ON WCG ASPLUND BANACH SPACES

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ABSTRACT. In this paper we give a sufficient and necessary condition for a Banach space to be Weakly Compactly Generated and Asplund. This condition links the weak* and the norm topologies of its dual .

1. INTRODUCTION

Weakly compactly generated (WCG) Asplund spaces are the non separable counterpart for Banach spaces with separable duals. They have been extensively studied in connection with smoothness, renorming and topological properties of weak-* compact subsets such as Corson compactness [2, 4]. In particular they coincide with Asplund spaces with Corson compact weak-* dual unit ball [12] and they have an equivalent locally uniformly rotund and Fréchet differentiable norm [2, 3]. The main tool for proving these properties has been the so-called projectional resolutions of the identity (PRI), a method to decompose a Banach space by means of well ordered collections ("long sequences") of projections which goes back to Amir and Lindenstrauss [1]. For X a WCG Asplund space the PRI will be shrinking and Troyanski's renorming technique can be applied to obtain locally uniformly rotund norms in both X and X^*, [14]. This method for renorming has been recently rebuilt in [6] with a transfer technique that works even in the non-linear case [8]. A main ingredient for that is a descriptive connection between approximations in two different metrics through a separable fibre. It is our aim here to study this linking property in full generality, even in the non-metric case and in particular in a Banach space with its norm and weak topologies and in a dual Banach space with its norm and weak-* topologies. It turns out that this property characterizes the WCG Asplund spaces. In the course of the proof we shall use PRI and it will be shown how the linking property is a topological device, gluing separable pieces, an alternative to the "long sequences" of a PRI. From the topological point of view, the linking property goes back to Srivatsa's selection theorem [13], [6].

The linking property we are going to deal with was introduced by Moltó, Orihuela and Troyanski in [6] and it corresponds with the following:

Definition 1.1. *Let X be a set and τ_1, τ_2 be two topologies on it. We shall say that X has $\mathcal{L}(\tau_1, \tau_2)$ if for any $x \in X$ there exists a countable set $S(x)$ containing x so that if $A \subset X$ then $\overline{A}^{\tau_2} \subset \overline{\cup\{S(x); x \in A\}}^{\tau_1}$.*

Definition 1.2. *Let A be a set in a topological vector space X and τ_1, τ_2 two topologies on X (the second may only be defined on A). We shall say that A has span-$\mathcal{L}(\tau_1, \tau_2)$ if for any $x \in A$ there exists a countable set $S(x)$ containing x so that if $B \subset A$ then $\overline{B}^{\tau_2} \subset \overline{\text{span} \cup \{S(x); x \in B\}}^{\tau_1}$.*

We shall also need the following definition.

Definition 1.3. *Let (X, τ) be a topological space. We shall say that X has the Linking Separability Property (LSP, for short) if there exists a metric d defined on X, with the metric topology finer than τ, such that X has $\mathcal{L}(d, \tau)$.*

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In the second section of the paper we shall study some properties of LSP spaces. In the third section we will focus on property $\mathcal{L}(\|\cdot\|, weak)$ for a Banach space and $\mathcal{L}(\|\cdot\|, weak^*)$ in the dual and we obtain our Main Theorem below. The study of compact spaces (K,τ) with property $\mathcal{L}(d,\tau)$ for a metric d is done in [9, 10].

Main Theorem: *Let X be a Banach space. Then the following conditions are equivalent:*
i) *X is WCG and Asplund;*
ii) B_{X^*} *has* $\mathcal{L}(\|\cdot\|, w^*)$*;*
iii) X^* *has* $\mathcal{L}(\|\cdot\|, w^*)$*;*
iv) X^* *has* $\mathcal{L}(w, w^*)$.

2. LSP Topological spaces.

Our first result provides us with a slight change in the definition of property $\mathcal{L}$ which will be convenient in many proofs.

