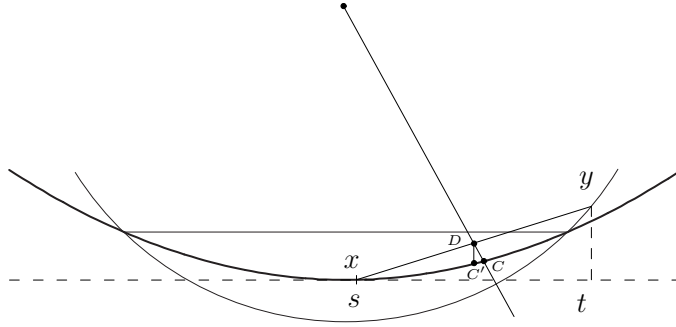


FIGURE 2

Then it holds $\varphi(x, y) = |DC|$. Clearly,

$$|DC'| \geq \frac{y_j(s) + y_j(t)}{2} - y_j\left(\frac{s+t}{2}\right) = \frac{\delta(\varepsilon^j)}{\varepsilon^{2j}}(s-t)^2,$$

and we are going to prove that there exists a constant $W(\varepsilon)$ such that $|DC| \geq W(\varepsilon)|DC'|$. In order to show it, we consider the following right-angled triangle: let us take the straight line passing through D , which is perpendicular to the x -axis, and the orthogonal line to DC passing through C ; these two lines intersect in a point that we denote by G ; then, we consider the triangle with vertices D , C and G . Now, it is clear that $\frac{|DC|}{|DG|} = \cos \alpha$, where α is the angle of our triangle corresponding to the point D . But this angle satisfies $|\alpha| \leq \arcsin(\varepsilon/2)$, and then $\cos \alpha \geq \cos(\arcsin(\varepsilon/2))$. If we denote by

$$W(\varepsilon) = \cos\left(\arcsin\frac{\varepsilon}{2}\right),$$

and noting that $|DG| \geq |DC'|$, we finally obtain that $|DC| \geq W(\varepsilon)|DC'|$.

On the other hand we have that the quotient $|t - s|/\|x - y\| = \cos \beta$, where β is the angle determined by the segment $[x, y]$ with respect to the x -axis. Since $\varepsilon \leq 1/2$ we have that the absolute value $|\beta|$ is always strictly

less than $\pi/2$. Therefore, we can assure that there exists a constant $A(\varepsilon)$ such that $|t - s| \geq A(\varepsilon) \|x - y\|$. Finally, we have that

$$\begin{aligned} \varphi(x, y) &= |DC| \geq W(\varepsilon)|DC'| \geq W(\varepsilon) \frac{\delta(\varepsilon^j)}{\varepsilon^{2j}} (s - t)^2 \\ &\geq W(\varepsilon)A(\varepsilon)^2 \frac{\delta(\varepsilon^j)}{\varepsilon^{2j}} \|x - y\|^2 = W(\varepsilon)A(\varepsilon)^2 \delta(\varepsilon^j) \varepsilon^{2(i-j)}, \end{aligned}$$

and since δ is a modulus function, we obtain that

$$\delta(\varepsilon^i) = \delta(\varepsilon^{i-j} \varepsilon^j) \leq L \delta(\varepsilon^j) \varepsilon^{2(i-j)}$$

for a certain positive constant L . Then

$$\varphi(x, y) \geq \frac{W(\varepsilon)A(\varepsilon)^2}{L} \delta(\varepsilon^i).$$

Now, if we take $L'(\varepsilon) = \min \left\{ (1 - \varepsilon^2), \frac{W(\varepsilon)A(\varepsilon)^2}{L} \right\} > 0$ we have that

$$\eta(\varepsilon^i) \geq L'(\varepsilon) \delta(\varepsilon^i),$$

as we want to show.

The first part of the inequality in the theorem is obvious since

$$\varphi(a_i, b_i) = \delta(\varepsilon^i). \quad \square$$

2.2. Main Theorem. The main theorem of this work is just a corollary of the previous results. Indeed, we can deduce it from Proposition ??.

Theorem 2.8. *A function δ is a modulus function as in definition 2.2 if and only if it is equivalent, in the sense of definition 2.1, to the modulus of convexity of a Banach space.*

Proof. Let δ be a modulus function, that is, a non-negative function satisfying the properties in definition 2.2, if we show that for some $0 < \varepsilon < 1/2$ we have $\eta \sim \delta$, by lemma ?? we have finished the proof. Thus, let us fix a certain $0 < \varepsilon < 1/2$. By proposition ??, it holds that

$$\delta(\varepsilon^i) \geq \eta(\varepsilon^i) \geq L' \delta(\varepsilon^i)$$

for every $i \in \mathbb{N}$.

Let us take an arbitrary $t \in (0, \varepsilon]$ and consider i in such a way $t \in (\varepsilon^{i+1}, \varepsilon^i]$, then we have

$$(2.1) \quad \eta(t) \geq \eta(\varepsilon^{i+1}) \geq L' \delta(\varepsilon^{i+1}) = L' \delta(\varepsilon \varepsilon^i) \geq L' \delta(\varepsilon t).$$

Now, for every fixed $t \in (0, \varepsilon^2]$, there exists i such that $t \in (\varepsilon^{i+1}, \varepsilon^i]$ and finally we have

$$(2.2) \quad \eta(t) \leq \eta(\varepsilon^i) \leq \delta(\varepsilon^i) = \delta(\varepsilon^{-1} \varepsilon^{i+1}) \leq \delta(\varepsilon^{-1} t).$$

Combining (??) and (??) we conclude that $\eta \sim \delta$. This finishes the proof. \square

3. APPLICATIONS

An immediate consequence of our main theorem is that any given uniformly convex Banach space can be renormed having any "arbitrarily bad" modulus of convexity.

