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Continuity properties up to a countable partition

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Abstract. Approximation and rigidity properties in renorming constructions are characterized with some classes of simple maps. Those maps describe continuity properties up to a countable partition. The construction of such kind of maps can be done with ideas from the First Lebesgue Theorem. We present new results on the relationship between Kadec and locally uniformly rotund renormability as well as characterizations of the last one with the simple maps used here.

Resumen. Las propiedades de aproximación y rigidez en construcciones de renormamiento son caracterizadas con clases de aplicaciones simples. Dichas aplicaciones describen propiedades de continuidad módulo particiones numerables. La construcción de este tipo de aplicaciones puede hacerse con ideas del Primer Teorema de Lebesgue. Presentamos nuevos resultados sobre la relación entre renormamientos de Kadec y localmente uniformemente convexos así como caracterizaciones de los últimos con las aplicaciones simples aquí utilizadas.

1. Introduction

The σ–continuity property for a map from a topological space $(X, \mathcal{T})$ into a metric space (Y, d) is an extension of the concept of continuity suitable to deal with countable decompositions of the domain space X as well as with pointwise cluster points of sequences of continuous functions $\Phi_n : X \to Y$, $n = 1, 2, \ldots$ in the pointwise topology, [29]. The precise definitions are as follows:

Definition 1 A map Φ from a topological space $(X, \mathcal{T})$ into a metric space (Y, d) is said to be *σ-continuous* if for every $\varepsilon > 0$ we can decompose X as $X = \bigcup_{n=1}^{\infty} X_{n,\varepsilon}$ such that for every $n \in \mathbb{N}$ and every $x \in X_{n,\varepsilon}$ there is a neighbourhood U of x such that

$$\text{osc}\ \left(\Phi_{\restriction_{U \cap X_{n,\varepsilon}}}\right) := sup\left\{d(\Phi(x), \Phi(y)) : x, y \in X_{n,\varepsilon}\right\} < \varepsilon.$$

Let us remark that a natural example of σ–continuous maps are the so called piecewise continuous, or even less the piecewise locally constant maps according with the following:

Definition 2 A map Φ from a topological space $(X, \mathcal{T})$ into a set Y (resp. a topological space) is said to be *piecewise locally constant* (resp. *piecewise continuous*) if X can be decomposed as the union of a

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sequence of sets $\{X_n : n \in \mathbb{N}\}$ such that the restriction of Φ to each X_n is a locally constant map, (resp. continuous), where *locally constant* means that for every positive integer n and every $x \in X_n$ there exists an open set W with $x \in W$ and $\Phi(x) = \Phi(y)$ for every $y \in W \cap X_n$.

Since every map from every topological space $(X, \mathcal{T})$ into a separable metric space (Y, d) will be σ–continuous the concept is relevant only when the range space (Y, d) is a non separable metric space. When $(X, \mathcal{T})$ is a subset of a locally convex topological vector space we have refined our study to deal with σ–*slicely continuous maps*, [29]. With the word *slice* we stress the fact that the continuity property required must be described with open half spaces only, so the neighbourhoods used will be slices on the pieces $X_{n,\varepsilon}$ we decompose the space $(X, \mathcal{T})$ in definition 1, i.e. the neigbourhood U must be an open half space. We have developed a systematic study of this kind of maps in connection with renorming properties of Banach spaces, [29]. Indeed, if a normed space E has a norm with a unit sphere S_E such that the weak and the norm topologies coincide on it, i.e. what is called a Kadec norm, then the identity map on E from the weak to the norm topologies is σ–continuous [12, 17, 18, 29]. Nevertheless it is an open problem whether the reverse implication holds: if the identity map from the weak to the norm topology of the normed space $(E, \|\cdot\|)$ is σ–continuous we do not know if there is an equivalent Kadec norm on $(E, \|\cdot\|)$. The best result in this direction is the existence of a positively homogeneous map $F : E \to \mathbb{R}$ with $\|\cdot\| \leq F(\cdot) \leq 2\|\cdot\|$ such that on the pseudo sphere $\{x \in E : F(x) = 1\}$ the weak and the norm topologies coincide whenever the identity map from the weak to the norm topology of the normed space $(E, \|\cdot\|)$ is σ–continuous , [31]. Until know, it has been not possible to decide if such an F exists satisfying the triangle inequality. Nevertheless, the identity map on the normed space E from the weak to the norm topologies is σ-slicely continuous if, and only if, there exits an equivalent locally uniformly rotund norm on E, [25]. Thus we see that the former function F can be constructed as the locally uniformly rotund norm on E when we deal with open half spaces to describe the σ-continuity of the identity map. Let us recall that a norm $\|\cdot\|$ in a normed space is locally uniformly rotund (**LUR** for short) if $\lim_k \|x_k - x\| = 0$ whenever $\lim_k \left(2\|x_k\|^2 + 2\|x\|^2 - \|x_k + x\|^2\right) = 0$.

