THE QUANTITATIVE DIFFERENCE BETWEEN COUNTABLE COMPACTNESS AND COMPACTNESS

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ABSTRACT. We establish here some inequalities between distances of pointwise bounded subsets H of $\mathbb{R}^X$ to the space of real-valued continuous functions $C(X)$ that allow us to examine the quantitative difference between (pointwise) countable compactness and compactness of H relative to $C(X)$. We prove, amongst other things, that if X is a countably K-determined space the *worst* distance of the pointwise closure $\overline{H}$ of H to $C(X)$ is at most 5 times the *worst* distance of the sets of cluster points of sequences in H to $C(X)$: here distance refers to the metric of uniform convergence in $\mathbb{R}^X$. We study the quantitative behavior of sequences in H approximating points in $\overline{H}$. As a particular case we obtain the results known about angelicity for these $C_p(X)$ spaces obtained by Orihuela. We indeed prove our results for spaces $C(X, Z)$ (hence for Banach-valued functions) and we give examples that show when our estimates are sharp.

1. INTRODUCTION

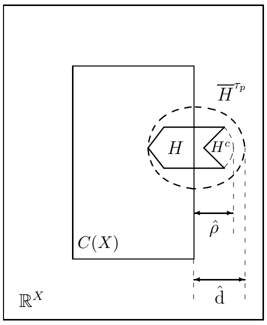

Figure 1

The type of problems that we study in this paper are illustrated in the Figure 1. Take X a topological space and let $C(X)$ be the space of real-valued continuous functions defined on X. Now consider $C(X) \subset \mathbb{R}^X$ as the figure shows and let d be the metric of uniform convergence on $\mathbb{R}^X$. In order to fix ideas we start with a pointwise bounded set $H \subset C(X)$ (a bit later we will allow H to be a subset of $\mathbb{R}^X$ as in the figure): if τ_p is the topology of pointwise convergence on $\mathbb{R}^X$, then Tychonoff's theorem says that $\overline{H}^{\mathbb{R}^X}$ is τ_p-compact. Therefore in order for H to be τ_p-relatively compact in $C(X)$ the only thing we must worry about is to have $\overline{H}^{\mathbb{R}^X} \subset C(X)$. Notice that if $\hat{d}$ is the *worst* distance from $\overline{H}^{\mathbb{R}^X}$ to $C(X)$ then $\hat{d} = 0$ if, and only, if $\overline{H}^{\mathbb{R}^X} \subset C(X)$ if, and only if, H is τ_p-relatively compact in $C(X)$. In general $\hat{d} \geq 0$ gives us a measure of non τ_p-compactness for H relative to $C(X)$. Hence the question is:

(A) *for which kind of spaces X can we theoretically compute $\hat{d}$?*

and moreover

(B) *are there useful estimates for $\hat{d}$ that are equivalent to qualitative properties of the sets H's?*

Here is a simplified case in the framework of (B) that we picture in Figure 1: take a pointwise bounded set $H \subset \mathbb{R}^X$ and let H^c be the set of those elements in $\overline{H}^{\mathbb{R}^X}$ that are cluster points of sequences in H (H^c is likely to be strictly smaller than

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$\overline{H}^{\mathbb{R}^X}$). If $\hat{\rho}$ is the *worst* distance from H^c to $C(X)$, the inclusion $H^c \subset \overline{H}^{\mathbb{R}^X}$ clearly implies $\hat{\rho} \leq \hat{d}$. We study the existence of a universal constant M such that for all pointwise bounded sets $H \subset \mathbb{R}^X$ we have that $\hat{d} \leq M\hat{\rho}$. We succeed by finding that this constant can be taken as $M = 2$ if $X = K$ is a compact space and the sets H are taken uniformly bounded and seated inside $C(K)$. More generally, for the very general class of countably K-determined spaces X (for topologists the class of Lindelöf Σ-spaces X) we prove that the universal constant can be taken as $M = 5$. The inequality

$$\hat{\rho} \leq \hat{d} \leq 5\hat{\rho}, \tag{1.1}$$

somehow quantifies the fact that τ_p-relatively countably compact and τ_p-relatively compact subsets of $C(X)$ are the same. To properly say so we prove in Theorem 3.2 an inequality sharper than (1.1) (when reading this theorem notice that $ck(H) \leq \hat{\rho}$) that really says about *the quantitative difference between countable compactness and compactness* for these $(C(X), \tau_p)$ spaces.

An answer to question (A) can be given using the result below:

Theorem 1.1 ([4, Proposition 1.18])**.** *Let X be a normal space. If $f \in \mathbb{R}^X$, then*

$$d(f, C(X)) = \frac{1}{2}\operatorname{osc}(f)$$

where $\operatorname{osc}(f) = \sup_{x\in X} \operatorname{osc}(f, x)$ *and*

$$\operatorname{osc}(f, x) := \inf_U \{ \sup_{y,z\in U} |f(y) - f(z)| : U \subset X \text{ open}, x \in U\}.$$

In the cited reference, the theorem is stated under more restrictive conditions: X is paracompact and f is uniformly bounded on X. A careful reading of the proof in the reference should be enough to convince the reader that the uniform boundedness of f is not needed and that paracompactness can be replaced by X being normal if one takes into account [6, Exercise 1.7.5 (b)].

This paper is organized as follows. Section 2 is devoted to establishing the already introduced inequalities for spaces $C(K)$, K compact, and bounded sets $H \subset C(K)$, see Theorem 2.3: here we use techniques about distances to spaces $C(K)$, distances of iterated limits and oscillations of functions introduced in [5] by W. Marciszewski, M. Raja and the second named author. In Example 2.4 it is shown that the constants involved in Theorem 2.3 are sharp.

