Math. Nachr. **xxx** (2001), xxx – yyy

Measurable selectors for the metric projection

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(Received ???, 2001; revised version ???, 2001; accepted ???, 2001)

To Professor M. Valdivia on the occasion of his seventieth birthday

Abstract. Let Y and Z be two topological spaces and $F : Y \times Z \to \mathbb{R}$ a function that is upper semi-continuous in the first variable and lower semi-continuous in the second variable. If Z is Polish and for every $y \in Y$ there is a point $z \in Z$ with $F(y,z) = \inf_{w \in Z} F(y,w)$ we prove that there is a *nice* measurable function $h : Y \to Z$ satisfying $F(y,h(y)) = \inf_{z \in Z} F(y,z)$ for every $y \in Y$. As an application we obtain the existence of universally measurable selectors for the metric projection onto weakly $\mathcal{K}$-analytic convex proximinal subsets of a Banach space, that allows us to prove then that $L^p(\mu, Y)$ is proximinal in $L^p(\mu, X)$ for every proximinal weakly $\mathcal{K}$-analytic subspace Y of a Banach space X.

1. Introduction

Let Y and Z two topological spaces, $F : Y \times Z \to \mathbb{R}$ a function that is upper semi-continuous in the first variable and lower semi-continuous in the second variable. In the second section of this paper we are concerned with the study of the structure of the sets of attaining points of the kind

$$G = \{y \in Y : \text{ there is } z \in H \text{ such that } F(y,z) = \inf_{w \in Z} F(y,w)\}$$

for a given subset H of Z, and we prove, see theorem 2.1, that when H is *descriptive* then G is *descriptive* too. By doing so we are then able to prove, see corollary 2.5, that when Z is Polish and for every $y \in Y$ there is a point $z \in Z$ such that $F(y,z) = \inf_{w \in Z} F(y,w)$ then there is a *nice* measurable function $h : Y \to Z$ that selects points where the inf is attained, that is, satisfying

$$F(y,h(y)) = \inf_{z \in Z} F(y,z)$$

for every $y \in Y$. These results are particularized afterwards to several different situations in Banach spaces to find out, among other things, a *descriptive* structure for Bishop-Phelps sets, see corollary 2.3, and universally measurable selectors for the metric projection, see theorem 3.8.

2000 *Mathematics Subject Classification.* Primary: 41A65. 46B20 and 46E40.
Keywords and phrases. Proximinal subsets, $\mathcal{K}$-analytic spaces, Bochner integrable functions.

Throughout this paper X will be a real Banach space with a fixed norm $\| \ \|$, B_X its closed unit ball and X^* its dual; (Ω, Σ, μ) will be a complete probability space and for $1 \leq p \leq \infty$, $L^p(\mu, X)$ is the space of all Bochner p-integrable (measurable and essentially bounded for $p = \infty$) functions on Ω with values in X endowed with the usual p-norm. Given a subset Y of X and $x \in X$ we write, as usual, $d(x, Y) := \inf_{y \in Y} \|x - y\|$ for the distance from x to Y; for this given subset Y we naturally have the multi-valued map (maybe with some empty values) given by

$$P_Y(x) = \{y \in Y : \|x - y\| = d(x, Y)\}$$

for every $x \in X$, that we call the metric projection onto Y. A convex subset Y of X is said to be proximinal in X provided that for every $x \in X$ the set of its best approximation points $P_Y(x)$ is non empty. Proximinal subsets are natural objects in Approximation Theory and selectors for the metric projection have been widely studied in the literature, see [14, 15] and the references therein.

When $Y \subset X$ is proximinal we consider, in the third section of this paper, the map in two variables $F : Y \times X \to \mathbb{R}$ given by $F(y, z) := \|y - z\|$ and apply to it the result about the selector previously found in the second section to prove

Theorem 3.8 *Let X be a Banach space and Y a proximinal and weakly $\mathcal{K}$-analytic convex subset of X. Then for every separable closed subset $M \subset X$ the metric projection $P_Y|_M : M \to 2^Y$ has an analytic measurable selector with separable range.*

All concepts needed to properly understand the statements in theorem 3.8 can be found in section 3 below. To really feel the scope of the theorem it suffices to say that the class of proximinal vector subspaces Y which are $\mathcal{K}$-analytic for the weak topology includes the classes of reflexive subspaces, proximinal separable subspaces, proximinal weakly compactly generated subspaces, proximinal quasi-reflexive subspaces, etc.