In the different approaches for those renormings always appears somewhere in the construction two properties. The first one is a property of *approximation* of the renorming structure, the second one is the so called *rigidity condition*. Let us describe both of them for the Banach space

$$c_0(\Gamma) := \left\{x \in \mathbb{R}^\Gamma : \forall\epsilon > 0 \#\{\gamma \in \Gamma : |x(\gamma)| \geq \varepsilon\} < \infty\right\},$$

a characteristic object in the study of renormings. For a given $x \in c_0(\Gamma)$ and $\epsilon > 0$ we can associate the finite subset of Γ given by $L^\varepsilon(|x|) := \{\gamma \in \Gamma : |x(\gamma)| \geq \varepsilon\}$. The *aproximation* here follows from the fact that $\|x - x \cdot 1_{L^\varepsilon(|x|)}\| < \epsilon$. The cardinality of the set $L^\varepsilon(|x|)$ together with a positive integer p such that $|x(\gamma)| \leq \epsilon - 1/p$, for all $\gamma \notin L^\varepsilon(|x|)$, are two integers associated to x and ϵ that we denote with $r(x, \epsilon)$. For any y with the same associated integers, i.e. with $r(x, \epsilon) = r(y, \epsilon)$, we will have that $L^\varepsilon(|x|) = L^\varepsilon(|y|)$ whenever $|x(\gamma) - y(\gamma)| < p^{-1}$ for every $\gamma \in L^\varepsilon(|x|)$, i.e when y belongs to the pointwise open neighbourhood of x given by $U := \left\{y \in c_0(\Gamma) : |x(\gamma) - y(\gamma)| < p^{-1} \forall\gamma \in L^\varepsilon(|x|)\right\}$. The numbers $r(x, \epsilon)$ describe the *rigidity condition* for the function $x \in c_0(\Gamma)$. In the example 1 of section 4 we will see how to deal with them to get an open half space instead of the U above. An equivalent **LUR** norm $|||\cdot|||$can be constructed on $c_0(\Gamma)$ if we force on the new norm that the condition $\lim_k \left(2\,|||x_k|||^2 + 2\,|||x|||^2 - |||x_k + x|||^2\right) = 0$ must imply, for every fixed $\epsilon > 0$, the equality $r(x_n, \epsilon) = r(x, \epsilon)$ for n big enough. In that way the rigidity condition tell us that the distance of x_n to the finite dimensional subspace $\{y \in c_0(\Gamma) : y(\gamma) = 0 \forall\gamma \notin L^\varepsilon(|x|)\}$ is less or equal than ϵ for n big enough, thus the sequence $\{x_n : n = 1, 2, ...\}$ is a relatively compact subset of the Banach space $c_0(\Gamma)$ and x can be forced to be its limit too.

When we have a biorthogonal system $\{(x_\gamma, f_\gamma) : \gamma \in \Gamma\}$ in the normed space E, then we always have a locally uniformly rotund equivalent norm on the linear subspace $span\,\{x_\gamma : \gamma \in \Gamma\}$, but in the closure $\overline{span\,\{x_\gamma : \gamma \in \Gamma\}}$ it may not have such a renorming. Indeed, in a space with a Markushevich basis, i.e a Banach space E with a biortoghonal system $\{(x_\gamma, f_\gamma) : \gamma \in \Gamma\}$ such that $\overline{span\,\{x_\gamma : \gamma \in \Gamma\}} = E$ and