In Section 3 we get rid of the constraints imposed in Theorem 2.3, namely: besides obtaining the results from compact spaces K by to countably K-determined topological space X, we deal with pointwise bounded sets $H \subset \mathbb{R}^X$ instead of uniformly bounded sets made up of continuous functions, see Theorems 3.1 and 3.2. To do so we have to prove a technical result for Z^X extending Proposition 2.2 to the case when (Z, d) is only assumed to be separable instead of compact and when the ε-interchanging of limits with the whole X is replaced by ε-interchanging of limits with some *distinguished* subsets of X: see Lemmas 1, 2 and 3. We note that all our results in this section are proved for spaces $C(X, Z)$ with X countably K-determined and (Z, d) metric and separable. We prove that our results here do imply the main result obtained by Orihuela in [12]. The paper ends up by showing that if X is a normal space with countable tightness (in particular metric space) then the constants involved in the proved inequalities can be sharpened, see Proposition 3.5 and Corollary 3.6.

A bit of terminology: by letters $T, X, Y, \ldots$ we denote here sets or completely regular topological spaces, (Z, d) is a metric space (Z if d is tacitly assumed); $\mathbb{R}$ is considered as a metric space endowed with the metric associated to $|\cdot|$. The space Z^X is equipped with the product topology τ_p. In Z^X we also consider the *standard supremum metric*, that abusively is also denoted by d and that we allow to take the value $+\infty$, i.e.,

$$d(f, g) = \sup\{d(f(x), g(x)) : x \in X\}$$

for functions $f, g : X \to Z$: we could have replaced the original metric in (Z, d) by a bounded one without changing the uniform structure of Z and thus providing us with a real uniform metric on Z^X; nonetheless we rather prefer to use the original metric of (Z, d) and then deal with the usual arithmetic with $+\infty$ and real numbers when needed. $C(X, Z)$ is the space of continuous maps from X into Z: $C(X)$ is the space of real-valued continuous functions and $C_b(X)$ stands for the subspace of $C(X)$ made up of uniformly bounded functions. With symbols $C_p(X, Z)$, and their like, we denote the space $C(X, Z)$ endowed with the topology induced by τ_p.

For A and B nonempty subsets of a metric space (Z, d), we consider the *usual distance* between A and B given by

$$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\},$$

and the *Hausdorff non-symmetrized distance* from A to B defined by

$$\hat{d}(A, B) = \sup\{d(a, B) : a \in A\}.$$

2. THE RESULT FOR $C(K)$ AND SHARPNESS OF THE CONSTANTS

The notion below introduced in [5] was first considered by Grothendieck in [9], for $\varepsilon = 0$. For $\varepsilon \geq 0$, this concept has also been used, in the framework of Banach spaces, in [3, 7, 11] amongst others.

Definition 1. *Let (Z, d) be a metric space, X a set and $\varepsilon \geq 0$.*

(i) *We say that a sequence $(f_m)_m$ in Z^X ε-interchanges limits with a sequence $(x_n)_n$ in X if*

$$d(\lim_n \lim_m f_m(x_n), \lim_m \lim_n f_m(x_n)) \leq \varepsilon$$

whenever all limits involved do exist.

(ii) *We say that a subset H of Z^X ε-interchanges limits with a subset A of X, if each sequence in H ε-interchanges limits with each sequence in A. When $\varepsilon = 0$ we simply say that H interchanges limits with A.*

The following two results appeared in [5]:

Proposition 2.1 ([5, Corollary 2.6])**.** *Let X be a topological space and let H be a uniformly bounded subset of $C_b(X)$. The following properties hold:*

(i) *if X is normal and H ε-interchanges limits with X, then*

$$\hat{d}(\overline{H}^{\mathbb{R}^X}, C_b(X)) \leq \varepsilon\,.$$

(ii) *if X is countably compact and $\hat{d}(\overline{H}^{\mathbb{R}^X}, C_b(X)) \leq \varepsilon$, then H 2ε-interchanges limits with X.*

Proposition 2.2 ([5, Proposition 5.2]). *Let (Z,d) be a compact metric space, K a set, and $H \subset Z^K$ a set which ε-interchanges limits with K. Then for any $f \in \overline{H}^{Z^K}$, there is a sequence $(f_n)_n$ in H such that*

$$\sup_{x\in K} d(g(x), f(x)) \leq \varepsilon$$

for any cluster point g of (f_n) in Z^K.

Let T a topological space. For a subset A of T, $A^{\mathbb{N}}$ is considered as the set of all sequences in A and the set of all cluster points in T of a sequence $\varphi \in A^{\mathbb{N}}$ is denoted by $\mathrm{clust}_T(\varphi)$. Recall that $\mathrm{clust}_T(\varphi)$ is a closed subset of T and it can be expressed as

$$\mathrm{clust}_T(\varphi) = \bigcap_{n\in\mathbb{N}} \overline{\{\varphi(m) : m > n\}}.$$

Combining the above two results we can prove now the result below.

Theorem 2.3. *Let K be a compact topological space and let H be a uniformly bounded subset of $C(K)$. If we define*

$$\mathrm{ck}(H) := \sup_{\varphi\in H^{\mathbb{N}}} d(\mathrm{clust}_{\mathbb{R}^K}(\varphi), C(K))$$

then

$$\mathrm{ck}(H) \leq \hat{d}(\overline{H}^{\mathbb{R}^K}, C(K)) \leq 2\,\mathrm{ck}(H), \tag{2.1}$$

and for any $f \in \overline{H}^{\mathbb{R}^K}$, there is a sequence $(f_n)_n$ in H such that

$$\sup_{x\in K} |g(x) - f(x)| \leq 2\,\mathrm{ck}(H) \tag{2.2}$$

for any cluster point g of (f_n) in $\mathbb{R}^K$.