Lemma 2.1. *Let* (X,τ) *be a topological space and* ϱ *a metric on X. Suppose that for every* $x \in X$*, there exists a* ϱ*-separable set,* $Z(x)$ *such that if* $A \subset X$ *then,* $\overline{A}^{\tau} \subset \overline{\cup\{Z(x); x \in A\}}^{\varrho}$.
Then X has $\mathcal{L}(\varrho,\tau)$.

Proof. For $x \in X$, let $\{y_n(x)\}$ be a ϱ-dense subset of $Z(x)$ and define $S(x) = \{y_n(x)\}$. This selection of $S(x)$ gives us property $\mathcal{L}(\varrho,\tau)$. ■

Now we will study some stability properties of LSP spaces.

Proposition 2.2. *(transitivity) Let X be a set,* τ_1, τ_2, τ_3 *three topologies on X. If X has* $\mathcal{L}(\tau_1,\tau_2)$ *and* $\mathcal{L}(\tau_2,\tau_3)$ *then it also has* $\mathcal{L}(\tau_1,\tau_3)$.

Proof. For $x \in X$, let $S_{1,2}(x)$ be the countable set given by $\mathcal{L}(\tau_1,\tau_2)$, and $S_{2,3}(x)$ the one given by $\mathcal{L}(\tau_2,\tau_3)$.

Define $S(x) = \cup\{S_{1,2}(y); y \in S_{2,3}(x)\}$. It is not difficult to show that $S(x)$ is the countable set we are looking for. ■

Proposition 2.3. *Let* $f : (X,\tau) \to (Y,\delta)$ *be a homeomorphism of topological spaces. If X has LSP then so does Y.*

Proof. Let d be the metric on X given by the LSP. Let us define on Y the metric: $\varrho(y_1, y_2) = d(f^{-1}(y_1), f^{-1}(y_2))$. First notice that the ϱ topology is finer than δ.

For any $x \in X$ let $S(x)$ be the countable set given by the $\mathcal{L}(d,\tau)$ of X. For $y \in Y$, define $Z(y) = f(S(f^{-1}(y)))$. It is easy to see that the countable set $Z(y)$ gives property $\mathcal{L}(\varrho,\delta)$ in Y.
■

Proposition 2.4. *Let* (X,τ) *be a LSP topological space, then any subspace of X is also LSP. In fact if* d *is a metric on X such that X has* $\mathcal{L}(d,\tau)$ *and* $H \subset X$ *then H has* $\mathcal{L}(d,\tau)$.

Proof. Let $H \subset X$ and d be the metric given by the LSP. For $x \in H$ there exists a countable set $S(x) = \{x^n(x)\}_{n\in\mathbb{N}}$ given by the $\mathcal{L}(d,\tau)$ of X.

Fix $n \in \mathbb{N}$. We denote by $B_d(x,r)$ the closed d-ball centered at x and of radius r. For $m \in \mathbb{N}$ set $A_{n,m}(x) = B_d(x^n(x), \frac{1}{m}) \cap H$. For those $n, m \in \mathbb{N}$ such that $A_{n,m}(x)$ is non-empty fix $y_{n,m}(x) \in A_{n,m}(x)$. Recall that $x \in S(x)$ and therefore some of these sets must be non-empty. Let us define $S'(x) = \{y_{n,m}(x)\}$ for those values of n, m such that $y_{n,m}(x)$ is defined.

One shows that if $A \subset H$ then $\overline{A}^{\tau_H} \subset \overline{\cup\{S'(x); x \in A\}}^{d}$. ■

In order to give further properties of these spaces we need to recall the following notions introduced by Jayne, Namioka and Rogers. Let (X,τ) be a topological space and let d be a metric on X.

The space X is said to be *σ-fragmented* by the metric d if, for each $\varepsilon > 0$, it is possible to write $X = \bigcup_{i=1}^{\infty} X_i^{\varepsilon}$, where each set X_i^{ε} has the property that each non-empty subset of X_i^{ε} has a non-empty relatively open subset of d-diameter less than ε.

We say that X has a *countable cover by sets of small local d-diameter* (d-SLD, for short) if for every $\varepsilon > 0$ there exists a decomposition $X = \bigcup_{n=1}^{\infty} X_n^{\varepsilon}$ such that for each $n \in \mathbb{N}$ every point of X_n^{ε} has a relatively τ-neighbourhood of d-diameter less than ε.

