ON THE EQUIVALENCE OF MCSHANE AND PETTIS INTEGRABILITY IN NON-SEPARABLE BANACH SPACES

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Dedicated to Professor Isaac Namioka with admiration.

Abstract. We show that McShane and Pettis integrability coincide for functions $f : [0,1] \to L^1(\mu)$, where μ is any finite measure. On the other hand, assuming the Continuum Hypothesis, we prove that there exist a weakly Lindelöf determined Banach space X, a scalarly null (hence Pettis integrable) function $h : [0,1] \to X$ and an absolutely summing operator u from X to another Banach space Y such that the composition $u \circ h : [0,1] \to Y$ is not Bochner integrable; in particular, h is not McShane integrable.

1. Introduction

Several methods of integration for functions taking values in Banach spaces have been studied over the years. Among these methods, those developed by Bochner [7] and Pettis [24, 25, 30] have been the most popular ones. McShane's [23] alternative approach to Lebesgue's integration theory has also been extended to the case of vector-valued functions, see e.g., [13], [17] and [18]. In general, the McShane integral lies strictly between the Bochner and Pettis integrals [13, 17], although for some classes of Banach spaces McShane and Pettis integrability coincide: this happens for separable spaces [13, 17, 18], super-reflexive (e.g., Hilbert) spaces [4] and $c_0(\Gamma)$ (where Γ is any non-empty set) [4]. The relationship between the McShane integral and others which are less known (e.g., the Henstock-Kurzweil, Birkhoff and Talagrand integrals) has been discussed in [11], [12], [27] and [29].

The following question was attributed to K. Musial in [4]: *Is every scalarly null Banach space-valued function McShane integrable?* Under the Continuum Hypothesis (CH), L. Di Piazza and D. Preiss [4] answered in the negative Musial's question by means of an $\ell^\infty(\omega_1)$-valued function. Recall that a Banach space X is *weakly measure compact* if and only if every X-valued scalarly measurable function is scalarly equivalent to a strongly measurable one [8]. This property holds true for any weakly Lindelöf Banach space (e.g., weakly compactly generated –WCG– or, more generally, weakly Lindelöf determined –WLD–). In view of the comments above, an affirmative answer to Musial's question for functions taking values in a particular weakly measure compact Banach space X would imply automatically

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that McShane and Pettis integrability coincide for X-valued functions. This approach was used by Di Piazza and Preiss to obtain the aforementioned results on the equivalence of both integrals in super-reflexive spaces and $c_0(\Gamma)$ for any non-empty set Γ. An important part of their argument relies on the existence of suitable projectional resolutions of the identity (PRI) in those spaces. The fact that every WCG space is weakly measure compact and admits a PRI led them to ask whether McShane and Pettis integrability are still equivalent for functions with values in arbitrary WCG spaces [4, p. 1178]. It is also natural to think about the same question within the more general class of WLD Banach spaces, since all of these are weakly measure compact and admit a PRI as well.

In this paper (summarized below) we discuss the coincidence of McShane and Pettis integrability in certain non-separable WLD Banach spaces.

In Section 2 we introduce the terminology and notation used throughout the paper. A few known lemmas on PRIs and the McShane integral are stated there for the convenience of the reader.

Section 3 is devoted to show that, *for an arbitrary finite measure* μ*, a function* $f : [0,1] \to L^1(\mu)$ *is McShane integrable if and only if it is Pettis integrable* (Theorem 3.5). Recall that $L^1(\mu)$ is always WCG for finite μ, while it may be non-separable. Besides the aforementioned reduction to the case of scalarly null functions, our proof of Theorem 3.5 makes use of Maharam's classification of measure algebras, a special type of separable projectional resolutions of the identity (SPRI) on $L^1(\mu)$-spaces and the already known equivalence of McShane and Pettis integrability for Hilbert space-valued functions (a proof is included here, see Corollary 3.2).

In Section 4 we present another example (also under CH) of a scalarly null function which is not McShane integrable (Example 4.1). The novelty of this example relies on the fact that the Banach space in the range is WLD, so we cannot expect a general result on the coincidence of McShane and Pettis integrability in WLD spaces. In Example 4.1, the key to distinguish Pettis integrability from McShane integrability has to do with the behavior of the composition of a vector-valued function with an absolutely summing operator, as we next explain.

Recall that an operator (i.e., linear and continuous map) $u : X \to Y$ between Banach spaces is absolutely summing if it takes unconditionally convergent series to absolutely convergent ones. As one may expect, absolutely summing operators also improve the integrability properties of vector-valued functions. This topic has been studied by several authors, see [3], [5], [19], [21], [22] and [28]. Given an X-valued Pettis integrable function f, the Y-valued composition $u \circ f$ is Bochner integrable in many cases (but not always): this happens whenever f is McShane, Birkhoff or Talagrand integrable, as well as whenever X is a subspace of a weakly Lindelöf $C(K)$ space, see [28]. The latter is the case if X is WCG or, more generally, if X is WLD and (B_{X^*}, w^*) has the so-called *property (M)* (i.e., every Radon probability on it has separable support). The results in [28] left open the question whether $u \circ f$ is Bochner integrable provided that f is scalarly null or X is WLD. It turns out that this is not true in general, since the composition of the function of Example 4.1 with certain absolutely summing operator is not Bochner integrable. Our construction is based on an example (under CH) of a WLD Banach space whose dual unit ball fails property (M), due to G. Plebanek and O. Kalenda [26].

2. Preliminaries

All unexplained notation and terminology can be found in our standard references [7], [9], [10] and [30]. The cardinality of a set A is denoted by $\mathrm{card}(A)$ and ω_1 stands for the first uncountable ordinal. Our Banach spaces X are assumed to be real. We write $\|\cdot\|_X$ (or simply $\|\cdot\|$) to denote the norm of X. The density character of X, denoted by $\mathrm{dens}(X)$, is the smallest cardinality of a dense subset of X. As usual, $B_X = \{x \in X : \|x\|_X \leq 1\}$ and X^* stands for the (topological) dual of X. We denote by w^* the weak* topology on X^*. Given a compact Hausdorff topological space K, we denote by $C(K)$ the Banach space of all real-valued continuous functions on K, equipped with the supremum norm. Given a finite measure space (Ω, Σ, μ), we write $L^1(\mu)$ to denote the Banach space of all (equivalence classes of) Σ-measurable and μ-integrable real-valued functions on Ω, equipped with the usual norm $\|f\|_{L^1(\mu)} = \int_\Omega |f| \, d\mu$.