Theorem 3.8 is a nice tool to study how proximinality is transferred to spaces of Bochner integrable functions. Our paper ends by proving

Theorem 3.9 *Let (Ω, Σ, μ) be a complete probability space, X a Banach space, Y a weakly $\mathcal{K}$-analytic subspace of X and $1 \leq p \leq \infty$. Then $L^p(\mu, Y)$ is proximinal in $L^p(\mu, X)$ if, and only if, Y is proximinal in X.*

which properly extends results about proximinality of $L^p(\mu, Y)$ in [12] for Y finite dimensional subspace of $L^1(\mu)$, in [8] for reflexive subspaces Y, in [9] for separable quasireflexive proximinal subspaces Y and in [13] for separable proximinal subspaces Y. Let us stress that after theorem 3.9, $L^p(\mu, Y)$ is proximinal in $L^p(\mu, X)$ even for all proximinal subspaces Y when $X = L^1(\mu)$ and for all quasireflexive proximinal subspaces Y of X without the separability assumption in [9].

On the other hand, Mendoza has built in [13] a Banach space X with a proximinal subspace Y such that $L^p(\mu, Y)$ is not proximinal in $L^p(\mu, X)$ where $1 \leq p \leq \infty$. This implies, in particular, that the selection theorem 3.8 is not true in general for arbitrary proximinal subspaces.

2. Structure of the set of attaining points: measurable selectors

We shall start by recalling some definitions from descriptive set theory. $\mathbb{N}^{\mathbb{N}}$ denotes the space of sequences of positive integers endowed with its product topology and $\mathbb{N}^{(\mathbb{N})}$ is the set of finite sequences of positive integers. Given $\sigma \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$, let $\sigma|n = (\sigma(1), \dots \sigma(n)) \in \mathbb{N}^{(\mathbb{N})}$. Given a topological space X and a family $\mathcal{A}$ of subsets of X, we will say that $A \subset X$ is Souslin-$\mathcal{A}$ in X if it can be written as

$$A = \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n=1}^{\infty} A(\sigma|n)$$

where $A(s) \in \mathcal{A}$ for every finite sequence $s \in \mathbb{N}^{(\mathbb{N})}$. When $\mathcal{A}$ is the family of the Borel subsets of X, A is said to be Čech-analytic in X; when $\mathcal{A} = \mathcal{F}$ is the family of closed subsets of X, we will simply say that A is a Souslin-$\mathcal{F}$ in X. In completely metrizable separable spaces the concepts of Čech-analytic subset and Souslin-$\mathcal{F}$ subset coincide with the classical notion of analytic subset as image of a Polish space. We say that a map F from a topological space T to the power set of X is upper semi-continuous if for each $t \in T$ and each open set G of X containing $F(t)$, there is a neighborhood U of t with $F(U) \subset G$. A subset A of X is said to be $\mathcal{K}$-analytic if there is an upper semi-continuous map F from $\mathbb{N}^{\mathbb{N}}$ to the family of compact subsets of X such that $A = \bigcup_{\sigma} F(\sigma)$. Every $\mathcal{K}$-analytic subset A of X can be written as

$$A = \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n=1}^{\infty} H(\sigma|n)$$

where the family $\{H(s) : s \in \mathbb{N}^{(\mathbb{N})}\}$ is made of subsets of A having the following property, that we shall call (CL):

> for every $\sigma \in \mathbb{N}^{\mathbb{N}}$ if we take $z_n \in H(\sigma|n)$, $n \in \mathbb{N}$, then the sequence $(z_n)_n$ has a cluster point in A.

Good references for analytic, $\mathcal{K}$-analytic and Čech-analytic spaces are [2, 4, 5, 7]. A Banach space that is $\mathcal{K}$-analytic for its weak topology will be referred as weakly $\mathcal{K}$ -analytic; the class of weakly $\mathcal{K}$-analytic Banach spaces contains the reflexive, the separable and the weakly compactly generated Banach spaces, among others, see [16].