$f_\gamma(x) \neq 0$ for some $\gamma \in \Gamma$ whenever $x \neq 0$, $x \in E$; we do not have always a **LUR** renorming on E. For instance, the space l^∞ doest not have an equivalent Kadec norm and we know, by a result of Plichko [4], that it is a complemented subspace of a Banach space E with Markushevich basis, thus for such a space E we cannot have an equivalent Kadec or **LUR** norm. In [29] we have studied conditions on the operator $T : E \to c_0(\Gamma)$, given by evaluation $T(x)(\gamma) := f_\gamma(x)$ for a given Markushevich basis in E, to transfer the **LUR** norm in the space $c_0(\Gamma)$ to a **LUR** norm on E. The case of a strong Markushevich basis is done in section 4. Another example is the space $c_0(\Upsilon)$ for the dyadic tree Υ, where the linear span of $\{1_{[0,\tau]} : \tau \in \Upsilon\}$ has an equivalent **LUR** norm and the closure $\overline{span\{1_{[0,\tau]} : \tau \in \Upsilon\}}$ doest not have an equivalent rotund norm on it [15]. Let us recall that a norm $\|\cdot\|$ is rotund (strictly convex) if the unit sphere does not contain non trivial segments i.e. $x = y$ whenever $\|x\| = \|y\| = \|(x+y)/2\| = 1$.

It is our intention in the present paper to show how both properties, approximation and rigidity, are in the core of the concepts of σ-slicely continuity and σ-continuity. The rigidity condition is described with the piecewise locally constant maps whereas the family of all σ-continuous maps will be generated by them if we operate with the limits of sequences:

Theorem 1 *Let $(X, \mathcal{T})$ be a topological space and let (Y, ρ) be a metric space. Given a map $\Phi : X \to Y$ the following are equivalent:*

i) Φ is σ–continuous;

ii) there exists a sequence $\{\Phi_n : X \to Y : n \in \mathbb{N}\}$ of piecewise locally constant functions such that $\lim_{n\to\infty} \Phi_n x = \Phi x$ uniformly on $x \in X$.

Corresponding results for maps with σ-relative discrete function basis, maps with property P, and σ-fragmentable maps has been showed in [13], [33], and [19] respectively. We will present our approach in the section 3 of the paper. There we shall deal with a subset of a locally convex space A instead of the topological space $(X, \mathcal{T})$ using always as neighbourhoods open half spaces[1], to prove the following the following:

Theorem 2 *Let A be a subset of a locally convex topological vector space and let (Y, ρ) be a metric space. For a map $\Phi : A \to Y$ the following assertions are equivalent:*

i) Φ is σ–slicely continuous.

ii) There exists a sequence $\Phi_n : A \to Y$, $n \in \mathbb{N}$ of piecewise slicely constant functions such that $\lim_{n\to\infty} \Phi_n x = \Phi x$ uniformly on $x \in A$.

In section 2 we present applications of the proof of the First Theorem of H. Lebesgue to the construction of σ-continuous maps in normed spaces where the weak and the norm topologies are involved. Another Lebesgue theorem says that a real function $f : I \to \mathbb{R}$ defined on an interval I of the real line can be represented as the limit of a sequence of continuous functions if, and only if, for each $\varepsilon > 0$ the domain I can be written as the union of a sequence $\{I_n : n = 1, 2, ...\}$ of closed subsets such that the oscilation of f in every I_n is less than ε, [23]. In the same direction, from theorem 5 and its corollary 7 in [19], we know that for a given map $\Phi : X \to Y$, where X is a metric space and Y a normed space, the condition of being σ-continuous with closed subsets, i.e. the sets $X_{n,\varepsilon}$ in definition 1 must be closed, is equivalent to the fact that Φ is the pointwise limit of a sequence of continuous functions from X to Y. Our main theorems above differs in the fact that our domain space here is an arbitrary topological space $(X, \mathcal{T})$ or an arbitrary subset A of a locally convex space. Moreover, we are not interested here in the topological nature of the pieces $X_{n\varepsilon}$, and our sequence of approximations Φ_n are going to be only piecewise locally constant maps.

[1] We will say that the maps are σ–slicely continuous and piecewise slicely constant instead of σ–continuous and piecewise locally constant in that case, [29].

Measurability properties for this kind of maps, where the structure of the pieces $X_{n\varepsilon}$ must be carefully analyzed, has been studied by Hansell in a series of papers [8, 9, 10, 11, 12, 13]. In section 4, we present the following application to **LUR** renormings, where the piecewise locally constant maps represents the rigidity while the approximation is provided by taking cluster points:

Theorem 3 *A normed space $(E, \|\cdot\|)$ with a norming subspace $F \subset E^*$ admits an equivalent $\sigma(E,F)$-lower semicontinuous* **LUR** *norm if, and only if, there is a sequence $\{I_n : E \to E, n = 1, 2, ...\}$ of piecewise slicely constant maps for the $\sigma(E,F)$-topology such that*

$$x \in \overline{\{I_n(x) : n = 1, 2, ...\}}^{\sigma(E,E^*)}, \forall x \in E$$

An excellent monograph of renorming theory up to 1993 is [1]. In order to have an up to date account of the theory we should add at least [15], [6], and [37].