A Banach space X is *WCG* if there is a weakly compact set $K \subset X$ such that $\overline{\mathrm{span}}(K) = X$. Standard examples of WCG spaces are the separable or reflexive ones, $c_0(\Gamma)$ (for any non-empty set Γ) and $L^1(\mu)$ (for any finite measure μ). The well known Amir-Lindenstrauss theorem [1] (cf. [10, Theorem 11.6]) asserts that every non-separable WCG Banach space X admits a *PRI*, i.e., a collection

$$\{P_\alpha : \ \omega \leq \alpha \leq \mathrm{dens}(X)\}$$

of bounded linear projections on X such that $P_\omega \equiv 0$, $P_{\mathrm{dens}(X)}$ is the identity on X and for every $\omega < \alpha \leq \mathrm{dens}(X)$ the following hold:

- $\|P_\alpha\| = 1$.
- $\mathrm{dens}(P_\alpha(X)) \leq \mathrm{card}(\alpha)$.
- $P_\alpha \circ P_\beta = P_\beta \circ P_\alpha = P_\beta$ whenever $\omega \leq \beta \leq \alpha$.
- $\bigcup_{\omega \leq \beta < \alpha} P_{\beta+1}(X)$ is dense in $P_\alpha(X)$.

As in [9, Definition 6.2.6], we say that a collection $\{P_\alpha : \omega \leq \alpha \leq \mathrm{dens}(X)\}$ of bounded linear projections on a non-separable Banach space X is a *SPRI* if $P_\omega \equiv 0$, $P_{\mathrm{dens}(X)}$ is the identity on X and for every $\omega \leq \alpha < \mathrm{dens}(X)$ we have:

- $(P_{\alpha+1} - P_\alpha)(X)$ is separable.
- $P_\alpha \circ P_\beta = P_\beta \circ P_\alpha = P_\beta$ whenever $\omega \leq \beta \leq \alpha$.
- $x \in \overline{\mathrm{span}}\{(P_{\alpha+1} - P_\alpha)(x) : \omega \leq \alpha < \mathrm{dens}(X)\}$ for every $x \in X$.

The last property also holds true for any PRI, see e.g., [9, Proposition 6.2.1]. In particular, if $\mathrm{dens}(X) = \omega_1$, then every PRI on X is also a SPRI.

A Banach space X is *WLD* if (B_{X^*}, w^*) is a *Corson compactum*, i.e., it is homeomorphic to some set $S \subset [-1,1]^\Gamma$ (endowed with the product topology) such that for each $s \in S$ the set $\{\gamma \in \Gamma : s(\gamma) \neq 0\}$ is countable. The class of WLD spaces is strictly bigger than that of WCG spaces and is made up of weakly Lindelöf spaces, see e.g., [9, Chapter 7] and [10, Chapters 11 and 12]. Every non-separable WLD space admits a PRI as well as a SPRI, see e.g., [9, Chapters 6 and 8]. The following folk lemma (which we did not find in print as stated below) will be useful in Section 3. The proof given here imitates that of [10, Proposition 12.51].

Lemma 2.1. *Let $\{P_\alpha : \omega \leq \alpha \leq \mathrm{dens}(X)\}$ be either a PRI or a SPRI on a non-separable WLD Banach space X. Then for each $x^* \in X^*$ the set*

$$\{\omega \leq \alpha < \mathrm{dens}(X) : \ x^*|_{(P_{\alpha+1} - P_\alpha)(X)} \not\equiv 0\}$$

is countable.

Proof. Let S be the set of all $x^* \in X^*$ for which

$$\Gamma(x^*) := \{\omega \leq \alpha < \mathrm{dens}(X) : \ x^*|_{(P_{\alpha+1}-P_\alpha)(X)} \not\equiv 0\}$$

is countable. It is clear that S is a linear subspace of X^*. We claim that S is w^*-closed. Indeed, by the Banach-Dieudonné theorem (cf. [10, Theorem 4.44]) it suffices to show that $mB_{X^*} \cap S$ is w^*-closed for every $m \in \mathbb{N}$. To this end, fix $m \in \mathbb{N}$ and take $x^* \in \overline{mB_{X^*} \cap S}^{w^*}$. Since mB_{X^*} is angelic (cf. [10, Exercise 12.55]), there is a sequence (x_n^*) in $mB_{X^*} \cap S$ which w^*-converges to x^*, so $\Gamma(x^*) \subset \bigcup_{n=1}^\infty \Gamma(x_n^*)$ and therefore $x^* \in mB_{X^*} \cap S$. This shows that S is w^*-closed.

Now take any $x_0 \in X$ satisfying $x^*(x_0) = 0$ for every $x^* \in S$. Since

$$x_0 \in \overline{\mathrm{span}}\{(P_{\alpha+1} - P_\alpha)(x_0) : \omega \leq \alpha < \mathrm{dens}(X)\}$$

and $y^* \circ (P_{\alpha+1} - P_\alpha) \in S$ for every $y^* \in (P_{\alpha+1} - P_\alpha)(X)^*$ and $\omega \leq \alpha < \mathrm{card}(X)$, we conclude that $x_0 = 0$. An appeal to the Hahn-Banach theorem ensures that $S = X^*$ and the proof is over. $\square$

The statement "every Corson compactum has property (M)" is undecidable in ZFC: it is true under Martin's Axiom and the negation of CH, whereas it is false under CH, see e.g., [26] and the references therein. It is known that for a Banach space X the following implications hold:

$$\begin{array}{ccccc} \text{WCG} & \Rightarrow & \text{WLD and } (B_{X^*}, w^*) \text{ has property (M)} & \Rightarrow & \text{WLD} \\ & & \Updownarrow & & \\ & & C(B_{X^*}) \text{ WLD} & & \end{array}$$

and no reverse arrow is true in general, see [2], [9, Chapter 7] and [26].