Given two sets Y and Z and a map $F : Y \times Z \to \mathbb{R}$ we will denote for every $z \in Z$ (resp. $y \in Y$) by F^z (resp. F_y) the partial function

$$F^z(y) := F(y, z), \quad \text{for } y \in Y,$$

$$(\text{resp. } F_y(z) := F(y, z), \quad \text{for } z \in Z).$$

The following result will play an important role in this paper.

Theorem 2.1. *Let Y and Z be topological spaces and $F : Y \times Z \to \mathbb{R}$ be a map satisfying:*

H1. *F^z is upper semi-continuous for every $z \in Z$;*

H2. *F_y is lower semi-continuous for every $y \in Y$.*

Then for every $\mathcal{K}$-analytic subset H of Z the set

$$G = \{y \in Y : \text{ there is } z \in H \text{ such that } F(y,z) = \inf_{w \in Z} F(y,w)\}$$

is Čech-analytic in Y.

Proof. Let us take a family $\{H(s) : s \in \mathbb{N}^{(\mathbb{N})}\}$ of subsets of H satisfying

$$H = \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n=1}^{\infty} H(\sigma|n)$$

and having property (CL). Let us define

$$G(s) = \{y \in Y : \inf_{z \in H(s)} F(y,z) = \inf_{z \in Z} F(y,z)\}$$

for every finite sequence $s \in \mathbb{N}^{(\mathbb{N})}$. Every $G(s)$ is a Borel subset of Y because it appears as the set where two upper semi-continuous functions coincide. We claim that

$$G = \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n=1}^{\infty} G(\sigma|n).$$

Indeed, if $y \in G$ we can take $z \in H$ such that $F(y,z) = \inf_{w \in Z} F(y,w)$; if $\sigma \in \mathbb{N}^{\mathbb{N}}$ has been taken in such a way that $z \in \bigcap_{n=1}^{\infty} H(\sigma|n)$, then we have that $y \in \bigcap_{n=1}^{\infty} G(\sigma|n)$. Conversely, if $y \in \bigcap_{n=1}^{\infty} G(\sigma|n)$ for some $\sigma \in \mathbb{N}^{\mathbb{N}}$ then there is $z_n \in H(\sigma|n)$, for every $n \in \mathbb{N}$, such that

$$F(y, z_n) \le \inf_{w \in Z} F(y,w) + 1/n.$$

Take now a cluster point $z \in H$ of the sequence $(z_n)_n$. The lower semicontinuity of F_y implies that $F(y,z) \le \inf_{w \in Z} F(y,w)$, and so $y \in G$ and the proof is concluded. ■

For a given set Z the product topology of $\mathbb{R}^Z$ will be denoted by $t_p(Z)$.

Corollary 2.2. *Let Z be a topological space, $H \subset Z$ a $\mathcal{K}$-analytic subset and Y a subset of the space $C_b(Z)$ of bounded continuous functions on Z. Then the set*

$$G = \{g \in Y : \text{ there is } z \in H \text{ such that } g(z) = \sup_{w \in Z} g(w)\}$$

is Čech-analytic in Y endowed with the topology induced by $t_p(Z)$. In particular, if Z is assumed to be metric and compact then G is analytic when endowed with the topology $t_p(Z)$.

Proof. To get the first part of the corollary it is enough to use theorem 2.1 for the map $F = Y \times Z \to \mathbb{R}$ given by $F(g,z) = -g(z)$ for $g \in Y$ and $z \in Z$. The second part follows from the first part because when Z is metric and compact, G is a Čech-analytic subset of the analytic space $(C(Z), t_p(Z))$, and therefore G itself is analytic too. ■

The previous corollary is in the same vein as the known result stating that the space of the continuous functions on $[0,1]$ which attain their maximum on the irrationals is analytic non Borel, [7]; let us observe that this last statement implies that even when H is Polish it is not possible to ensure that G in theorem 2.1 or its corollary 2.2 is Borel.

A particular case of the former corollary is the following one.