Proof. The first inequality in (2.1) straightforwardly follows from the definitions involved. We prove now that if $\mathrm{ck}(H) < +\infty$ then H $2\,\mathrm{ck}(H)$-interchanges limits with K. Indeed, let $(f_m)_m$ be a sequence in H and $(x_n)_n$ a sequence in K and let us assume that both iterated limits

$$\lim_n \lim_m f_m(x_n), \lim_m \lim_n f_m(x_n)$$

exist in $\mathbb{R}$. If we fix $\alpha \in \mathbb{R}$ with $\alpha > \mathrm{ck}(H)$ the sequence $(f_m)_m$ has a τ_p-cluster point $f \in \mathbb{R}^K$ such that $d(f, C(K)) < \alpha$. Take and fix now $f' \in C(K)$ such that

$$\sup_{x\in K} |f(x) - f'(x)| < \alpha. \tag{2.3}$$

Let us pick $x \in K$ a cluster point of $(x_n)_n$. Since f' and each f_m are continuous $f'(x)$ and $f_m(x)$ are, respectively, cluster points in $\mathbb{R}$ of $(f'(x_n))_n$ and $(f_m(x_n))_n$. Hence we can produce a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $\lim_k f'(x_{n_k}) = f'(x)$. Thus we have that

$$\begin{aligned} &|\lim_k f(x_{n_k}) - f(x)| \leq \\ &\leq |\lim_k f(x_{n_k}) - \lim_k f'(x_{n_k})| + |f'(x) - f(x)| \overset{(2.3)}{\leq} 2\alpha. \end{aligned} \tag{2.4}$$

We conclude that

$$\lim_m \lim_n f_m(x_n) = \lim_m f_m(x) = f(x)$$

and so

$$|\lim_n \lim_m f_m(x_n) - \lim_m \lim_n f_m(x_n)| =$$

$$= |\lim_n \lim_m f_m(x_n) - f(x)| = |\lim_k f(x_{n_k}) - f(x)| \overset{(2.4)}{\leq} 2\alpha.$$

Now, the second inequality in (2.1) follows from Proposition 2.1 and (2.2) follows from Proposition 2.2. The proof is over. $\square$

Recall that a subset M of a topological space T is said to be relatively compact (resp. relatively compact) if $\overline{M} \subset T$ is compact (resp. every sequence in M has a cluster point in T). Observe that if H is relatively countably compact in $C_p(K)$ then $\operatorname{ck}(H) = 0$ and that H is relatively compact in $C_p(K)$ if, and only if, $\hat{d}(\overline{H}^{\mathbb{R}^K}, C(K)) = 0$. Theorem 2.3 says about approximation of points in the pointwise closure of H by sequences from H and about the quantitative difference between τ_p-countable compactness and τ_p-compactness of H relative to $C(K)$.

Example 2.4. The following example communicated to us by Prof. Marciszewski shows that the constant 2 in the inequality (2.1) in Theorem 2.3 cannot be improved. Consider $[0, \omega_1]$ the compact set of all the ordinals smaller or equal to the first non countable ordinal ω_1. Put

$$K = (\{-1, 1\} \times [0, \omega_1])/R$$

where R is the relation defined as xRy if, and only if

$$x = y \text{ or } x, y \in \{(-1, \omega_1), (1, \omega_1)\}.$$

Clearly K is a compact set. For $\alpha \prec \omega_1$ define $f_\alpha : K \to \mathbb{R}$ as

$$f_\alpha(i, \gamma) = \begin{cases} 0 & \text{if } \gamma \succ \alpha, \\ i & \text{if } \gamma \preceq \alpha \end{cases}$$

and put $H = \{f_\alpha : \alpha \prec \omega_1\} \subset C(K)$. If $(f_{\alpha_n})_n$ is a sequence in H and $\alpha := \sup\{\alpha_n : n \in \mathbb{N}\}$ then $\alpha \prec \omega_1$ and $f_{\alpha_n}(i, \beta) = 0$ for all $n \in \mathbb{N}$ and $\beta \succ \alpha$. So for every $\beta \succ \alpha$ we have that $g(i, \beta) = 0$ for each cluster point g of $(f_{\alpha_n})_n$. If we define $h : K \to \mathbb{R}$ as $h(i, \beta) = 0$ if $\beta \succ \alpha$ and $h(i, \beta) = i/2$ if $\beta \preceq \alpha$ then $h \in C(K)$ and $d(h, g) \leq 1/2$ for each cluster point g of $(f_{\alpha_n})_n$. Thus we conclude that $\operatorname{ck}(H) \leq 1/2$. On the other hand, the function $h' : K \to \mathbb{R}$ defined as $h'(i, \beta) = 0$ if $\beta = \omega_1$ and $h'(i, \beta) = i$ if $\beta \neq \omega_1$ belongs to $\overline{H}^{\mathbb{R}^K}$ and clearly $\operatorname{osc}(h') = 2 = 2d(h', C(K))$, see Theorem 1.1. Then

$$\hat{d}(\overline{H}^{\mathbb{R}^K}, C(K)) \geq d(h', C(K)) = 1 \geq 2 \operatorname{ck}(H)$$

and therefore by Theorem 2.3 $d(\overline{H}^{\mathbb{R}^K}, C(K)) = 2 \operatorname{ck}(H)$. $\square$

3. Approximation by sequences in $C_p(X)$

In this section we provide several lemmata leading to Theorem 3.1, that is a fairly general result of approximation by sequences, and to Theorem 3.2, whose inequalities say about the quantitative difference between countable compactness and compactness in the spaces $C_p(X, Z)$ considered. Here we will present our results in its more general scope extending ideas from [12].

If X is a topological space, (Z, d) a metric space and H a relatively compact subset of the space (Z^X, τ_p) we define

$$\mathrm{ck}(H) := \sup_{\varphi \in H^{\mathbb{N}}} d(\mathrm{clust}_{Z^K}(\varphi), C(X, Z)). \tag{3.1}$$

Note that Tychonoff's theorem implies that in (Z^X, τ_p) each relatively countably compact set is relatively compact.

Lemma 1. *Let X be a topological space, (Z, d) a metric space and H a relatively compact subset of the space (Z^X, τ_p). If we define*

$$\varepsilon := \mathrm{ck}(H) + \hat{d}(H, C(X, Z)),$$

then H 2ε-interchanges limits with relatively countably compact subsets of X.