Throughout this paper we denote by λ the Lebesgue measure on the σ-algebra $\mathcal{L}$ of all Lebesgue measurable subsets of $[0,1]$. Let X be a Banach space and consider a function $f : [0,1] \to X$. Given $A \subset [0,1]$, we write $f\chi_A$ to denote the X-valued function defined on $[0,1]$ which agrees with f on A and vanishes outside A. Recall that f is said to be

(i) *scalarly null* if for each $x^* \in X^*$ the composition $x^* \circ f$ vanishes a.e. (the exceptional set depends on x^*);
(ii) *scalarly measurable* if $x^* \circ f$ is measurable for every $x^* \in X^*$;
(iii) *strongly measurable* if it is scalarly measurable and there is $E \in \mathcal{L}$ with $\lambda(E) = 1$ such that $f(E)$ is separable; equivalently, f is the a.e. limit of a sequence of simple functions, cf. [7, Theorem 2, p. 42];
(iv) *Bochner integrable* if it is strongly measurable and $\int_0^1 \|f\| \, d\lambda < \infty$;
(v) *Pettis integrable* if $x^* \circ f$ is integrable for every $x^* \in X^*$ and for each $E \in \mathcal{L}$ there is $x_E \in X$ (the *Pettis integral* of f over E) such that

$$\int_E (x^* \circ f) \, d\lambda = x^*(x_E) \quad \text{for every } x^* \in X^*.$$

Clearly, every scalarly null function is Pettis integrable. Recall also that a function $g : [0,1] \to X$ is *scalarly equivalent* to f if $f - g$ is scalarly null.

In order to introduce the McShane integral we need some extra terminology. A *gauge* on $[0,1]$ is a function $\delta : [0,1] \to \mathbb{R}^+$. A *McShane partition* of $[0,1]$ is a finite collection $\{(E_i, t_i)\}_{1 \leq i \leq p}$, where the E_i's are non-overlapping closed subintervals such that $\bigcup_{i=1}^p E_i = [0,1]$ and $t_i \in [0,1]$ for every $1 \leq i \leq p$. If the condition "$\bigcup_{i=1}^p E_i = [0,1]$" is dropped, then $\{(E_i, t_i)\}_{1 \leq i \leq p}$ is called a *partial McShane* *partition* of $[0,1]$. We say that $\{(E_i,t_i)\}_{1\leq i\leq p}$ is *subordinate* to δ provided that $E_i \subset (t_i-\delta(t_i), t_i+\delta(t_i))$ for every $1\leq i\leq p$. It is an easy exercise to show that for every gauge δ on $[0,1]$ there is a McShane partition of $[0,1]$ subordinate to δ.

The function $f:[0,1]\to X$ is *McShane integrable*, with *McShane integral* $x\in X$, if for every $\varepsilon>0$ there is a gauge δ on $[0,1]$ such that

$$\Big\|\sum_{i=1}^{p}\lambda(E_i)f(t_i)-x\Big\|_X \leq \varepsilon$$

for every McShane partition $\{(E_i,t_i)\}_{1\leq i\leq p}$ of $[0,1]$ subordinate to δ. As we have already mentioned in the introduction, in this case f is also Pettis integrable (and the respective integrals coincide), see [17, Theorem 2C].

The following two auxiliary results (Lemmas 1 and 2 in [4]) will be helpful when dealing with scalarly null McShane integrable functions.

Lemma 2.2. *Let X be a Banach space and $f:[0,1]\to X$ a function. Then f is scalarly null and McShane integrable if and only if for every $\varepsilon>0$ there is a gauge δ on $[0,1]$ such that*

$$\Big\|\sum_{i=1}^{p}\lambda(E_i)f(t_i)\Big\|_X \leq \varepsilon$$

for every partial McShane partition $\{(E_i,t_i)\}_{1\leq i\leq p}$ of $[0,1]$ subordinate to δ.

Lemma 2.3. *Let X be a Banach space and $f_n:[0,1]\to X$ a sequence of scalarly null McShane integrable functions converging pointwise to a function $f:[0,1]\to X$. Then f is scalarly null and McShane integrable.*

We stress that the McShane integral can also be set up in the more general case of functions defined on σ-finite outer regular quasi-Radon measure spaces, see [13]. It is worth pointing out that our results in Section 3 are valid in this setting as well. However, as in [4], we only work with functions defined on $[0,1]$ in order to avoid some non-interesting technicalities arising in the general case which would obscure the main ideas.

3. McShane and Pettis integrability for $L^1(\mu)$-valued functions

Recall that the *ℓ^p-sum* $(1\leq p<\infty)$ of a family $(X_i)_{i\in I}$ of Banach spaces is the linear space

$$\Big(\bigoplus_{i\in I}X_i\Big)_p := \Big\{(x_i)_{i\in I}\in\prod_{i\in I}X_i:\ \|(x_i)_{i\in I}\|_{(\bigoplus_{i\in I}X_i)_p} := \Big(\sum_{i\in I}\|x_i\|_{X_i}^p\Big)^{1/p}<\infty\Big\},$$

which becomes a Banach space when equipped with the norm $\|\cdot\|_{(\bigoplus_{i\in I}X_i)_p}$.

Following [4, Corollary 1], we denote by $\mathcal{P}$ the class of all Banach spaces X for which every scalarly null function $f:[0,1]\to X$ is McShane integrable. The stability of this class under ℓ^p-sums is discussed in Proposition 3.1 below. Although part (i) can be deduced from Corollary 1 (b) and Lemma 5 in [4], we prefer to give here a more direct proof which advances some of the ideas used later when dealing with $L^1(\mu)$-valued functions.

Proposition 3.1. *Let $(X_i)_{i\in I}$ be a family of Banach spaces belonging to $\mathcal{P}$.*

(i) *$(\bigoplus_{i\in I}X_i)_p$ belongs to $\mathcal{P}$ for every $2\leq p<\infty$.*
(ii) *If I is countable, then $(\bigoplus_{i\in I}X_i)_p$ belongs to $\mathcal{P}$ for every $1\leq p<\infty$.*

Proof. (i) Set $X := (\bigoplus_{i\in I} X_i)_p$. Let $f : [0,1] \to X$ be a scalarly null function. We will check that f is McShane integrable. Note first that we can assume without loss of generality that f is bounded. Indeed, it suffices to bear in mind Lemma 2.3 and the fact that there is a sequence (f_n) of bounded scalarly null functions converging pointwise to f (take $A_n := \{t \in [0,1] : \|f(t)\|_X \leq n\}$ and $f_n := f\chi_{A_n}$ for every $n \in \mathbb{N}$). Fix $M > 0$ such that $\|f(t)\|_X \leq M$ for every $t \in [0,1]$. We will show that f is the uniform limit of a sequence of scalarly null McShane integrable functions (an appeal to Lemma 2.3 will then finish the proof).

For each $i \in I$, write $\pi_i : X \to X_i$ to denote the canonical projection and set $g_i := \pi_i \circ f$. Since g_i is scalarly null and X_i belongs to $\mathcal{P}$, we infer that g_i is McShane integrable.