Corollary 2.3. *Let X be a weakly $\mathcal{K}$-analytic Banach space. Then the Bishop-Phelps set*

$$NA = \{x^* \in X^* : \text{ there is } x \in B_X \text{ such that } \|x^*\| = x^*(x)\}$$

is weak*-Čech-analytic in X^*.

Proof. Apply the former corollary to $Z = B_X$ endowed with the weak topology, $H = B_X$ and $Y = X^* \subset C_b(Z)$. ■

Theorem 2.1 also allows us to describe sets where minimum distances are attained.

Corollary 2.4. *Let X be a Banach space and C and M subsets of X. Then for any weakly $\mathcal{K}$-analytic subset H of C, the set*

$$G = \{x \in M : \text{ there is } y \in H \text{ such that } \|x - y\| = \inf_{w \in C} \|x - w\|\}$$

is a norm Souslin-$\mathcal{F}$ subset of M.

Proof. Consider $Y = M$ endowed with the topology induced by the norm topology, $Z = C$ endowed with the weak topology and let $F(y,z) := \|y - z\|$, for $y \in Y$ and $z \in Z$. For every $y \in Y$ the function F_y is lower semi-continuous on Z and for every $z \in Z$ the function F^z is continuous on Y. Therefore, theorem 2.1 allows us to ensure that G is a norm Čech-analytic subset of Y. Furthermore, in this case, the sets $G(s)$ in the proof of theorem 2.1 are norm closed in Y because they are sets where two continuous functions on Y coincide, namely, the distance functions to H and $H(s)$. ■

It should be noted that when H is closed, convex and weakly $\mathcal{K}$-analytic in corollary 2.4 we can then ensure that G is weakly Čech-analytic; to see this we have to take into account that under these hypothesis the sets $H(s)$ in the proof of theorem 2.1 can be taken to be convex and therefore each $G(s)$ appears as a set where two convex, hence weakly lower semi-continuous, functions coincide.

We will say that a map between topological spaces $f : Y \to Z$ is Čech-analytic (resp. Souslin-$\mathcal{F}$, analytic) measurable if the preimage of any Borel subset of Z belongs to the smallest σ-algebra containing the Čech-analytic (resp. Souslin-$\mathcal{F}$, analytic) subsets of Y. A selector for a multivalued map $\phi : Y \to 2^Z$ is a single-valued map $f : Y \to Z$ such that $f(y) \in \phi(y)$ for every $y \in Y$.

Theorem 2.1 and the Kuratowski and Ryll-Nardzewski selection theorem allow us to prove the following selection result,

Corollary 2.5. *Let Y be a topological space, Z Polish and $F : Y \times Z \to \mathbb{R}$ a map satisfying:*

H1. *F^z is upper semi-continuous for every $z \in Z$;*

H2. *F_y is lower semi-continuous for every $y \in Y$;*

H3. *for every $y \in Y$ there is $z \in Z$ such that $F(y,z) = \inf_{w\in Z} F(y,w)$.*

Then there is a Čech-analytic measurable map $h : Y \to Z$ such that

$$F(y, h(y)) = \inf_{z\in Z} F(y,z)$$

for every $y \in Y$.

Proof. Let $\phi : Y \to 2^Z$ be the multivalued map given by

$$\phi(y) := \{z \in Z : F(y,z) = \inf_{w\in Z} F(y,w)\}.$$

We want to find a Čech-analytic measurable selector $h : Y \to Z$ for ϕ. According to Kuratowski and Ryll-Nardzewski's theorem [11, Theorem on page 398], we have to prove that for every open set $H \subset Z$ the set

$$\phi^{-1}(H) := \{y \in Y : \phi(y) \cap H \neq \emptyset\}$$

is Čech-analytic in Y. To see this it is enough to bear in mind that if H is open in the Polish space Z then H is Polish too, in particular $\mathcal{K}$-analytic; now from the equality

$$\phi^{-1}(H) = \{y \in Y : \text{ there is } z \in H \text{ such that } F(y,z) = \inf_{w\in Z} F(y,w)\}$$

we get that $\phi^{-1}(H)$ is Čech-analytic after theorem 2.1, and the proof is therefore concluded. ■