The equivalence between $a)$ and $b)$ in our next result is in [9, 11]; $c)$ implies $b)$ is in [6] and that proof extends to proving $d)$ implies $b)$.

Theorem 2.5. *Let X be a set, ϱ and d be metrics defined on X. Then the following conditions are equivalent:*

a) (X, d) is σ-fragmented by ϱ;

b) (X, d) has ϱ-SLD;

c) X has $\mathcal{L}(\varrho, d)$.

Moreover, if (X, ϱ) is a topological vector space then any of the properties above is equivalent to

d) X has span-$\mathcal{L}(\varrho, d)$.

Proof. We need only to prove b)⇒c). Assume that (X, d) has ϱ-SLD. Given $\varepsilon > 0$, let $X_{n,\varepsilon}$ be subsets of X covering it, such that for every $x \in X_{n,\varepsilon}$ there is a $\delta_{x,n} > 0$ for which the ball $B_d(x, \delta_{x,n})$ meets $X_{n,\varepsilon}$ in a set of ϱ-diameter less than ε.

For a given positive integer m, we define $X_{m,n,\varepsilon} = \{x \in X_{n,\varepsilon}; \delta_{x,n} \geq \frac{1}{m}\}$, and we have $X_{n,\varepsilon} = \cup_m X_{m,n,\varepsilon}$ and $X = \cup_{m,n} X_{m,n,\varepsilon}$.

Let us fix $x \in X$, $m, n \in \mathbb{N}$ and $\delta > 0$. If $B_d(x, \delta)$ meets $X_{m,n,\varepsilon}$ we will choose an element $y(x, m, n, \varepsilon, \delta) \in B_d(x, \delta) \cap X_{m,n,\varepsilon}$. We claim that

$$x \in \overline{\{y(x, m, n, \frac{1}{p}, \frac{1}{q}); m, n, p, q \in \mathbb{N}\}}^{\varrho} .$$

Indeed, for a given positive integer p, let m_p and n_p be so that

$$x \in X_{m_p, n_p, \frac{1}{p}} \quad (*).$$

Since $x \in B_d(x, \frac{1}{m_p}) \cap X_{m_p, n_p, \frac{1}{p}}$ this set is non-void with ϱ-diameter less than $\frac{1}{p}$ and it makes sense to consider $y(x, m_p, n_p, \frac{1}{p}, \frac{1}{m_p})$ that belongs to this set, hence we have $\varrho(x, y(x, m_p, n_p, \frac{1}{p}, \frac{1}{m_p})) < \frac{1}{p}$.

For every x we define the set $S(x) = \overline{\{y(x, m, n, \frac{1}{p}, \frac{1}{q}); m, n, p, q \in \mathbb{N}\}}^{\varrho}$.

Let us now take a sequence (x_n) in X which is d-convergent to x. Given a positive integer p let m_p and n_p such that $(*)$ holds. For a fixed positive integer p there is an r_p such that if $k \geq r_p$ we have $d(x_k, x) < \frac{1}{2m_p}$. Thus x belongs to $B_d(x_k, \frac{1}{2m_p}) \cap X_{m_p, n_p, \frac{1}{p}}$ which must be a non-empty set for $k \geq r_p$. Now according to the definition of $y(x_k, m_p, n_p, \frac{1}{p}, \frac{1}{2m_p})$ we have

$$y(x_k, m_p, n_p, \frac{1}{p}, \frac{1}{2m_p}) \in B_d(x_k, \frac{1}{2m_p}) \cap X_{m_p, n_p, \frac{1}{p}}.$$

Since this set is contained in $B_d(x, \frac{1}{m_p}) \cap X_{m_p, n_p, \frac{1}{p}}$ and the latter has ϱ-diameter less than $\frac{1}{p}$, we have, for $k \geq r_p$, $\varrho(x, y(x_k, m_p, n_p, \frac{1}{p}, \frac{1}{2m_p})) < \frac{1}{p}$. Then $x \in \overline{\cup S(x_k)}^{\varrho}$ and (c) is proved. ■

A one-to-one map $\varphi : (X, d) \to (Y, \varrho)$ is called a *SLD* map (see [11]) if (X, φ_ϱ) has property d-SLD, where φ_ϱ denotes the topology (metric) given by the family $\{\varphi^{-1}(U); U$ is $\varrho -$ open $\}$. Our next result shows that SLD maps transfer the property $\mathcal{L}$.