Fix $\varepsilon > 0$. For each $t \in [0,1]$ there is a finite set $I(t) = \{i_1(t), \dots, i_{n(t)}(t)\} \subset I$ such that

$$\sum_{i\in I\setminus I(t)} \|g_i(t)\|_{X_i}^p \leq \varepsilon^p.$$

Then the function $\varphi : [0,1] \to X$ given by

$$\pi_i(\varphi(t)) = \begin{cases} g_i(t) & \text{if } i \in I(t), \\ 0 & \text{if } i \notin I(t), \end{cases}$$

satisfies $\|f(t) - \varphi(t)\|_X \leq \varepsilon$ for every $t \in [0,1]$. We claim that φ *is scalarly null and McShane integrable.*

For each $n \in \mathbb{N}$ we define a function $\varphi_n : [0,1] \to X$ by

$$\pi_i(\varphi_n(t)) = \begin{cases} g_i(t) & \text{if } n \leq n(t) \text{ and } i = i_n(t), \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\varphi = \sum_{n=1}^{\infty} \varphi_n$ pointwise. By Lemma 2.3, in order to check that φ is scalarly null and McShane integrable it suffices to show that the same holds for each φ_n.

To this end, fix $n \in \mathbb{N}$ and $\eta > 0$. For each $i \in I$, Lemma 2.2 applied to g_i ensures the existence of a gauge δ_i on $[0,1]$ such that

$$\Big\|\sum_{j=1}^{J} \lambda(F_j) g_i(s_j)\Big\|_{X_i} \leq \eta^p$$

for every partial McShane partition $\{(F_j, s_j)\}_{1\leq j\leq J}$ of $[0,1]$ subordinate to δ_i. Define a gauge δ on $[0,1]$ by $\delta(t) := \min\{\delta_i(t) : i \in I(t)\}$ for every $t \in [0,1]$.

Now let $\{(E_k, t_k)\}_{1\leq k\leq K}$ be a partial McShane partition of $[0,1]$ subordinate to δ. Given $i \in I$, set

$$A(i) := \{1 \leq k \leq K : \ n \leq n(t_k), \ i = i_n(t_k)\}$$

and observe that the collection $\{(E_k, t_k)\}_{k\in A(i)}$ is a partial McShane partition of $[0,1]$ subordinate to δ_i, hence

$$\Big\|\sum_{k\in A(i)} \lambda(E_k) g_i(t_k)\Big\|_{X_i} \leq \eta^p.$$

Therefore,

$$\begin{aligned}(1)\quad \Big\|\sum_{k=1}^{K}\lambda(E_k)\varphi_n(t_k)\Big\|_X &= \Big(\sum_{i\in I}\Big\|\sum_{k=1}^{K}\lambda(E_k)\pi_i(\varphi_n(t_k))\Big\|_{X_i}^p\Big)^{1/p}\\ &= \Big(\sum_{i\in I}\Big\|\sum_{k\in A(i)}\lambda(E_k)g_i(t_k)\Big\|_{X_i}^p\Big)^{1/p} \leq \eta\cdot\Big(\sum_{i\in I}\Big\|\sum_{k\in A(i)}\lambda(E_k)g_i(t_k)\Big\|_{X_i}^{p-1}\Big)^{1/p}.\end{aligned}$$

Since $p\geq 2$ and $A(i)\cap A(i')=\emptyset$ whenever $i\neq i'$, we have

$$\begin{aligned}\sum_{i\in I}\Big\|\sum_{k\in A(i)}\lambda(E_k)g_i(t_k)\Big\|_{X_i}^{p-1} &\leq M^{p-1}\cdot\Big(\sum_{i\in I}\lambda\Big(\bigcup_{k\in A(i)}E_k\Big)^{p-1}\Big)\\ &\leq M^{p-1}\cdot\Big(\sum_{i\in I}\lambda\Big(\bigcup_{k\in A(i)}E_k\Big)\Big)\leq M^{p-1},\end{aligned}$$

which combined with (1) yields

$$\Big\|\sum_{k=1}^{K}\lambda(E_k)\varphi_n(t_k)\Big\|_X \leq \eta\cdot M^{(p-1)/p}.$$

As $\eta>0$ is arbitrary, φ_n is scalarly null and McShane integrable (by Lemma 2.2). This completes the proof of (i).

(ii) We use the notations X, π_i and g_i as in the proof of part (i). Enumerate $I=\{i_1,i_2,\dots\}$. For each $n\in\mathbb{N}$, the function $f_n:[0,1]\to X$ defined by

$$\pi_i(f_n(t))=\begin{cases}g_i(t) & \text{if } i\in\{i_1,\dots,i_n\},\\ 0 & \text{otherwise,}\end{cases}$$

is scalarly null and McShane integrable, as can be easily seen (use Lemma 2.2 and the fact that each g_i is scalarly null and McShane integrable). Since $f_n(t)\to f(t)$ as $n\to\infty$ for every $t\in[0,1]$, an appeal to Lemma 2.3 ensures that f is McShane integrable. The proof is over. □

As an immediate consequence of Proposition 3.1 (i) (and the weak measure compactness of Hilbert spaces) we get the following:

Corollary 3.2 (Di Piazza-Preiss)**.** *Let X be a Hilbert space. Then a function $f:[0,1]\to X$ is McShane integrable if and only if it is Pettis integrable.*

The previous corollary will be helpful when proving that $L^1(\mu)$ belongs to $\mathcal{P}$ for any finite measure μ (Theorem 3.5 below). Roughly speaking, we will do that by "approximating" $L^1(\mu)$-valued scalarly null functions by $i(L^2(\mu))$-valued ones, where $i:L^2(\mu)\to L^1(\mu)$ is the "inclusion" operator. Every $L^1(\mu)$-valued scalarly null function whose range is contained in $i(L^2(\mu))$ is McShane integrable, as a consequence of the following lemma.