Proof. The proof goes like the one in Theorem 2.3 but with some further precautions: in order to avoid a possible confusion of the reader we repeat some of the arguments already presented. We only have to take care of the case $\varepsilon < +\infty$. Let $(f_m)_m$ be a sequence in H and $(x_n)_n$ a sequence in a relatively countably compact subset of X and let us assume that both iterated limits

$$\lim_n \lim_m f_m(x_n), \lim_m \lim_n f_m(x_n)$$

exist. If we fix $\alpha \in \mathbb{R}$ with $\alpha > \mathrm{ck}(H)$, then the sequence $(f_m)_m$ has a τ_p-cluster point $f \in Z^X$ such that $d(f, C(X, Z)) < \alpha$. We observe that

$$\lim_n \lim_m f_m(x_n) = \lim_n f(x_n). \tag{3.2}$$

Take and fix now $f' \in C(X, Z)$ such that

$$d(f, f') < \alpha. \tag{3.3}$$

On the other hand, if we fix $\beta \in \mathbb{R}$ with $\beta > \hat{d}(H, C(X, Z))$, then for every $m \in \mathbb{N}$ there is $f'_m \in C(X, Z)$ such that

$$d(f_m, f'_m) < \beta. \tag{3.4}$$

Let us fix now $x \in X$ a cluster point of $(x_n)_n$. Since f' and each f'_m are continuous $f'(x)$ and $f'_m(x)$ are, respectively, cluster points of $(f'(x_n))_n$ and $(f'_m(x_n))_n$ in the metric space (Z, d); hence we can produce a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $\lim_k f'(x_{n_k}) = f'(x)$ and $\lim_k f'_m(x_{n_k}) = f'_m(x)$ for every $m \in \mathbb{N}$. Thus we have that

$$\begin{aligned} & d(\lim_k f(x_{n_k}), f(x)) \leq \\ & \qquad \leq d(\lim_k f(x_{n_k}), \lim_k f'(x_{n_k})) + d(f'(x), f(x)) \overset{(3.3)}{\leq} 2\alpha \end{aligned} \tag{3.5}$$

and that

$$\begin{aligned} & d(\lim_k f_m(x_{n_k}), f_m(x)) \leq \\ & \qquad \leq d(\lim_k f_m(x_{n_k}), \lim_k f'_m(x_{n_k})) + d(f'_m(x), f_m(x)) \overset{(3.4)}{\leq} 2\beta. \end{aligned} \tag{3.6}$$

We take now a subsequence $(f_{m_j})_j$ of $(f_m)_m$ such that $f(x) = \lim_j f_{m_j}(x)$ and we conclude that

$$d(\lim_m \lim_n f_m(x_n), f(x)) = d(\lim_j \lim_k f_{m_j}(x_{n_k}), \lim_j f_{m_j}(x)) \overset{(3.6)}{\leq} 2\beta$$

and

$$d(\lim_n \lim_m f_m(x_n), f(x)) \overset{(3.2)}{=} d(\lim_k f(x_{n_k}), f(x)) \overset{(3.5)}{\leq} 2\alpha.$$

The last two inequalities imply that

$$d(\lim_n \lim_m f_m(x_n), \lim_m \lim_n f_m(x_n)) \leq 2\varepsilon,$$

and the proof is over. □

Next easy lemma will be used repeatedly in the proof of Lemma 3.

Lemma 2. *Suppose that (Z, d) is a separable metric space and let X be a set. Given functions $f_1, \dots, f_n \in Z^X$ and $D \subset X$ there is a countable subset $L \subset D$ such that for every $x \in D$*

$$\inf_{y \in L} \max_{1 \leq k \leq n} d(f_k(y), f_k(x)) = 0.$$

Proof. The metric

$$d_\infty\big((t_k), (s_k)\big) := \sup_{1 \leq k \leq n} d(t_k, s_k),$$

$(t_k), (s_k) \in Z^n$, defines the product topology of the space Z^n. (Z^n, d_∞) is a separable metric space and consequently its subspace

$$H = \{(f_1(x), f_2(x), \dots, f_n(x)) : x \in D\}$$

is separable too. Thus, for some countable set $L \subset D$ we have $H \subset \overline{G}^{Z^n}$ where

$$G := \{(f_1(y), f_2(y), \dots, f_n(y)) : y \in L\}.$$

In other words, for each $x \in D$ we have

$$(f_1(x), f_2(x), \dots, f_n(x)) \in \overline{G}^{Z^n},$$

that precisely means

$$0 = \inf_{g \in G} d_\infty(g, (f_1(x), \dots, f_n(x))) = \inf_{y \in L} \max_{1 \leq k \leq n} d(f_k(y), f_k(x)).$$

□

Let $\mathbb{N}^{\mathbb{N}}$ be the space of all sequences of positive integers and let $\mathbb{N}^{(\mathbb{N})}$ be the set of all finite sequences of positive integers. As a topological space $\mathbb{N}^{\mathbb{N}}$ always carries its product topology τ_p of the discrete spaces $\mathbb{N}$. We use the following conventions: if $\alpha = (a_1, a_2, \dots) \in \mathbb{N}^{\mathbb{N}}$ and if $n \in \mathbb{N}$, then $\alpha|n := (a_1, a_2, ..., a_n)$. Let Σ be a subset of $\mathbb{N}^{\mathbb{N}}$: we denote by $\mathrm{F}(\Sigma)$ the subset of the set of finite sequences of positive integers $\mathbb{N}^{(\mathbb{N})}$ defined by

$$\mathrm{F}(\Sigma) = \{(a_1, a_2, \dots, a_n) \in \mathbb{N}^{(\mathbb{N})} : \text{ there exists } \alpha \in \Sigma, \alpha|n = (a_1, a_2, \dots, a_n)\}.$$

Let $\{A_\alpha : \alpha \in \Sigma\}$ be a family of non-void subsets of the set X. Given $\alpha = (a_1, a_2, \dots) \in \Sigma$ and $n \in \mathbb{N}$ we write

$$C_{\alpha|n} = \bigcup \{A_\beta : \beta \in \Sigma \text{ and } \beta|n = \alpha|n\}.$$

As usual, for a given set $C \subset X$ and a sequence $(x_n)_n$ in X we say that $(x_n)_n$ is eventually in C if there is $m \in \mathbb{N}$ such that $x_n \in C$ for $n \geq m$.