Proposition 2.6. *Let $\varphi : (X, \tau) \to (Y, \delta)$ be a continuous one-to-one map. Let d and ϱ be metrics defined on X and Y respectively, with d finer than τ and ϱ finer than δ. Assume that $\varphi : (X, d) \to (Y, \varrho)$ is a SLD map. If Y has $\mathcal{L}(\varrho, \delta)$, then X has $\mathcal{L}(d, \tau)$.*

Proof. First of all, let us show that X has the $\mathcal{L}(\varphi_\varrho, \varphi_\delta)$. By Proposition 2.4, $\varphi(X)$ has $\mathcal{L}(\varrho, \delta)$. So for any $y \in \varphi(X)$ there exists a countable set $S(y) \subset \varphi(X)$ satisfying the $\mathcal{L}$ condition.

For any $x \in X$ define $Z(x) = \varphi^{-1}(S(\varphi(x)))$. Let us see that the countable set $Z(x)$ does the job. To do that, take $(x_\gamma)_\gamma$ converging to x in (X, φ_δ), i.e. $\varphi(x_\gamma) \to \varphi(x)$ in δ. So we have

$\varphi(x) \in \{\overline{\bigcup_\gamma S(\varphi(x_\gamma))}^{\varrho} \cap \varphi(X)\}$, hence $x \in \overline{\bigcup_\gamma \varphi^{-1}(S(\varphi(x_\gamma)))}^{\varphi_\varrho} = \overline{\bigcup_{\gamma<\beta} Z(x_\gamma)}^{\varphi_\varrho}$ On the other hand, by the τ-δ continuity of the map φ, φ_δ is coarser than τ, so for any $A \subset X$,

$$\overline{A}^{\tau} \subset \overline{A}^{\varphi_\delta} \subset \overline{\bigcup_{x\in A} Z(x)}^{\varphi_\varrho}.$$

Therefore X has $\mathcal{L}(\varphi_\varrho, \tau)$ (although φ_ϱ may not be finer than τ).

Since the map φ is SLD, we have (X, φ_ϱ) is d-SLD which is equivalent to X has $\mathcal{L}(d, \varphi_\varrho)$. We just have to apply the transitivity of $\mathcal{L}$ to obtain X has $\mathcal{L}(d, \tau)$. ∎

Transitivity of the property $\mathcal{L}$ and its coincidence, for metric topologies, with the SLD property and σ-fragmentability yield several results

Proposition 2.7. *Let (X, τ) have LSP and let ϱ be any metric on X finer than τ. If (X, τ) is σ-fragmented by ϱ, then X has $\mathcal{L}(\varrho, \tau)$.*

Proof. Let d a metric on X such that X has $\mathcal{L}(d, \tau)$. Since (X, τ) is ϱ-σ-fragmented and d is finer than τ, (X, d) is σ-fragmented by ϱ. Then Theorem 2.5 yields X has $\mathcal{L}(\varrho, d)$. Now applying transitivity we obtain $\mathcal{L}(\varrho, \tau)$. ∎

Proposition 2.8. *Let (X, τ) be σ-fragmented by a metric d finer than τ (resp. d-SLD). If ϱ is another metric such that X has $\mathcal{L}(\varrho, \tau)$, then (X, τ) is σ-fragmented by ϱ (resp. ϱ-SLD.)*

Proof. Since X has $\mathcal{L}(\varrho, \tau)$ and d is finer than τ, for any $A \subset X$ we have $\overline{A}^{d} \subset \overline{A}^{\tau} \subset \overline{\cup\{S(x); x \in A\}}^{\varrho}$, i. e. X has $\mathcal{L}(\varrho, d)$. Theorem 2.5 gives (X, d) has ϱ-SLD and that implies by Theorem 2.4 in [11] that (X, τ) is σ-fragmented by ϱ (resp ϱ-SLD). ∎

When we have a vector space, Theorem 2.5 and transitivity will allow us to give some results.