Lemma 3.3. *Let X be a Banach space and $f:[0,1]\to X$ a scalarly null function. Suppose there exist a Banach space Y belonging to $\mathcal{P}$ and an operator $T:Y\to X$ such that $T^*(X^*)$ is sequentially w^*-dense in Y^* and $f([0,1])\subset T(Y)$. Then f is McShane integrable.*

Proof. Let $g : [0,1] \to Y$ be a function such that $T \circ g = f$. It suffices to check that g is McShane integrable. To this end, we will show that g is scalarly null. Fix $y^* \in Y^*$. There is a sequence (x_n^*) in X^* such that $(x_n^* \circ T)$ is w^*-convergent to y^*. Since each $x_n^* \circ f = (x_n^* \circ T) \circ g$ vanishes a.e., the same holds for $y^* \circ g$. As $y^* \in Y^*$ is arbitrary, g is scalarly null and the proof is finished. □

Given an infinite ordinal α, we denote by Σ_α the product σ-algebra on $\{0,1\}^\alpha$, i.e., the σ-algebra generated by all the sets of the form $\prod_{\beta<\alpha} A_\beta$, where $A_\beta \subset \{0,1\}$ and $A_\beta = \{0,1\}$ for all but finitely many $\beta < \alpha$. We write λ_α to denote the usual product probability on Σ_α. It is well known that $\mathrm{card}(\alpha) = \mathrm{dens}(L^1(\lambda_\alpha)) =$ smallest cardinal κ for which there is a family $\mathcal{C} \subset \Sigma_\alpha$ with $\mathrm{card}(\mathcal{C}) = \kappa$ such that $\inf\{\lambda_\alpha(A \triangle C) : C \in \mathcal{C}\} = 0$ for every $A \in \Sigma_\alpha$. For basic information on the probability space $(\{0,1\}^\alpha, \Sigma_\alpha, \lambda_\alpha)$ we refer the reader to [14, §254].

Recall that a *Schauder basis* in a (necessarily separable) Banach space X is a sequence (x_n) in X such that every $x \in X$ can be written in a unique way as $x = \sum_{n=1}^\infty a_n(x) x_n$ for some sequence $(a_n(x))$ in $\mathbb{R}$. In this case, for each $m \in \mathbb{N}$ the mapping $x \mapsto \sum_{n=1}^m a_n(x) x_n$ is a bounded linear projection, cf. [10, Lemma 6.4].

Lemma 3.4. *Let κ be an uncountable cardinal. Then $L^1(\lambda_\kappa)$ admits a SPRI*

$$\{P_\alpha : \ \omega \le \alpha \le \kappa\}$$

such that, for each $\omega \le \alpha < \kappa$, the subspace $(P_{\alpha+1} - P_\alpha)(L^1(\lambda_\kappa))$ has a Schauder basis made up of Σ_κ-simple functions.

Proof. We divide the proof into several steps. The first one is an easy observation.

Step 1. Let α and β be infinite ordinals with $\mathrm{card}(\alpha) = \mathrm{card}(\beta)$. Then there is an isometric isomorphism from $L^1(\lambda_\alpha)$ onto $L^1(\lambda_\beta)$ which maps Σ_α-simple functions to Σ_β-simple ones.

Step 2. Construction of a PRI on $L^1(\lambda_\kappa)$. For each ordinal $\omega < \alpha \le \kappa$, let $\mathcal{F}_\alpha$ be the σ-algebra on $\{0,1\}^\kappa$ generated by the family $\{\pi_\beta : \beta < \alpha\}$, where $\pi_\beta : \{0,1\}^\kappa \to \mathbb{R}$ stands for the β-th coordinate projection. It is clear that $\mathcal{F}_\alpha$ is exactly the family of all subsets of the form $A \times \{0,1\}^{\kappa\setminus\alpha}$, where $A \in \Sigma_\alpha$. Let $Q_\alpha : L^1(\lambda_\kappa) \to L^1(\lambda_\kappa)$ be the norm 1 linear projection that maps each $f \in L^1(\lambda_\kappa)$ to its *conditional expectation* with respect to $\mathcal{F}_\alpha$, usually denoted by $E(f|\mathcal{F}_\alpha)$, cf. [7, Lemma 3, p. 122]. In particular, Q_κ is the identity on $L^1(\lambda_\kappa)$. Set $Q_\omega \equiv 0$. The basic properties of conditional expectations and martingales ensure that the collection $\{Q_\alpha : \omega \le \alpha \le \kappa\}$ is a PRI on $L^1(\lambda_\kappa)$. Indeed:

- From the definitions it follows that $Q_\alpha \circ Q_\beta = Q_\beta \circ Q_\alpha = Q_\alpha$ whenever $\omega \le \alpha \le \beta \le \kappa$.
- $\mathrm{dens}(Q_\alpha(L^1(\lambda_\kappa))) \le \mathrm{card}(\alpha)$ for every $\omega < \alpha \le \kappa$. To check this, take $\mathcal{C} \subset \Sigma_\alpha$ with $\mathrm{card}(\mathcal{C}) = \mathrm{card}(\alpha)$ such that $\inf\{\lambda_\alpha(A \triangle C) : C \in \mathcal{C}\} = 0$ for every $A \in \Sigma_\alpha$. Since $\mathrm{span}\{\chi_{C \times \{0,1\}^{\kappa\setminus\alpha}} : C \in \mathcal{C}\}$ is dense in $Q_\alpha(L^1(\lambda_\kappa))$, we get the desired inequality.
- For each $\omega < \alpha \le \kappa$, the set $\bigcup_{\omega \le \beta < \alpha} Q_{\beta+1}(L^1(\lambda_\kappa))$ is dense in $Q_\alpha(L^1(\lambda_\kappa))$. To prove this, fix $f \in Q_\alpha(L^1(\lambda_\kappa))$ and note that

  $$Q_{\beta+1}(f) = E(Q_{\beta'+1}(f)|\mathcal{F}_{\beta+1}) \quad \text{for every } \omega \le \beta \le \beta' < \alpha.$$

  So $(Q_{\beta+1}(f), \mathcal{F}_{\beta+1})_{\beta \in T}$ is a *martingale*, where T is the directed set $[\omega, \alpha)$. Since $f \in L^1(\lambda_\kappa)$ satisfies $E(f|\mathcal{F}_{\beta+1}) = Q_{\beta+1}(f)$ for every $\beta \in T$ and the

σ-algebra generated by $\bigcup_{\beta\in T}\mathcal{F}_{\beta+1}$ is $\mathcal{F}_\alpha$, we can apply [7, Corollary 2, p. 126] to conclude that $Q_{\beta+1}(f)\to E(f|\mathcal{F}_\alpha)=f$ in $L^1(\lambda_\kappa)$.