Lemma 3. *Let (Z, d) be a separable metric space, X a set and H a subset of the space (Z^X, τ_p) and $\varepsilon \geq 0$. We assume that:*

(i) *there is* $\Sigma \subset \mathbb{N}^{\mathbb{N}}$ *and a family* $\{A_\alpha : \alpha \in \Sigma\}$ *of non-void subsets of the set* X *such that* $X = \bigcup\{A_\alpha : \alpha \in \Sigma\}$*;*

(ii) *for every* $\alpha = (a_1, a_2, \dots) \in \Sigma$ *the set* H ε*-interchanges limits in* Z *with every sequence* $(x_n)_n$ *in* X *that is eventually in each set* $C_{\alpha|m}$*,* $m \in \mathbb{N}$*.*

Then for any $f \in \overline{H}^{Z^X}$ *there exists a sequence* $(f_n)_{n\in\mathbb{N}}$ *in* H *such that*

$$\sup_{x\in X} d(g(x), f(x)) \leq \varepsilon$$

for any cluster point g *of* $(f_n)_{n\in\mathbb{N}}$ *in* Z^X*.*

Proof. Define $f_0 := f$. Since $\mathrm{F}(\Sigma)$ is countable and infinite there is a bijection $\varphi : \mathbb{N} \to \mathrm{F}(\Sigma)$. We define $D_n := C_{\varphi(n)}$ for each $n \in \mathbb{N}$. We claim that there are a sequence of functions $f_0, f_1, \dots, f_n, \dots$ and a sequence of sets $L_1, L_2, \dots, L_n, \dots$ with the properties:

(a) $L_n = \{l_1^n, l_2^n, \dots, l_m^n, \dots\}$ is a countable subset of D_n for every $n \in \mathbb{N}$;

(b) for each $n \in \mathbb{N}$ and every $x \in D_n$ we have

$$\inf_{y\in L_n} \max_{0\leq k<n} d(f_k(y), f_k(x)) = 0; \tag{3.7}$$

(c) for each $n \in \mathbb{N}$ the function f_n belongs to H and

$$d(f_n(y), f_0(y)) < \frac{1}{n} \text{ for every } y \in \{l_k^j : 1 \leq k \leq n, 1 \leq j \leq n\}. \tag{3.8}$$

We prove the existence of the above sequences of functions and sets by recurrence. *FIRST STEP.* Applying Lemma 2 to $D := D_1$ and f_0 we obtain a countable subset $L_1 = \{l_1^1, l_2^1, \dots, l_m^1, \dots\}$ of D_1 such that

$$\inf_{y\in L_1} d(f_0(y), f_0(x)) = 0 \text{ for every } x \in D_1.$$

Since $f \in \overline{H}^{Z^X}$, there is $f_1 \in H$ such that

$$d(f_1(l_1^1), f_0(l_1^1)) < 1.$$

INDUCTION STEP. Assuming we have produced $f_1, f_2, \dots, f_n$ and $L_1, L_2, \dots, L_n$ satisfying (3.7) and (3.8) we use Lemma 2 for $D := D_{n+1}$ and $f_0, f_1, \dots, f_n$ to obtain $L_{n+1} \subset D_{n+1}$ satisfying

$$\inf_{y\in L_{n+1}} \max_{0\leq k<n+1} d(f_k(y), f_k(x)) = 0 \text{ for every } x \in D_{n+1}.$$

Once again, since $f \in \overline{H}^{Z^X}$ we can take a function $f_{n+1} \in H$ satisfying

$$d(f_{n+1}(y), f_0(y)) < \frac{1}{n+1} \text{ for every } y \in \{l_k^j : 1 \leq k \leq n+1, 1 \leq j \leq n+1\}.$$

The constructed sequences $f_0, f_1, \dots, f_n, \dots$ and $L_1, L_2, \dots, L_n, \dots$ satisfy (a), (b) and (c) above.

We shall prove now that $(f_n)_{n\in\mathbb{N}}$ has the property required in the thesis in the lemma: fix a cluster point g of $(f_n)_n$ in Z^X and fix a point $x \in X$ and let us prove that $d(g(x), f(x)) \leq \varepsilon$. We note first that inequality (3.8) implies that

$$\lim_n f_n(y) = f(y) \text{ for every } y \in L = \bigcup_{n\in\mathbb{N}} L_n. \tag{3.9}$$

Now, we pick $\alpha = (a_1, a_2, \dots) \in \Sigma$ such that $x \in A_\alpha$ and define

$$P := \varphi^{-1}(\{\alpha|n : n \in \mathbb{N}\}) \subset \mathbb{N}.$$

P is an infinite subset because φ is a bijection. Since the point $x \in \bigcap_{p\in P} D_p$, (3.7) applied to each $p \in P$ allows us to pick $y_p \in L_p$ with the property

$$d(f_k(y_p), f_k(x)) < \frac{1}{p} \text{ for } 0 \leq k < p. \tag{3.10}$$

Being P infinite we can and do fix $p_1 < p_2 < \cdots < p_j < \cdots \nearrow +\infty$ a strictly increasing sequence in P. We claim that the sequence $(y_{p_j})_j$ is eventually in $C_{\alpha|n}$ for every $n \in \mathbb{N}$. Indeed, for a given $n \in \mathbb{N}$ take $p_{j(n)}$ an element of the sequence $(p_j)_j$, with $p_{j(n)} > \varphi^{-1}(\alpha|i)$, $i = 1, 2, \ldots, n$. Therefore, if $j > j(n)$ then $p_j \neq \varphi^{-1}(\alpha|i)$ for $i = 1, 2, \ldots, n$ and consequently $\varphi(p_j) = \alpha|n(p_j)$ for some $n(p_j) > n$. The latter implies

$$y_{p_j} \in D_{p_j} = C_{\alpha|n(p_j)} \subset C_{\alpha|n}, \text{ for } j > j(n),$$

proving that $(y_{p_j})_j$ is eventually in each $C_{\alpha|n}$.