Similar arguments to those used in the previous proof together, now, with corollary 2.2 would allow us to show that when (Z,d) is a metric compact space the multivalued map $\psi : C(Z) \to 2^Z$ given by

$$\psi(g) := \{z \in Z : g(z) = \sup_{w\in Z} g(w)\}$$

for every $g \in C(Z)$, has a $t_p(Z)$-analytic measurable selector. Nonetheless, in this case of Z being a metric compact space ψ has a selector $h : C(Z) \to Z$ which is the pointwise limit of a sequence of continuous functions for the supremum norm of $C(Z)$ and the metric of Z, that is, h is a norm-to-d first Baire class function, [3, Theorem I.4.7].

3. Selectors for the metric projection: applications

As said in the introduction, the metric projection on a subset Y of a Banach space $(X, \|\ \|)$ is the multivalued map given by $P_Y(x) = \{y \in Y : \|x - y\| = d(x,Y)\}$ for every $x \in X$. In what follows we shall denote by

$$Q(Y) = \{x \in X : P_Y(x) \neq \emptyset\}.$$

With this notation, a subset $Y \subset X$ is proximinal in X when $Q(Y) = X$. From corollary 2.4 it follows that $Q(Y)$ is a norm Souslin-$\mathcal{F}$ subset of X if Y is weakly $\mathcal{K}$-analytic.

It is easily checked that when $Y \subset X$ is a reflexive subspace (and therefore proximinal!), the proximinal map $P_Y : X \to 2^Y$ is norm-to-weak upper semicontinuos with non empty weakly compact values; thus Theorem 8 in [6] can be applied to get that P_Y has a selector $h : X \to Y$ that is of the first Baire class with respect to the norm of X and Y.

In this section we will take advantage of our work already done and find other measurable selectors for the metric projection when Y is proximinal but not necessarily reflexive. The first result we can get in this setting is the following corollary.

Corollary 3.6. *Let X be a Banach space and $Y \subset X$ a closed separable subset. If $M \subset Q(Y)$, then the metric projection $P_Y|_M : M \to 2^Y$ has a norm Souslin-$\mathcal{F}$ selector $h : M \to Y$. In particular, when Y is assumed to be separable and proximinal then the metric projection $P_Y : X \to 2^Y$ does have a norm Souslin-$\mathcal{F}$ selector.*

Proof. The second part of the corollary obviously follows from the first part and the proof for this latter one goes as the one for corollary 2.5. Indeed, $M \subset Q(Y)$ means that the map $P_Y|_M : M \to 2^Y$ takes no empty values and corollary 2.4 applies to tell us that for every open set $H \subset Y$ the set

$$
\begin{aligned}
(3.1) \quad P_Y^{-1}(H) &:= \{x \in M : P_Y(x) \cap H \neq \emptyset\} = \\
&= \{x \in M : \text{ there is } y \in H \text{ such that } \|x - y\| = \inf_{w \in Y} \|x - w\|\}
\end{aligned}
$$

is norm Souslin-$\mathcal{F}$ in M. Again, an appeal to Kuratowski and Ryll-Nardzewski's theorem, [11, Theorem on page 398], finishes the proof. ∎

Let us observe that when in the previous theorem M is assumed norm closed and separable then $P_Y^{-1}(H)$ is even analytic for H being either open or closed.

Corollary 3.6 naturally leads to the following now easy equivalence that is worth of being labelled as a lemma.

Lemma 3.7. *Let X be a Banach space, $Y \subset X$ closed and $M \subset Q(Y)$ closed and separable. The following statements are equivalent:*

(i) *There is $Y_0 \subset Y$ closed, separable and such that for every $x \in M$ there is $y_0 \in Y_0$ for which*

$$\|x - y_0\| = d(x, Y) = d(x, Y_0).$$

(ii) *The metric projection $P_Y|_M : M \to 2^Y$ has a norm analytic selector with separable range.*

Under the Continuum Hypothesis every analytic measurable map from a Polish space into a metric space has automatically separable range (a proof for this goes as the one in [10, Page 398] for Borel measurable maps); so if we assume (CH) we do not need to stress separable range in condition *(ii)* in the former lemma.