Step 3. Fix $\omega\leq\alpha<\kappa$. We claim that

$$(Q_{\alpha+1}-Q_\alpha)(L^1(\lambda_\kappa))=\{(\chi_{E_\alpha}-1/2)f:\ f\in Q_\alpha(L^1(\lambda_\kappa))\},\tag{2}$$

where $E_\alpha=\pi_\alpha^{-1}(\{1\})$. Indeed, since $\mathcal{F}_{\alpha+1}$ is exactly the σ-algebra generated by $\mathcal{F}_\alpha\cup\{E_\alpha\}$, it is easy to see that

$$\mathcal{F}_{\alpha+1}=\{(B\cap E_\alpha)\cup(B'\setminus E_\alpha):\ B,B'\in\mathcal{F}_\alpha\}.$$

The previous equality, the fact that $\mathcal{F}_{\alpha+1}$-simple (resp. $\mathcal{F}_\alpha$-simple) functions are dense in $Q_{\alpha+1}(L^1(\lambda_\kappa))$ (resp. $Q_\alpha(L^1(\lambda_\kappa))$) and the equality $Q_\alpha(\chi_{E_\alpha})=1/2$ allow us to deduce that (2) holds.

Notice that there is an isometric isomorphism

$$\phi_\alpha:L^1(\lambda_\alpha)\to Q_\alpha(L^1(\lambda_\kappa))$$

such that $\phi_\alpha(\chi_C)=\chi_{C\times\{0,1\}^{\kappa\setminus\alpha}}$ for every $C\in\Sigma_\alpha$. Thus, in view of (2), we can define an isomorphism

$$\varphi_\alpha:L^1(\lambda_\alpha)\to(Q_{\alpha+1}-Q_\alpha)(L^1(\lambda_\kappa)),\quad \varphi_\alpha(f):=(\chi_{E_\alpha}-1/2)\phi_\alpha(f).$$

Step 4. The case $\kappa=\omega_1$. Then $\{Q_\alpha:\omega\leq\alpha\leq\omega_1\}$ is a SPRI on $L^1(\lambda_{\omega_1})$. Let us check that it satisfies the required property. Let $f\in L^1(\lambda_\omega)$ be the function defined by $f(z):=1$ if $\pi_0(z)=0$, $f(z):=-1$ if $\pi_0(z)=1$. For each $n\in\mathbb{N}$ and each $z_0,\dots,z_{n-1}\in\{0,1\}$, define $f_{(z_0,\dots,z_{n-1})}\in L^1(\lambda_\omega)$ by

$$f_{(z_0,\dots,z_{n-1})}(z):=\begin{cases}1 & \text{if } \pi_k(z)=z_k \text{ for every } 0\leq k<n \text{ and } \pi_n(z)=0,\\ -1 & \text{if } \pi_k(z)=z_k \text{ for every } 0\leq k<n \text{ and } \pi_n(z)=1,\\ 0 & \text{otherwise.}\end{cases}$$

It is well known that the sequence of Σ_ω-simple functions

$$1,f,f_{(0)},f_{(1)},f_{(0,0)},f_{(0,1)},f_{(1,0)},f_{(1,1)},\dots$$

(with this order!) is a Schauder basis of $L^1(\lambda_\omega)$: it is just the Haar basis of $L^1[0,1]$ (cf. [10, p. 164]) viewed through the usual measure space isomorphism between $([0,1],\mathcal{L},\lambda)$ and $(\{0,1\}^\omega,\Sigma_\omega,\lambda_\omega)$ (cf. [14, 254K]).

Fix $\omega\leq\alpha<\omega_1$. In view of the preceding paragraph and the observation isolated in Step 1, $L^1(\lambda_\alpha)$ has a Schauder basis made up of Σ_α-simple functions. Since the isomorphism φ_α of Step 3 maps Σ_α-simple functions to Σ_{ω_1}-simple ones, we conclude that $(Q_{\alpha+1}-Q_\alpha)(L^1(\lambda_{\omega_1}))$ has a Schauder basis made up of Σ_{ω_1}-simple functions. This finishes the proof of the case $\kappa=\omega_1$.

Step 5. We prove the statement of the lemma by transfinite induction on κ. The case $\kappa=\omega_1$ has already been considered in Step 4. So assume that $\kappa>\omega_1$ and that for every cardinal $\omega_1\leq\kappa'<\kappa$ the space $L^1(\lambda_{\kappa'})$ admits a SPRI $\{Q_\beta^{\kappa'}:\omega\leq\beta\leq\kappa'\}$ such that all the subspaces $(Q_{\beta+1}^{\kappa'}-Q_\beta^{\kappa'})(L^1(\lambda_{\kappa'}))$ have a Schauder basis made up of $\Sigma_{\kappa'}$-simple functions. It is now clear (by Steps 1 and 3) that for every $\omega_1\leq\alpha<\kappa$ the space $(Q_{\alpha+1}-Q_\alpha)(L^1(\lambda_\kappa))$ admits a SPRI $\{P_\beta^\alpha:\omega\leq\beta\leq\mathrm{card}(\alpha)\}$ such that, for each $\omega\leq\beta<\mathrm{card}(\alpha)$, the subspace

$$(P_{\beta+1}^\alpha-P_\beta^\alpha)\Big((Q_{\alpha+1}-Q_\alpha)(L^1(\lambda_\kappa))\Big)\tag{3}$$

has a Schauder basis made up of Σ_κ-simple functions. By [9, Proposition 6.2.7], the whole space $L^1(\lambda_\kappa)$ admits a SPRI $\{P_\alpha : \omega \leq \alpha \leq \kappa\}$. Moreover, a glance at the proof of [9, Proposition 6.2.7] reveals that for each $\omega \leq \alpha < \kappa$ the subspace $(P_{\alpha+1} - P_\alpha)(L^1(\lambda_\kappa))$ coincides with a space of the form (3) and so it has the required property. The proof of the lemma is now complete. □

As a consequence of Maharam's theorem on the classification of measure algebras, for any finite measure μ the space $L^1(\mu)$ is isometrically isomorphic to

$$\Big(\ell^1(\Gamma) \oplus \Big(\bigoplus_{i\in I} L^1(\lambda_{\kappa_i})\Big)_1\Big)_1$$

where Γ and I are countable (maybe empty) sets, each κ_i is an infinite cardinal and $\kappa_i \neq \kappa_{i'}$ whenever $i \neq i'$, cf. [20, Theorem 9, p. 127]. Bearing in mind this fact, we are now ready to prove the main result of this section.