Proposition 2.9. *Let (X, τ) be a LSP vector space and let ϱ be a metric, with the ϱ topology being a vector topology finer than τ, such that X has span-$\mathcal{L}(\varrho, \tau)$. Then X has $\mathcal{L}(\varrho, \tau)$.*

Proof. Let d a metric on X with $\mathcal{L}(d, \tau)$. Since d is finer than τ and we have span-$\mathcal{L}(\varrho, \tau)$, we obtain span-$\mathcal{L}(\varrho, d)$. Hence Theorem 2.5 gives $\mathcal{L}(\varrho, d)$ on X. $\mathcal{L}(\varrho, \tau)$ on X follows now by transitivity. ∎

Proposition 2.10. *Let (X, τ) a vector space σ-fragmented by a metric d (resp. d-SLD) finer than τ. If ϱ is another metric, with the ϱ topology being a vector topology, such that X has span-$\mathcal{L}(\varrho, \tau)$, then (X, τ) is σ-fragmented by ϱ (resp. ϱ-SLD).*

Proof. Since d is finer than τ, as in the previous proof, we have span-$\mathcal{L}(\varrho, d)$, hence $\mathcal{L}(\varrho, d)$. As in the proof of Proposition 2.8 we obtain (X, τ) is σ-fragmented by ϱ (resp. has ϱ-SLD.) ∎

3. BANACH SPACES

We turn now our attention to the class of Banach spaces. We shall say that a Banach space X *has the LSP* when $(X, weak)$ does. In this case we will give the precise metric for the LSP; indeed it must be the norm metric.

Theorem 3.1. *Let X be a Banach space. Then X has the LSP if, and only if X has $\mathcal{L}(\|\cdot\|, weak)$.*

Proof. By the Theorem of Hahn-Banach, for any $A \subset X$, $\overline{A}^{weak} \subset \overline{\text{span } A}^{\|\cdot\|}$. Hence X has span-$\mathcal{L}(\|\cdot\|, weak)$ with $S(x) = \{x\}$. So if X has the LSP, applying Proposition 2.9 we get $\mathcal{L}(\|\cdot\|, weak)$. ∎

Let us now give some examples of Banach spaces with the LSP.

Theorem 3.2. *Let E be a Banach space of density character at most ω_1. Then E has the LSP.*

Proof. Let $\{x_\gamma; 0 \le \gamma < \omega_1\}$ be a $\|\cdot\|$-dense subset of E. For any γ, $0 \le \gamma < \omega_1$, set $E_\gamma = \mathbb{Q} - \text{span}\{x_\beta; \beta < \gamma\}$.

Given $x \in E$ we can find a countable set $\{x_{\gamma_n}\}_{n\in\mathbb{N}} \subset E$ with $\gamma_n < \omega_1$, such that $x \in \overline{\{x_{\gamma_n}\}_{n\in\mathbb{N}}}^{\|\cdot\|}$. Set $\gamma(x) = \sup_n \gamma_n$. Now define $S(x) = \overline{E_{\gamma(x)}}^{\|\cdot\|}$ which is a separable vector space and contains x and gives us the LSP in E.

Let $A \subset X$ and $x \in \overline{A}^w$, due to the countable tightness of the weak topology of a Banach space (Kaplansky), we can find a sequence (x_n) in A such that $x \in \overline{\{x_n\}}^w$. Hence $x \in \overline{\text{span } \cup_n S(x_n)}^{\|\cdot\|} = \overline{\cup_n S(x_n)}^{\|\cdot\|}$, since the set $\cup_n S(x_n)$ is actually a vector space. ■

We will make use of the so-called *Projectional Resolutions of the Identity*. For properties we refer the reader to [2, 4].