Observe also that (3.10) implies that

$$\lim_j f_k(y_{p_j}) = f_k(x) \text{ for } k = 0, 1, 2, \ldots. \tag{3.11}$$

Since $g(x)$ is a cluster point of $(f_n(x))_n$ in the metric space (Z, d) we can choose a subsequence $(f_{n_k})_k$ of $(f_n)_n$ such that $\lim_k f_{n_k}(x) = g(x)$. With all the above we have

$$\lim_k \lim_j f_{n_k}(y_{p_j}) \overset{(3.11)}{=} \lim_k f_{n_k}(x) = g(x),$$

$$\lim_j \lim_k f_{n_k}(y_{p_j}) \overset{(3.9)}{=} \lim_j f(y_{p_j}) \overset{(3.11)}{=} f(x).$$

Being the sequence $(y_{p_j})_j$ eventually in every $C_{\alpha|n}$ the assumption (ii) in the lemma ensures us that H ε-interchanges limits with $(y_{p_j})_j$, consequently

$$d(g(x), f(x)) = d(\lim_k \lim_j f_{n_k}(y_{p_j}), \lim_j \lim_k f_{n_k}(y_{p_j})) \leq \varepsilon,$$

and the proof is over. □

Recall that a topological space X is said to be *countably K-determined* if there is a subspace $\Sigma \subset \mathbb{N}^{\mathbb{N}}$ and an upper semi-continuous set-valued map $T : \Sigma \to 2^X$ such that $T(\alpha)$ is compact for each $\alpha \in \Sigma$ and $T(\Sigma) := \bigcup\{T(\alpha) : \alpha \in \Sigma\} = X$. Here the set-valued map T is called *upper semi-continuous* if for each $\alpha \in \Sigma$ and for any open subset U of X such that $T(\alpha) \subset U$ there exists a neighborhood V of α with $T(V) \subset U$. A good reference for countably K-determined spaces is [2] where they appear under the name *Lindelöf Σ-spaces*: notice that this class of spaces does properly contain the classes of K-analytic and (so) the σ-compact spaces. The paper [14] is a milestone when speaking about Banach spaces which are countably K-determined when endowed with their weak topologies.

Theorem 3.1. *Let X be a countably K-determined space, (Z, d) a separable metric space and H a relatively compact subset of the space (Z^X, τ_p). Then, for any $f \in \overline{H}^{Z^X}$ there exists a sequence $(f_n)_n$ in H such that*

$$\sup_{x\in X} d(g(x), f(x)) \overset{(a)}{\leq} 2\,\mathrm{ck}(H) + 2\hat{d}(H, C(X, Z)) \overset{(b)}{\leq} 4\,\mathrm{ck}(H)$$

for any cluster point g of $(f_n)_n$ in Z^X.

Proof. We define $\varepsilon := \mathrm{ck}(H) + \hat{d}(H, C(X,Z))$. Let $T : \Sigma \to 2^X$ be the set-valued map giving the countably K-determined structure to X and let us write $A_\alpha := T(\alpha)$ for every $\alpha \in \Sigma$. Then, the family $\{A_\alpha : \alpha \in \Sigma\}$ covers X. We start by proving the following:

CLAIM.- For every $\alpha \in \Sigma$ the set H 2ε-interchanges limits with every sequence $(x_n)_n$ in X that is eventually in each set $C_{\alpha|m}$, $m \in \mathbb{N}$.

To prove this we only have to use Lemma 1 after noticing that any such a sequence $(x_n)_n$ lies in a compact subset of X, namely

$$K := \{x_n : n \in \mathbb{N}\} \cup T(\alpha).$$

That such a K is compact is a well known fact about compact-valued upper-semicontinuous maps but we include a short proof for the sake of completeness. Let $\{U_i : i \in I\}$ be an open cover of K in X. Since $T(\alpha)$ is compact there are finitely many $U_{i_1}, U_{i_2}, \dots, U_{i_p}$ such that $T(\alpha) \subset U = \bigcup_{k=1}^p U_{i_k}$. Now the upper-semicontinuity of T applies to provide $m \in \mathbb{N}$ with the property that

$$C_{\alpha|m} = \bigcup \{T(\beta) : \beta \in \Sigma \text{ and } \beta|m = \alpha|m\} \subset U.$$

Since $(x_n)_n$ is eventually in every set $C_{\alpha|m}$, there is $n(m) \in \mathbb{N}$ such that $x_n \in U$ for all $n > n(m)$. If we take $U_{i_{p+1}}, \dots, U_{i_{p+n(m)}}$ from $\{U_i : i \in I\}$ such that $x_k \in U_{i_{p+k}}$ for $k = 1, 2, \dots, n(m)$, then $K \subset \bigcup_{k=1}^{p+n(m)} U_{i_k}$ and consequently K is compact and the proof of the claim is over.

Once the claim is proved inequality (a) follows from Lemma 3. To finish we observe that given $f \in H$ if we take $\varphi(n) := f$, $n \in \mathbb{N}$, then $\mathrm{clust}_{Z^K}(\varphi) = \{f\}$ and therefore we have that

$$\hat{d}(H, C(X,Z)) \leq \mathrm{ck}(H), \tag{3.12}$$

leading to inequality (b). $\square$

Theorem 3.2. *Let X be a countably K-determined space, (Z,d) a separable metric space and H a relatively compact subset of the space (Z^X, τ_p). Then*

$$\mathrm{ck}(H) \overset{(a)}{\leq} \hat{d}(\overline{H}^{Z^X}, C(X,Z)) \overset{(b)}{\leq} 3\,\mathrm{ck}(H) + 2\hat{d}(H, C(X,Z)) \overset{(c)}{\leq} 5\,\mathrm{ck}(H).$$

Proof. Inequality (a) follows from the very definitions of the notions involved. When $\mathrm{ck}(H) = +\infty$ all inequalities become trivial equalities. So we only take care of the case when $\mathrm{ck}(H) < +\infty$. Inequality (c) follows from (3.12). To prove (b) we define ε like we did in Theorem 3.1 as $\varepsilon := \mathrm{ck}(H) + \hat{d}(H, C(X,Z))$. We fix $\alpha \in \mathbb{R}$ with

$$\alpha > \mathrm{ck}(H). \tag{3.13}$$

Pick now any $f \in \overline{H}^{Z^X}$. Theorem 3.1 ensures the existence of a sequence $(f_n)_n$ in H such that

$$\sup_{x \in X} d(g(x), f(x)) \leq 2\varepsilon \tag{3.14}$$

for any cluster point g of (f_n) in Z^X. Now inequality (3.13) ensures the existence of such a cluster point g with $d(g, C(X,Z)) < \alpha$ that together with inequality (3.14) finishes the proof of (b). $\square$

Observe that if $H \subset C(X,Z)$ then $\hat{d}(H, C(X,Z)) = 0$ and consequently the constant 5 can be replaced by the constant 3 in inequality (c) in the previous result. Observe also that Theorems 3.1 and 3.2 are selfcontained and really strengthen Theorem 2.3.