Theorem 3.5. *Let μ be a finite measure. Then a function $f : [0,1] \to L^1(\mu)$ is McShane integrable if and only if it is Pettis integrable.*

Proof. Since $L^1(\mu)$ is WCG, it is weakly measure compact and so it suffices to show that $L^1(\mu)$ belongs to $\mathcal{P}$. Moreover, in view of the comments preceding the theorem and Proposition 3.1 (ii), we can suppose without loss of generality that $\mu = \lambda_\kappa$ for some uncountable cardinal κ (recall that all separable Banach spaces belong to $\mathcal{P}$).

Fix a scalarly null function $f : [0,1] \to L^1(\lambda_\kappa)$. By Lemma 2.3, in order to prove that f is McShane integrable we only have to check that f is the pointwise limit of a sequence of scalarly null McShane integrable functions. In fact, we will show that for each $\varepsilon > 0$ there is a scalarly null McShane integrable function $g : [0,1] \to L^1(\lambda_\kappa)$ such that $\|f(t) - g(t)\|_{L^1(\lambda_\kappa)} \leq \varepsilon$ for every $t \in [0,1]$.

According to Lemma 3.4, $L^1(\lambda_\kappa)$ admits a SPRI $\{P_\alpha : \omega \leq \alpha \leq \kappa\}$ such that, for each $\omega \leq \alpha < \kappa$, the subspace $(P_{\alpha+1} - P_\alpha)(L^1(\lambda_\kappa))$ has a Schauder basis (x_n^α) made up of elements of $i(L^2(\lambda_\kappa))$, where $i : L^2(\lambda_\kappa) \to L^1(\lambda_\kappa)$ is the "inclusion" operator. Write $R_\alpha := P_{\alpha+1} - P_\alpha$ and let

$$R_{\alpha,m} : R_\alpha(L^1(\lambda_\kappa)) \to L^1(\lambda_\kappa)$$

be the canonical projection onto $\mathrm{span}\{x_1^\alpha, \dots, x_m^\alpha\}$ for every $m \in \mathbb{N}$.

Fix $\varepsilon > 0$. For each $t \in [0,1]$ there exist a finite set $\{\alpha_1(t), \dots, \alpha_{n(t)}(t)\} \subset \kappa$ and real numbers $a_1(t), \dots, a_{n(t)}(t)$ such that

$$\Big\| f(t) - \sum_{i=1}^{n(t)} a_i(t) R_{\alpha_i(t)}(f(t)) \Big\|_{L^1(\lambda_\kappa)} \leq \varepsilon.$$

Set $g(t) := \sum_{i=1}^{n(t)} a_i(t) R_{\alpha_i(t)}(f(t))$ for every $t \in [0,1]$. We will see that *the function $g : [0,1] \to L^1(\lambda_\kappa)$ is scalarly null and McShane integrable.*

Fix $n \in \mathbb{N}$ and consider the function $f_n : [0,1] \to L^1(\lambda_\kappa)$ given by

$$f_n(t) = \begin{cases} a_n(t) R_{\alpha_n(t)}(f(t)) & \text{if } n(t) \geq n, \\ 0 & \text{if } n(t) < n. \end{cases}$$

Claim. f_n is scalarly null and McShane integrable. Indeed, fix $m \in \mathbb{N}$ and define a function $f_{n,m} : [0,1] \to L^1(\lambda_\kappa)$ by

$$f_{n,m}(t) = \begin{cases} a_n(t)(R_{\alpha_n(t),m} \circ R_{\alpha_n(t)})(f(t)) & \text{if } n(t) \geq n, \\ 0 & \text{if } n(t) < n. \end{cases}$$

We next prove that $f_{n,m}$ *is scalarly null.* To this end, fix $x^* \in L^1(\lambda_\kappa)^*$. According to Lemma 2.1, the set $\{\omega \leq \alpha < \kappa : x^*|_{R_\alpha(L^1(\lambda_\kappa))} \not\equiv 0\}$ is countable, so we can enumerate it as $\{\alpha_1, \alpha_2, \dots\}$. For each $l \in \mathbb{N}$, set

$$B_l := \{t \in [0,1] : \ n(t) \geq n, \ \alpha_n(t) = \alpha_l\}.$$

Then $(x^* \circ f_{n,m})\chi_{B_l}$ vanishes a.e., because f is scalarly null and

$$(x^* \circ f_{n,m})(t) = a_n(t)(x^* \circ R_{\alpha_l,m} \circ R_{\alpha_l} \circ f)(t) \quad \text{for every } t \in B_l.$$

Writing $B := \bigcup_{l=1}^{\infty} B_l$, we infer that $(x^* \circ f_{n,m})\chi_B$ vanishes a.e. Since $x^* \circ f_{n,m}$ vanishes on $[0,1] \setminus B$, we conclude that $x^* \circ f_{n,m}$ vanishes a.e., as required.

Since $f_{n,m}$ is scalarly null and $f_{n,m}([0,1]) \subset i(L^2(\lambda_\kappa))$, an appeal to Lemma 3.3 establishes that $f_{n,m}$ *is McShane integrable* (bear in mind that i^* has norm dense range and that $L^2(\lambda_\kappa)$ belongs to $\mathcal{P}$, by Corollary 3.2). Finally, the fact that $f_{n,m} \to f_n$ pointwise as $m \to \infty$ allows us to apply Lemma 2.3 to infer that f_n is scalarly null and McShane integrable, as claimed.

Since $g(t) = \sum_{n=1}^{\infty} f_n(t)$ for every $t \in [0,1]$, another appeal to Lemma 2.3 ensures us that g is scalarly null and McShane integrable. The proof is over. □

Theorem 3.5 can be seen as an strengthening of the equivalence of McShane and Pettis integrability in Hilbert spaces (Corollary 3.2), because $\ell^2(\kappa)$ is isomorphic to a closed subspace of $L^1(\lambda_\kappa)$ for any infinite cardinal κ, see e.g., [10, Theorem 6.28] (case $\kappa = \omega$) and [20, Theorem 12, p. 128] (general case).

Observe that the conclusion of Theorem 3.5 is also valid when μ is σ-finite, since in this case $L^1(\mu)$ is isometrically isomorphic to $L^1(\mu')$ for some finite measure μ'.

4. Another example of a scalarly null function which is not McShane integrable

As we mentioned in the introduction, the following example involves the WLD Banach space whose dual unit ball fails property (M) constructed (under CH) by G. Plebanek and O. Kalenda [26].