2. Lebesgue First Theorem

In the contribution paper [35], Walter Rudin writes about the question of meausurability of a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which is separately continuous saying: "Several years ago, I used to pose this question to randomly selected analysts. The typical answer was something like this: 'Hmm-well- probably not- why should it be? The only group that did a little better were the probabilists. And there was just one person who said: 'Let see -yes, it is- and it is of Baire class one- and....' He knew." In Lebesgue's first published paper [22] the question is answered affirmatively, and in his wonderful paper of 1905 [24] he proves that for functions on $\mathbb{R}^k$ to be a Baire $k-1$ class is the sharpest result possible. Let us follow here the proof of Lebesgue First Theorem as it is presented by W. Rudin, [35], since it will introduce us to the kind of maps we are using in renorming theory, [29].

Let us assume that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a separately continuous function in the plane, then Lebesgue defined the sequence of jointly continuous functions $f_n : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as follow: $f_n(x,y) := f(x,y)$ if $(x,y) \in \{(j/n, y) : j \in \mathbb{Z}, y \in \mathbb{R}\}$, and by linear interpolation in the the first variable x in other case. The fact that $f(x,\cdot)$ is continuous for fixed x implies that the functions f_n are continuous as functions on the whole plane, and the fact that $f(\cdot,y)$ is continuous for fixed y tell us that the $lim_n f_n(x,y) = f(x,y)$ for every $(x,y) \in \mathbb{R} \times \mathbb{R}$. We can give a more precise formula for the functions f_n using the following families of continuous functions on the real line:

$\Xi_n := \left\{h_j^n(x) := max(0, 1 - |nx - j|), j \in \mathbb{Z}\right\}$ for $n = 1, 2, ...$

and setting $f_n(x,y) = \sum \left\{h_j^n(x) f(j/n, y) : j \in \mathbb{Z}\right\}$.

Let us observe the fact that the families Ξ_n are partitions of unity in $\mathbb{R}$ with the support of h_j^n contained in the interval $[(j-1)/n, (j+1)/n]$, consequently the length of the supports of the functions in the partition Ξ_n going to zero when n goes to infinity. This property is required for the proof of the convergence of the sequence $\{f_n : n = 1, 2, ...\}$ to the separately continuous function f. Indeed, the only requeriment we need, to express the idea of Lebesgue in full generality, is the existence of the partitions of unity associated with covers giving decreasing sequences of subsets with diameter going to zero, i.e the Lebesgue's idea can be translated for a map $f : X \times Y \to \mathbb{R}$ when X is a metric space and Y is any topological space:

Theorem 4 (W. Rudin [35]) *Let (X,d) be a metric space and $(Y,\mathcal{T})$ a topological space. If*

$$f : X \times Y \to \mathbb{R}$$

is a map such that:

- *$f(\cdot,y) : X \to \mathbb{R}$ is continuous for every y fixed in Y.*

- $f(x,\cdot) : Y \to \mathbb{R}$ *is continuous for every x fixed in a dense subset* D *of* X

Then f *is the limit of a sequence of continuous functions* $f_n : X \times Y \to \mathbb{R}$.

Proof. Let us consider for the metric space (X,d) continuous functions giving partitions of unity $\{h_{\alpha,n} : X \to [0,1] : \alpha \in A\}$ with $\{support(h_{\alpha,n}) : \alpha \in A\}$ a locally finite covering of sets with diameter less than $1/n$ in the metric space (X,d). Let us choose points $x_{\alpha,n}$ in D such that $h_{\alpha,n}(x_{\alpha,n}) > 0$ for every $\alpha \in A$ and every $n \in \mathbb{N}$. If we define the functions $f_n(x,y) := \sum h_{\alpha,n}(x) f(x_{\alpha,n}, y)$, for every $n \in \mathbb{N}$, it now easily follows that f_n is continuous on the product space $X \times Y$ and that the sequence $\{f_n(x,y) : n \in \mathbb{N}\}$ converges to $f(x,y)$ when n goes to infinity for every $(x,y) \in X \times Y$. ■

We can apply the former theorem to the duality of any locally convex topological vector space to obtain the following:

Corollary 1 *Let* A *be a subset of the locally convex topological vector space* $E[\mathcal{T}]$ *which is metrizable for the weak topology* $\sigma(E,E')$ *of* E *induced on it. Then there exist a sequence of continuous functions* $f_n : A[\sigma(E,E')] \to E[\mathcal{T}]$ *such that the* $\sigma(E,E')$ *limit of* $f_n(x)$ *is equal to* x *for every* x *in* A.