Remark 3.3. Note that if X and (Z,d) are as in Theorem 3.2 and $H \subset C(X,Z)$ is relatively compact in Z^X then the following conditions are equivalent:

(i) $\mathrm{ck}(H) = 0$,
(ii) H is a relatively countably compact subset of $C(X,Z)$,
(iii) H is a relatively compact subset of $C(X,Z)$.

Whereas (ii)⇒(i) is obvious and (ii)⇔(iii) was known [12], the implication (i)⇒(ii) seems to require indeed the inequalities in Theorem 3.2. □

A topological space T is said to be *angelic* if, whenever H is a relatively countably compact subset of T, its closure $\overline{H}$ is compact and each element of $\overline{H}$ is a limit of a sequence in H. Our references for angelic spaces are [8] and [12]. Theorems 3.1 and 3.2 are the quantitative versions of the angelicity of spaces $C_p(X,Z)$ established as the main result in [12] that we obtain as a corollary below.

Corollary 3.4 (Orihuela). *Let X be a countably K-determined space and (Z,d) a metric space. Then $C_p(X,Z)$ is an angelic space.*

Proof. A result by Fremlin states that $C_p(X,Z)$ is angelic for any metric space if, and only if, $C_p(X,\mathbb{R})$ is angelic, [8, Theorem 3.5]. We prove the latter. If we take $H \subset C(X)$ a τ_p-relatively countably compact set in $C(X)$, then $\mathrm{ck}(H) = 0$. This implies that the right hand side of inequality (c) in Theorem 3.2 is zero and therefore we have

$$\hat{d}(\overline{H}^{\mathbb{R}^X}, C(X)) = 0,$$

that says that $\overline{H}^{\mathbb{R}^X} \subset C(X)$ and consequently $\overline{H}^{\mathbb{R}^X}$ is compact in $C_p(X)$. On the other hand, if we pick $f \in \overline{H}^{\mathbb{R}^X}$ an application of Theorem 3.1 produces a sequence $(f_n)_n$ in H such that for any τ_p-cluster point g of $(f_n)_n$ we have

$$\sup_{x \in X} d(g(x), f(x)) = 0.$$

This means that the sequence $(f_n)_n$ actually converges to f because it lies in the τ_p-compact set $\overline{H}^{\mathbb{R}^X}$ and has f as its unique τ_p-cluster point. □

We point out that our main results, Theorems 3.1 and 3.2, can be proved (same proofs and difficulty) in the more general setting of spaces X being *web-compact, quasi-Souslin, etc.* as studied in [12]. Nonetheless we have preferred to stick to countably K-determined spaces X because this case already carries all the main ideas, is powerful enough for applications and this class of spaces X is already pretty interesting for both topologists and analysts.

We end up the section and the paper studying some other kind of spaces X for which estimates of the kind that we have presented in 3.2 can be proved.

A topological space T is said to be *countably tight* if, whenever S is a subset of T and $t \in \overline{S}$, then for some countable subset A of S, $t \in \overline{A}$. Note that the simplest examples of spaces *countably tight* are the first countable spaces (in particular metric spaces). There are spaces which are non first countable but countably tight as for instance Banach spaces endowed with their weak topologies, see [10, §24.1.6].

If K is a compact space such that $C_p(K)$ is Lindelöf then K is countably tight: therefore Talagrand, Gulko and Corson compact spaces are countably tight, [14].

Oscillations for real functions have been defined in Theorem 1.1. Similarly for $f \in (Z,d)^X$ the oscillation of f at $x \in X$ is defined by

$$\operatorname{osc}(f,x) = \inf_U \sup_{y,z\in U} d(f(y), f(z))$$

where the infimum is taken over the neighborhoods U of x in X. The overall oscillation of f is given by $\operatorname{osc}(f) = \sup_{x\in X} \operatorname{osc}(f,x)$.

Proposition 3.5. *Let X be a first countable space, (Z,d) a metric space and H a pointwise relatively compact subset of (Z^X, τ_p). Then*

$$\sup_{f\in\overline{H}} \operatorname{osc}(f) = \sup_{\varphi\in H^{\mathbb{N}}} \inf\{\operatorname{osc}(f) : f \in \operatorname{clust}_{Z^X}(\varphi)\}. \tag{3.15}$$

For $Z = \mathbb{R}$ the equality (3.15) *holds when X is countably tight.*

Proof. Let α be the right hand side of (3.15). Clearly

$$\beta := \sup_{f\in\overline{H}} \operatorname{osc}(f) \geq \alpha.$$

If $\beta = 0$ we are done. Otherwise, Otherwise, the equality (3.15) will be established if we prove that each time $\beta > \varepsilon > 0$ we also have $\alpha \geq \varepsilon$. Pick $f \in \overline{H}$ such that $\operatorname{osc}(f) > \varepsilon$ and then fix $x_0 \in X$ such that

$$\operatorname{osc}(f, x_0) > \varepsilon. \tag{3.16}$$

Let $\mathcal{U}$ be a basis of neighborhoods for $x_0 \in X$ and let us distinguish the two cases stated in the statement of the Proposition.

A. X is a first countable space.- We assume that $\mathcal{U} = \{U_n\}_n$ is countable to inductively use inequality (3.16) and choose $x_n, y_n \in U_n$ such that $d(f(x_n), f(y_n)) > \varepsilon$, for every $n \in \mathbb{N}$. Let us write $D := \{x_n : n \in \mathbb{N}\} \cup \{y_n : n \in \mathbb{N}\}$. Since $D \subset X$ is countable and $f \in \overline{H}$ there exists a sequence $\varphi \in H^{\mathbb{N}}$ such that $\lim_n \varphi(n)(x) = f(x)$ for every $x \in D$. Therefore, if g is an arbitrary τ_p-cluster point of φ then $g|_D = f|_D$ and we have in particular that

$$d(g(x_n), g(y_n)) > \varepsilon, \text{ for every } n \in \mathbb{N}, \tag{3.17}$$

and so $\operatorname{osc}(g, x_0) \geq \varepsilon$. Since g is an arbitrary τ_p-cluster point of φ we have proved that

$$\inf\{\operatorname{osc}(g) : g \in \operatorname{clust}_{Z^X}(\varphi)\} \geq \varepsilon$$

and therefore $\alpha \geq \varepsilon$ and the proof for this case is complete.