Definition 3.3. *Let X be a Banach space. We denote by μ the smallest ordinal such that its cardinality $|\mu|$ coincides with the density character of X (dens(X)). A projectional resolution of the identity, PRI for short, is a collection $\{P_\alpha : \omega_0 \le \alpha \le \mu\}$ of projections from X into X that satisfy:*
for every α with $\omega_0 \le \alpha \le \mu$
i) $\|P_\alpha\| = 1$;
ii) $P_\alpha \circ P_\beta = P_\beta \circ P_\alpha = P_\alpha$ if $\omega_0 \le \alpha \le \beta \le \mu$;
iii) $dens(P_\alpha(X)) \le |\alpha|$;
iv) $\bigcup\{P_{\beta+1}(X) : \omega_0 \le \beta < \alpha\}$ is norm dense in $P_\alpha(X)$;
v) $P_\mu = Id_X$.

Let us show that any Banach space with "good" PRI has the LSP. To do so we shall need the following Lemma from [2] (Lemma VI.1.3).

Lemma 3.4. *Assume that every member X of a given class $\mathcal{P}$ of Banach spaces admits a PRI $\{P_\alpha\}$ such that all $(P_{\alpha+1} - P_\alpha)(X)$ belong to $\mathcal{P}$. Then, given $X \in \mathcal{P}$ with dens$(X) = |\mu|$, there is a collection $\{Q_\gamma; \omega_0 \le \gamma \le \mu\}$ of projections of X into X such that, if we let $R_\gamma = \frac{(Q_{\gamma+1} - Q_\gamma)}{(\|Q_{\gamma+1}\| + \|Q_\gamma\|)}$, we have the following:*
i) $Q_\gamma Q_\delta = Q_\delta Q_\gamma = Q_\gamma$ if $\omega_0 \le \gamma \le \delta \le \mu$;
ii) Q_{ω_0} and $(Q_{\gamma+1} - Q_\gamma)(X)$ is separable for every $\omega_0 \le \gamma \le \mu$;
iii) $Q_\mu = Id$;
iv) for every $x \in X$, $\{\|R_\alpha(x)\|; \alpha \in [\omega_0, \mu)\} \in c_0([\omega_0, \mu))$;
iv) for every $x \in X$ and $\gamma \in [\omega_0, \mu]$, $Q_\gamma(x)$ belongs to the norm-closed linear span of $\{R_\alpha(x); \alpha < \gamma\} \cup \{Q_{\omega_0}(x)\}$.

Proposition 3.5. *Let $\mathcal{P}$ be a class of Banach spaces such that conditions of Lemma 3.4 hold. If $X \in \mathcal{P}$, then X has LSP, that is $\mathcal{L}(\|\cdot\|, weak)$.*

Proof. Given $x \in X$ let us define $supp(x) = \{\alpha \in [\omega_0, \mu); R_\alpha(x) \ne 0\}$ and set $S(x) = span(\cup\{Q_{\omega_0}(X) \cup R_\alpha(X); \alpha \in supp(x)\}$. $S(x)$ is $\|\cdot\|$-separable since $supp(x)$ is countable.

We want to see that if $A \subset X$ then $\overline{A}^{weak} \subset \overline{\cup\{S(x); x \in A\}}^{\|\cdot\|}$. So take $x \in \overline{A}^{weak}$ and $\{x_\gamma : \gamma \in \Gamma, \ge\}$ $weak$-convergent to x. Given $\varepsilon > 0$, since $x \in \overline{span\{Q_{\omega_0}(x) \cup R_\alpha(x)\}_{\alpha<\mu}}^{\|\cdot\|}$, there exist $\alpha_1, \ldots, \alpha_k \in supp(x)$ such that