This observation together with J. Borwein and J. Vanderwerff characterization of spaces with a modulus of convexity of power type 2, allows us to answer a question of Godefroy-Zizler described below, and also point out an error in Asplund's work [?].

Recall an alternatively definition of uniform convexity.

Definition 3.1. Let $\|\cdot\|$ be a norm on a Banach space X , we say that $\|\cdot\|$ is uniformly convex (UC) if for every pair of sequences $\{x_n\}_n, \{y_n\}_n$ in B_X satisfying

$$\lim_{n \rightarrow \infty} 2(\|x_n\|^2 + \|y_n\|^2) - \|x_n + y_n\|^2 = 0$$

we have that $\lim_n \|x_n - y_n\| = 0$.

This definition is very useful in renorming theory due to its homogeneity, which greatly simplifies the necessary calculations. However, this definition is not "completely homogeneous" since we assume that $x_n, y_n \in B_X$ (or more generally, we need boundedness). Theorem 3 from [?] claims that the additional assumption $x_n, y_n \in B_X$ is redundant. Godefroy and Zizler, in the course of writing their book [?] have come across this problem and being unable to verify Asplund's argument (which in fact, is sketched in the local setting, when it is true) asked if the redundancy of boundedness of x_n, y_n is indeed true. We answer this question in the negative.

Indeed, J. Borwein and J. Vanderwerff obtained the next theorem (whose proof is included here with their kind permission).

Definition 3.2. A norm $\|\cdot\|$ on a Banach space X is said to have power type p , for p real and positive, if there exists a positive constant C such that $\delta_{(X, \|\cdot\|)}(\varepsilon) \geq C\varepsilon^p$.

Theorem 3.3. Let $(X, \|\cdot\|)$ be a Banach space, then the following conditions are equivalent.

- (i) $\|x_n - y_n\| \rightarrow 0$ whenever $2(\|x_n\|^2 + \|y_n\|^2) - \|x_n + y_n\|^2 \rightarrow 0$.
- (ii) $\|\cdot\|$ is UC with modulus of convexity of power type 2.

Proof. (i) \Rightarrow (ii): Suppose $\|\cdot\|$ does not have modulus of convexity of power type 2. Then there exist $u_n, v_n \in S_X$ such that $\|u_n - v_n\| \geq \frac{1}{n}$ while $\|u_n + v_n\| \geq 2 - \frac{\varepsilon_n}{n^2}$ where $\varepsilon_n \rightarrow 0^+$. Now let $x_n = nu_n, y_n = nv_n$; then

$$\|x_n - y_n\| \geq 1, \quad \text{and} \quad \|x_n + y_n\| \geq 2n - \frac{\varepsilon_n}{n}.$$

Consequently,

$$\begin{aligned} 2(\|x_n\|^2 + \|y_n\|^2) - \|x_n + y_n\|^2 &\leq 4n^2 - (2n - \frac{\varepsilon_n}{n})^2 \\ &= 4\varepsilon_n - \frac{\varepsilon_n^2}{n^2} \end{aligned}$$

and so $2(\|x_n\|^2 + \|y_n\|^2) - \|x_n + y_n\|^2 \rightarrow 0$ which means (i) fails.

(ii) \Rightarrow (i): Suppose $\|\cdot\|$ has modulus of convexity of power type 2. Now suppose that $\|x_n - y_n\| \geq \delta$ but $2(\|x_n\|^2 + \|y_n\|^2) - \|x_n + y_n\|^2 \rightarrow 0$. Then $(\|x_n\| - \|y_n\|)^2 \rightarrow$

0. Let $\|x_n\| = \alpha_n$, because $\|\cdot\|$ is uniformly convex, we know $\alpha_n \rightarrow \infty$. Thus, by replacing y_n with $\alpha_n \frac{y_n}{\|y_n\|}$, we have

$$\liminf \|x_n - y_n\| \geq \delta \text{ and } \|y_n\| = \|x_n\| = \alpha_n.$$

Thus we may assume $\|x_n - y_n\| > \eta$ for all n where $\eta = \frac{\delta}{2}$. Now let $u_n = \frac{x_n}{\alpha_n}$ and $v_n = \frac{y_n}{\alpha_n}$. Then $\|u_n - v_n\| \geq \frac{\eta}{\alpha_n}$, and because the modulus of convexity of $\|\cdot\|$ is of power type 2, there is a $C > 0$ so that $\|u_n + v_n\| \leq 2 - C(\frac{\eta}{\alpha_n})^2$. Therefore, $\|x_n + y_n\| \leq 2\alpha_n - \frac{C\eta^2}{\alpha_n}$. Consequently, we compute

$$\begin{aligned} 2(\|x_n\|^2 + \|y_n\|^2) - \|x_n + y_n\|^2 &\geq 4\alpha_n^2 - (2\alpha_n - \frac{C\eta^2}{\alpha_n})^2 \\ &= 4C\eta^2 - \frac{C^2\eta^4}{\alpha_n^2} \rightarrow 0 \end{aligned}$$

This contradiction completes the proof. \square

Thus by our main theorem, l_2^2 has an equivalent UC renorming with a modulus of convexity not satisfying Borwein-Vanderwerff condition. However, being uniformly convex it does satisfy the alternatively UC definition.

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