B. X is countably tight and $Z = \mathbb{R}$.- If we define

$$f_1(x_0) = \inf_{U\in\mathcal{U}} \sup_{y\in U} f(y) \text{ and } f_2(x_0) = \sup_{V\in\mathcal{U}} \inf_{z\in V} f(z),$$

then for every $U, V \in \mathcal{U}$ we have that

$$+\infty \geq \sup_{y\in U} f(y) \geq f_1(x_0) \geq f(x_0) \geq f_2(x_0) \geq \inf_{z\in V} f(z) \geq -\infty. \tag{3.18}$$

We prove now the claim:

> *CLAIM.- for each $U \in \mathcal{U}$ there are elements $y_U, z_U \in U$ such that for every pair U, V in $\mathcal{U}$ we have that*
>
> $$f(y_U) - f(z_V) > \varepsilon. \tag{3.19}$$

To prove the claim we distinguish three cases:
B1. The values $f_1(x_0)$ and $f_2(x_0)$ are real.- In this case we clearly have

$$\operatorname{osc}(f, x_0) = f_1(x_0) - f_2(x_0) \overset{(3.16)}{>} \varepsilon.$$

For some $\gamma \in \mathbb{R}$ inequality (3.18) can be rephrased as

$$\sup_{y\in U} f(y) - \frac{\varepsilon}{2} \geq f_1(x_0) - \frac{\varepsilon}{2} > \gamma > f_2(x_0) + \frac{\varepsilon}{2} \geq \inf_{z\in V} f(z) + \frac{\varepsilon}{2}$$

for each $U, V \in \mathcal{U}$. Hence for every $U, V \in \mathcal{U}$ we can pick $y_U \in U$ and $z_V \in V$ such that

$$f(y_U) - \frac{\varepsilon}{2} > \gamma > f(z_V) + \frac{\varepsilon}{2}$$

and the claim is proved.
B2. $f_1(x_0) = +\infty$.- In this case inequality (3.18) can be rewritten for each $U, V \in \mathcal{U}$ as

$$+\infty = \sup_{y\in U} f(y) > f(x_0) + 2\varepsilon > f(x_0) + \varepsilon > f_2(x_0) \geq \inf_{z\in V} f(z) \geq -\infty.$$

Hence we can choose $y_U \in U$ and $z_V \in V$ such that

$$f(y_U) > f(x_0) + 2\varepsilon > f(x_0) + \varepsilon > f(z_V).$$

and the CLAIM is proved in this case.
B3. $f_1(x_0) = -\infty$.- It is similar to case *B2*.

Now we finish the proof of *B*. Observe that $x_0 \in \overline{\{y_U : U \in \mathcal{U}\}} \cap \overline{\{z_V : V \in \mathcal{U}\}}$. Since X is countably tight there are countable sets $B \subset \{y_U : U \in \mathcal{U}\}$ and $C \subset \{z_V : V \in \mathcal{U}\}$ such that

$$x_0 \in \overline{B} \cap \overline{C}. \tag{3.20}$$

Now $D := B \cup C \subset X$ is countable and proceeding as we did in Case A there exists a sequence $\varphi \in H^{\mathbb{N}}$ such that $\lim_n \varphi(n)(x) = f(x)$ for every $x \in D$. Therefore, if g is any τ_p-cluster point of φ then $g|_D = f|_D$ and if $U \in \mathcal{U}$ is arbitrary equation (3.20) applies to provide us with $b \in B \cap U$ and $c \in C \cap U$ that gives us when

$$d(g(b), g(c)) > \varepsilon$$

because of (3.19) and $f(b) = g(b)$ and $f(c) = g(c)$. Thus $\operatorname{osc}(g, x_0) \geq \varepsilon$ and since g is an arbitrary τ_p-cluster point of φ the proof of this case concludes as we concluded the proof of Case A. □

Corollary 3.6. *Let X be a metric space, E a Banach space and H a τ_p-relatively compact subset of E^X. Then*

$$\operatorname{ck}(H) \leq \hat{d}(\overline{H}^{E^X}, C(X, E)) \leq 2\operatorname{ck}(H). \tag{3.21}$$

In the particular case when $E = \mathbb{R}$ the space X can be taken normal and countably tight and we have

$$\hat{d}(\overline{H}^{\mathbb{R}^X}, C(X)) = \operatorname{ck}(H). \tag{3.22}$$

Proof. In [5, Lemma 2.7] it has been proved that if X is paracompact space, E is normed and $f \in E^X$ is bounded then

$$\frac{1}{2}\operatorname{osc}(f) \leq d(f, C_b(X, E)) \leq \operatorname{osc}(f)\,.$$

Where $C_b(X, E)$ stands for the family of bounded continuous functions from X to E: the reader can check that the the same proof of [5, Lemma 2.7] provides us with the estimates

$$\frac{1}{2}\operatorname{osc}(f) \leq d(f, C(X, E)) \leq \operatorname{osc}(f)$$

for arbitrary $f \in E^X$. The latter together with the first part of Proposition 3.5 give us inequalities (3.21). On the other hand, the equality (3.21) follows from Theorem 1.1 and the second part of Proposition 3.5. $\square$

Note that without extra hypothesis of countable tightness for X we cannot expect to have the equality $\hat{d}(\overline{H}^{\mathbb{R}^X}, C(X)) = \operatorname{ck}(H)$ as the Example 2.4 shows.

At this point we should credit the Ph. D. dissertation [13]: some ideas for the proof of Proposition 3.5 when $Z = \mathbb{R}$ have been inspired by the reading of a result in [13] that is sharpened by our Corollary 3.6.

We would like to finish this paper noting that the reference [1] is a recent paper dealing with cluster points of an arbitrary family of functions in the pointwise convergence topology.

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