$$\|x - \sum_{i=1}^{k} \lambda_i R_{\alpha_i}(x)\| < \varepsilon.$$

(Where one of the R_{α_i} could actually be Q_{ω_0}.) On the other hand $R_\alpha(x_\gamma) \to R_\alpha(x)$ weakly for $\alpha \in [\omega_0, \mu)$. So for $1 \le i \le k$ there exists $\gamma_i \in \Gamma$ such that $R_{\alpha_i}(x_\gamma) \ne 0$ for $\gamma >> \gamma_i$. Let us take $\gamma \ge \gamma_i$ for $1 \le i \le k$. It is clear that $\alpha_i \in supp(x_\gamma)$. Therefore

$$\sum_{i=1}^{k} \lambda_i R_{\alpha_i}(x) \in span(\cup\{Q_{\omega_0}(X) \cup R_\alpha(X); \alpha \in supp(x_\gamma)\}) = S(x_\gamma).$$

Thus $x \in \overline{\cup\{S(x_\gamma); \gamma \in \Gamma\}}^{\|\cdot\|}$ as we wanted. ■

Corollary 3.6. *Let X be a Banach space:*
i) If X is WCD (Weakly Countably Determined), then X has LSP.
ii) If X is Asplund, then X^* has LSP.

Proof. i) The fact that the class of WCD Banach spaces admits a PRI with the properties as in Lemma 3.4 can be found in [2], Theorem VI.2.5.

ii) The construction of a PRI satisfying the properties in Lemma 3.4 in the dual of an Asplund space can also be found in [2], Theorem VI.3.4. and Remark VI.3.5. ■

Remark 3.7. *The projections $\{P_\alpha\}$ built in a dual of an Asplund space are not in general $w^* - w^*$-continuous. In the case where they can be done $w^* - w^*$-continuous, Lemma 3.4 and Proposition 3.5 can be easily modified to obtain that X^* has $\mathcal{L}(\|\cdot\|, weak^*)$.*

An example of a Banach space without PRI but having LSP can be found in [2], p. 260. It is a $C(K)$ space that embeds into a $c_0(\Gamma)$ space by means of a SLD map.

Let us turn now our attention to duals of Banach spaces.

Definition 3.8. *Let X^* is a dual Banach space, we shall say that a subset of X^* has *LSP when it has $\mathcal{L}(\|\cdot\|, weak^*)$.*

We shall say that X^* has span-*LSP, if it has span-$\mathcal{L}(\|\cdot\|, weak^*)$.

We have seen that PRI on a Banach space enables us to prove the LSP. A starting point for a construction of a PRI will be the so-called Projectional Generator.

Definition 3.9. *Let X be a Banach space, W a one-norming subset of X^*, that is, for every $x \in X$ we have $\|x\| = \sup\{|f(x)|; f \in B_{X^*} \cap W\}$, and let us assume that $\overline{W}$ is linear. Let $\Phi : W \to 2^X$ be an at most countable valued mapping such that: for any $B \subset W$ with $\overline{B}$ linear we have*

$$\Phi(B)^\perp \cap \overline{B_{X^*} \cap B}^{w^*} = \{0\}.$$

Then we call the pair (W, Φ) a projectional generator on X.

In fact (see Proposition 6.1.7 in [4]), any non-separable Banach space with a projectional generator admits a PRI. The construction of projectional generators can be found in [4].

In order to give the proof of the Main Theorem in the introduction we shall need some results.

Lemma 3.10. *Let X be a Banach space such that any weak*-separable subset of its dual unit ball is $\|\cdot\|$-separable, then X is Asplund.*

Proof. Let $A \subset X$ be separable subspace. Let us show that A^* is $\|\cdot\|^*$-separable. Indeed, let us consider the restriction map $R_A : X^* \to A^*$, which is onto and weak* and $\|\cdot\|^*$ continuous.

Let $F \subset B_{X^*}$ be a countable set such that $R_A(F)$ is w*-dense in (B_{A^*}, w^*) (since (B_{A^*}, w^*) is metrizable).