We are now ready to get rid of the separability assumption on the proximinal subset Y in corollary 3.6. However we need that the set M should be separable

Theorem 3.8. *Let X be a Banach space and Y a proximinal and weakly $\mathcal{K}$-analytic convex subset of X. Then for every separable closed subset $M \subset X$ the metric projection $P_Y|_M : M \to 2^Y$ has an analytic measurable selector with separable range.*

Proof. It is enough to check that condition *(i)* in lemma 3.7 is satisfied. To this end, let us fix $\{x_n : n \in \mathbb{N}\}$ a countable dense subset of M and let us take a family $\{H(s) : s \in \mathbb{N}^{(\mathbb{N})}\}$ of subsets of Y satisfying

$$Y = \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n=1}^{\infty} H(\sigma|n)$$

and having property (CL). For every pair $n, m \in \mathbb{N}$ and $s \in \mathbb{N}^{(\mathbb{N})}$ take $y(n,m,s) \in H(s)$ such that

$$\|x_n - y(n,m,s)\| \leq d(x_n, H(s)) + 1/m. \tag{3.2}$$

The set

$$Y_0 := \overline{\text{conv}}\{y(n,m,s) : n, m \in \mathbb{N}, s \in \mathbb{N}^{(\mathbb{N})}\}$$

is norm closed and separable. We shall prove that it fulfills the condition *(i)* in lemma 3.7. For a given $x \in M$ we take a sequence of integers $(n_k)_{k\in\mathbb{N}}$ such that $(x_{n_k})_{k\in\mathbb{N}}$ converges in the norm to x. Let now $y \in Y$ and $\sigma \in \mathbb{N}^{\mathbb{N}}$ be such that $\|x-y\| = d(x,Y)$ and $y \in H(\sigma|k)$ for every $k \in \mathbb{N}$. Then, according the inequality (3.2) we have

$$\|x_{n_k} - y(n_k, k, \sigma|k)\| \leq d(x_{n_k}, H(\sigma|k)) + 1/k \leq \|x_{n_k} - y\| + 1/k \tag{3.3}$$

for a certain $y(n_k, k, \sigma|k) \in H(k)$ for every $k \in \mathbb{N}$. If $y_0 \in Y_0$ is a weak cluster point of $(y(n_k, k, \sigma|k))_{k\in\mathbb{N}}$, inequality (3.3) implies $\|x-y_0\| \leq \|x-y\|$ because of the weak lower semicontinuity of the norm of X. It is clear that we have $\|x-y_0\| = d(x,Y) = d(x,Y_0)$ and therefore the proof is concluded. ■

When in the former theorem $P_Y|_M(x)$ is a singleton for every x in M (for instance when the norm in X is strictly convex) its unique selector $h : M \to Y$ with separable range is Borel measurable. Indeed, if $Y_0 = \overline{h(M)}^{\|\cdot\|}$ and we look at h as a function from M into Y_0 then for every open subset H of Y_0 the inverse images $h^{-1}(H) = P_Y|_M^{-1}(H)$ and $M \setminus h^{-1}(H) = P_Y^{-1}(Y_0 \setminus H)$ are analytic as it was remarked after corollary 3.6; therefore the Souslin separation theorem, see [10, Page 486], can be applied to get that $h^{-1}(H)$ is Borel.

As an application of the selection result above we finally prove

Theorem 3.9. *Let (Ω, Σ, μ) be a complete probability space, X a Banach space, Y a weakly $\mathcal{K}$-analytic subspace of X and $1 \leq p \leq \infty$. Then $L^p(\mu, Y)$ is proximinal in $L^p(\mu, X)$ if, and only if, Y is proximinal in X.*

Proof. A proof of the fact that $L^p(\mu, Y)$ being proximinal in $L^p(\mu, X)$ implies that Y is proximinal in X can be found in [19, Theorem 3.3] and also in [13, Corollary 2.6].

To prove the converse we will first prove the following claim.