Recall (see e.g., [15, §311]) that the *Stone space* of a Boolean algebra $\mathcal{A}$ is the set $\mathrm{Ult}(\mathcal{A})$ of all ultrafilters on $\mathcal{A}$, equipped with the compact Hausdorff topology generated by the sets of the form $\widehat{A} = \{\mathcal{U} \in \mathrm{Ult}(\mathcal{A}) : A \in \mathcal{U}\}$, where $A \in \mathcal{A}$.

Example 4.1 (Under CH)**.** *There exist a WLD Banach space X, a scalarly null function $h : [0,1] \to X$ and an absolutely summing operator u from X to another Banach space Y such that $u \circ h : [0,1] \to Y$ is not Bochner integrable. In particular, h is not McShane integrable.*

Proof. Let $\mathcal{A} \subset \Sigma_{\omega_1}$ be the sub-algebra constructed in the proof of [26, Theorem 3.1]. It is shown there that $\mathcal{A}$ satisfies the following properties:

(a) $K := \mathrm{Ult}(\mathcal{A})$ is a Corson compactum.
(b) $\lambda_{\omega_1}(A) > 0$ for every non-empty $A \in \mathcal{A}$.
(c) $\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$, where each $\mathcal{A}_\alpha$ is a sub-algebra and $\mathcal{A}_\alpha \subset \mathcal{A}_\xi$ whenever $\alpha < \xi < \omega_1$.
(d) For each $\xi < \omega_1$ there exist $H_\xi^0, H_\xi^1 \in \mathcal{A}_\xi$ non-empty and disjoint such that

$$\lambda_{\omega_1}(C \cap H_\xi^0) = \lambda_{\omega_1}(C \cap H_\xi^1) \quad \text{for every } C \in \bigcup_{\alpha<\xi} \mathcal{A}_\alpha.$$

(e) Define $g_\xi := \chi_{\widehat{H^0_\xi}} - \chi_{\widehat{H^1_\xi}} \in C(K)$ for every $\xi < \omega_1$. Then for each Radon probability ν on K the set $\{\xi < \omega_1 : \int_K g_\xi \, d\nu \neq 0\}$ is countable.

Let μ be the unique Radon probability on K satisfying $\mu(\widehat{A}) = \lambda_{\omega_1}(A)$ for every $A \in \mathcal{A}$, cf. [16, Proposition 416Q]. As shown in the proof of [26, Theorem 3.1], property (d) ensures that $L^1(\mu)$ is non-separable. In fact, a similar computation yields the following property:

(f) No uncountable subset of $\{g_\xi : \xi < \omega_1\}$ is $\|\cdot\|_{L^1(\mu)}$-separable.

Indeed, observe first that for each $\alpha < \xi < \omega_1$ we have $|g_\alpha - g_\xi| \geq 1$ on $\widehat{H^0_\alpha} \triangle \widehat{H^0_\xi}$, hence (d) can be applied to get

$$\begin{aligned} (4) \quad \int_K |g_\alpha - g_\xi| \, d\mu &\geq \mu(\widehat{H^0_\alpha} \triangle \widehat{H^0_\xi}) = \mu(\widehat{H^0_\alpha} \setminus \widehat{H^0_\xi}) + \mu(\widehat{H^0_\xi} \setminus \widehat{H^0_\alpha}) \\ &\geq \mu(\widehat{H^0_\alpha} \cap \widehat{H^1_\xi}) + \mu(\widehat{H^0_\xi} \setminus \widehat{H^0_\alpha}) = \mu(\widehat{H^0_\alpha} \cap \widehat{H^0_\xi}) + \mu(\widehat{H^0_\xi} \setminus \widehat{H^0_\alpha}) = \mu(\widehat{H^0_\xi}). \end{aligned}$$

In order to prove (f), fix $I \subset \omega_1$ such that $\{g_\alpha : \alpha \in I\}$ is $\|\cdot\|_{L^1(\mu)}$-separable. Take $J \subset I$ countable such that $\{g_\alpha : \alpha \in J\}$ is $\|\cdot\|_{L^1(\mu)}$-dense in $\{g_\alpha : \alpha \in I\}$. Since J is countable, there is $\xi' < \omega_1$ such that $\alpha < \xi'$ for every $\alpha \in J$. In view of (4), we have

$$\inf_{\alpha \in J} \int_K |g_\alpha - g_\xi| \, d\mu \geq \mu(\widehat{H^0_\xi}) > 0 \quad \text{for every } \xi' \leq \xi < \omega_1,$$

thus $g_\xi \notin \overline{\{g_\alpha : \alpha \in J\}}^{\|\cdot\|_{L^1(\mu)}}$ for every $\xi' \leq \xi < \omega_1$. It follows that $I \subset \xi'$, so I is countable and the proof of (f) is finished.

Let $X := \overline{\mathrm{span}}\{g_\xi : \xi < \omega_1\} \subset C(K)$. Kalenda proved in [26, Corollary 4.4] that X is a WLD Banach space such that (B_{X^*}, w^*) fails property (M).

Fix any bijection $\varphi : [0,1] \to \omega_1$ and set

$$h : [0,1] \to X, \quad h(t) := g_{\varphi(t)}.$$

Let $i : X \to C(K)$ be the inclusion operator and $j : C(K) \to L^1(\mu)$ the "identity" operator (that sends each function to its equivalence class). It is well known that j is absolutely summing (see e.g., [6, 2.9]), hence the same holds for the composition $u = j \circ i : X \to L^1(\mu)$. We claim that h and u satisfy the required properties.

Clearly, in order to check that h *is scalarly null* we only have to show that for each Radon probability ν on K we have $\int_K h(t) \, d\nu = 0$ for a.e. $t \in [0,1]$. Given such a ν, property (e) ensures that the set $\{t \in [0,1] : \int_K g_{\varphi(t)} \, d\nu \neq 0\}$ is countable (bear in mind that φ is a bijection) and so it has Lebesgue measure 0, as required.

On the other hand, $u \circ h : [0,1] \to L^1(\mu)$ *is not Bochner integrable.* Indeed, take any $\Omega \in \mathcal{L}$ with $\lambda(\Omega) = 1$. Since Ω is uncountable and φ is a bijection, the set $(u \circ h)(\Omega) = \{g_{\varphi(t)} : t \in \Omega\}$ is not $\|\cdot\|_{L^1(\mu)}$-separable, by property (f). It follows that $u \circ h$ is not strongly measurable and, therefore, it is not Bochner integrable.

Finally, since the composition of a McShane integrable function with an absolutely summing operator is always Bochner integrable (see [22, Theorem 5] or [28, Theorem 3.13]), h cannot be McShane integrable. The proof is complete. □

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