Set $W = \overline{\text{co}\,(F)}^{w^*}$. By hypothesis, W is $\|\cdot\|^*$-separable and therefore $R_A(W)$ is also $\|\cdot\|^*$-separable. Now by compactness it is easy to show that $R_A(W) = B_{A^*}$. Hence A^* is $\|\cdot\|^*$-separable. ■

Lemma 3.11. *Let X be a Banach space such that B_{X^*} has *LSP. Then there exists a projectional generator on X defined in all of X^*.*

Proof. Let us define $\Phi : B_{X^*} \to 2^X$ as follows. Given $x^* \in X^*$, let $S(x^*)$ be the countable set given by the *LSP. Now for any $y^* \in B_{X^*}$, let $\{x_n(y^*)\} \subset B_X$ be such that $\|y^*\| = sup\{| < y^*, x_n(y^*) > | : n \in \mathbb{N}\}$. Set $\Psi(y^*) = \{x_n(y^*)\}$ and finally define $\Phi(x^*) = \Psi(S(x^*))$. For $x^* \notin B_{X^*}$, $\Phi(x^*) = 0$.

It is clear that Φ is countably valued. We shall show that for any subset A of X^*, then $\Phi(A)^\perp \cap \overline{A \cap B_{X^*}}^{w^*} = \{0\}$. So let $x^* \in \Phi(A)^\perp \cap \overline{A \cap B_{X^*}}^{w^*}$. Since $x^* \in \overline{A \cap B_{X^*}}^{w^*}$ and B_{X^*} has *LSP,

$$x^* \in \overline{\cup\{S(y^*) : y^* \in A \cap B_{X^*}\}}^{\|\cdot\|} \equiv \overline{S(A \cap B_{X^*})}^{\|\cdot\|}.$$

Hence there exists a sequence $y_n^* \in S(A \cap B_{X^*})$ with $y_n^* \to x^*$ in the norm topology.

Now given $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $\|y_m^* - x^*\| \leq \frac{\varepsilon}{2}$. But, since $\Phi(A)$ norms $S(A)$ and $x^*_{|\Phi(A)} \equiv 0$, we should have

$$\|y_m^*\| = \sup_{n\in\mathbb{N}}\{| < y_m^*, x_n(y_m^*) > |\} = \sup_{n\in\mathbb{N}}\{| < y_m^* - x^*, x_n(y_m^*) > |\} \leq$$

$$\leq \sup\{| < y_m^* - x^*, x > | : x \in B_X\} = \|y_m^* - x^*\|.$$

So $\|x^*\| \leq \|x^* - y_m^*\| + \|y_m^*\| \leq 2\|x^* - y_m^*\| \leq \varepsilon$. The former reasoning being valid for every $\varepsilon > 0$, therefore $x^* = 0$ and the proof is done. ■

Now, using the previous results we are ready to prove the Main Theorem in the introduction.

Proof. i)⇒iii) Take $\mathcal{P}$ the class of Banach spaces consisting of duals of WCG Asplund spaces. By a result in [4] (Proposition 6.1.10), if X is a Banach space with a projectional generator defined in all of X^* (actually in all B_{X^*}) and X^* admits a projectional generator defined in X (i.e. X is Asplund [4] (Proposition 8.2.1)), then X admits a *shrinking PRI*, i.e., a PRI such that the adjoint maps form a PRI on X^*. Now Remark 3.7 allows us to apply Proposition 3.5 to $X^* \in \mathcal{P}$ and to obtain that X^* has *LSP.

iii)⇒ii) Obvious.

ii)⇒i) Since B_{X^*} has *LSP, by Lemma 3.10 X is Asplund and together with Lemma 3.11 (see comments in i)⇒iii) above), they provide us with the existence of a w^*-weak continuous injection from X^* into $c_0(\Gamma)$, and by a result of Amir and Lindenstrauss, (see [2], Corollary VI.5.2), X must be Weakly Compactly Generated.

iii)⇒iv) Obvious.

iv)⇒iii) By transitivity X has span-*LSP. Hence Lemma 3.10 can be applied to get that X is Asplund. So by Proposition 3.6 X^* has the LSP, i.e., $\mathcal{L}(\|\cdot\|, weak)$. By transitivity, X^* has *LSP. ■

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