Claim.- *If Y is a proximinal and weakly $\mathcal{K}$-analytic subspace of X, then for every function $f \in L^p(\mu, X)$ there is $g : \Omega \to Y$ Bochner measurable such that $g(w)$ is a best approximation for $f(w)$ for μ-almost all w in Ω.*

Given $f \in L^p(\mu, X)$ there is $\Omega_0 \in \Sigma$ such that $\mu(\Omega \setminus \Omega_0) = 0$ and $f(\Omega_0) \subset X$ is separable. Let M be the norm closure of $f(\Omega_0)$ in X and let $h : M \to Y$ be the norm analytic selector with separable range of $P_Y|_M : M \to Y$ ensured by theorem 3.8. The function $g : \Omega \to Y$ defined on Ω_0 as the composition $h \circ f$ and 0 otherwise, obviously satisfies by construction that $g(\omega)$ is a best approximation of $f(\omega)$ for μ-almost all w in Ω and it is Bochner measurable; measurability follows from the fact that $f|_{\Omega_0}$ is $\Sigma|_{\Omega_0}$-analytic measurable because a complete probability space is stable by the Souslin operation, [7, Theorem 29.16], and h is analytic measurable.

Let us now finish the proof of the theorem. Take $f \in L^p(\mu, X)$ and let $g : \Omega \to Y$ be the Bochner measurable function ensured in the claim; for every $h \in L^p(\mu, Y)$ we have $\|f(w) - g(w)\| \leq \|f(w) - h(w)\|$ μ-almost every w in Ω which implies that $g \in L^p(\mu, X)$ and thus g is a best approximation for f in $L^p(\mu, Y)$. ■

Mendoza has shown in [13] that there exists a Banach space X having a proximinal subspace Y such that $L^p(\mu, Y)$ is not proximinal in $L^p(\mu, X)$ where $1 \leq p \leq \infty$. This example also tell us that the selection theorem 3.8 is not true for an arbitrary proximinal subspaces.

We finish the paper with some remarks about the previous results containing some applications and possible extensions of them.

Remark 3.10. In theorem 3.9 the hypothesis Y weakly $\mathcal{K}$-analytic can be replaced by the weaker hypothesis of Y being weakly countably determined; Y is said to be weakly countably determined, briefly WCD, if there is $\Sigma \subset \mathbb{N}^{\mathbb{N}}$ and an upper semi-continuous compact valued map for the weak topology $F : \Sigma \to 2^Y$ such that $Y = \cup_{\sigma \in \Sigma} F(\sigma)$. The separable reduction argument in the proof of theorem 3.8 works when Y is only assumed to be WCD. The class of WCD Banach spaces properly contains the class of the weakly $\mathcal{K}$-analytic Banach spaces and the latter contains the class of weakly compactly generated Banach spaces which at the same time contains the separable Banach spaces and the reflexive Banach spaces [16, 17]. In [18] it was proved that if Y is a Banach space such that Y^{**}/Y is separable, then Y can be represented as a direct sum of closed linear subspaces $Y = A \oplus B$, where A is separable and B is reflexive, and therefore Y is weakly $\mathcal{K}$-analytic; in this way if Y is quasi-reflexive, that is of finite co-dimension in its bidual, then Y is weakly $\mathcal{K}$-analytic.

Remark 3.11. A question considered by several authors, see [14, 15] and the references therein, is when the metric projection does have a continuous selector. If the norm of the Banach space X is locally uniformly rotund, see [3, Definition 1.1], and $Y \subset X$ is a proximinal subspace, then it is not difficult to show that the metric projection is single valued and continuous. But even for finite dimensional Banach spaces the existence of a continuous selector for the metric projection cannot be assured, [1]. Nonetheless, if we change the question to,

> *How big is the set of the points of continuity for selectors for the metric projection?*

we can say something else using the results proved in this paper. It was already said at the beginning of section 3 that when $Y \subset X$ is a reflexive subspace the metric projection has a first Baire class selector; consequently, we know that in this case the selector is continuous on the complement of a set of first category, [7, Section 8]. The selectors found in corollary 3.6 and theorem 3.8 satisfy a weaker property: there is a set of first category such that the restriction of the selector to its complement is continuous; this last statement is a consequence of the fact that the σ-algebra of subsets with the Baire property, see [7, Section 29], contains the Souslin subsets.

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Acknowledgements

Research partially supported by the D.G.E.S. grant PB 98-0381.

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