Proof. With the same notations as in the proof of the former theorem, where the separately continuous function f will be here the duality map:

$$< \cdot, \cdot >: A[\sigma(E,E')] \times E'[\sigma(E',E)] \to \mathbb{R}$$

defined by the evaluation: $< x, \varphi > := \varphi(x)$, we have for the approximating sequence of jointly continuous maps the functions: $f_n(x,\cdot) := \sum h_{\alpha,n}(x) < x_{\alpha,n}, \cdot >$. Assuming the representation of every vector in the locally convex space E as a linear form on the dual space through the duality map we can write the former map as $f_n(x) := \sum h_{\alpha,n}(x) x_{\alpha,n}$ for every $x \in A$ which is clearly continuous from the weak topology on A to $E[\mathcal{T}]$, moreover for every $y \in E'$ we have that $lim_n f_n(x)(y) =< x, y >$, i.e. $lim_n f_n(x) = x$ in the weak topology $\sigma(E,E')$ for every $x \in A$. ■

As a consequence we have the following relationship between the weak and the norm topology of any normed space on weakly metrizable subsets of it:

Corollary 2 *Let* $(E, \|\cdot\|)$ *be a normed space and let* A *be a subset of* E *metrizable for the weak topology* $\sigma(E,E^*)$. *Then for every positive* ϵ *there is a decomposition of* A *as a countable union, i.e.* $A = \bigcup\{A_{n,\epsilon} : n = 1,2,...\}$, *such that for every positive integer* n *and every* x *in* $A_{n,\epsilon}$ *there is a* $\sigma(E,E^*)$*-open subset* W *of* E *such that* $x \in W$ *and the norm diameter of the set* $W \cap A_{n,\epsilon}$ *is less than* ϵ, *i.e the identity map in* A *from the weak to the norm topologies is* σ*–continuous.*

Proof. Let us denote by $\{g_n : A \to E, n = 1,2,...\}$ the family of all rational convex combinations of the sequence $\{f_n : A \to E\}$ given by the former corollary that weakly and pointwise converges to the identity map on A. By the Hahn Banach theorem it now follows that for every $x \in A$ we have

$$x \in conv\overline{\{f_n(x) : n = 1,2,...\}}^{\|\cdot\|}$$

from where it now follows the splitting

$$A = \bigcup\{A_{n,\epsilon} : n = 1,2,...\}$$

where $A_{n,\epsilon} = \{x \in A : \|g_n(x) - x\| < (\epsilon/3)\}$.
The required property now follows from the continuity of the maps g_n from the weak to the norm topologies. Indeed, if $x \in A_{n,\epsilon}$ and W is a weak open neigbourhood of x such that the oscillation of g_n on $W \cap A$ is less than $\epsilon/3$, we have for every $y \in A_{n,\epsilon} \cap W$ that

$$\|x - y\| \le \|x - g_n(x)\| + \|g_n(x) - g_n(y)\| + \|g_n(y) - y\| < \epsilon.$$

■

Remark 1 The property described in the last corollary has been called countable cover by sets of small local diameter in [16] and it coincides with the so called *descriptiveness* for the Banach spaces in [12], see our paper [29] as well as [30] for a complete description of the relationship between both concepts. Reciprocally, if the normed space E has this property of countable cover by sets of small local diameter it follows that there is a function $F : E \to \mathbb{R}$, positively homogeneous, with $\|\cdot\| \leq F(\cdot) \leq (1+\epsilon)\|\cdot\|$ and such that on the pseudo-sphere $\{x \in E : F(x) = 1\}$ the weak and the norm topology do coincide, [31].

Our approach to the descriptive property of Banach spaces has been always related with renorming properties (Kadec, locally uniformly rotundity, etc.) Let us show here, for instance, and as an straight forward application of the former corollaries the following consequence when we deal with a weakly uniformly rotund normed space $(E, \|\cdot\|)$, i.e. for the norm of E we have the following property: when the sequences $\{x_n : n = 1, 2, ...\}$ and $\{y_n : n = 1, 2, ...\}$ are in the unit sphere S_E and $1 - \|(x_n + y_n)/2\| \to 0$ when $n \to \infty$, then the sequence $\{(x_n - y_n) : n = 1, 2, ...\}$ goes to zero in the weak topology of E:

Corollary 3 *A normed space E with a weakly uniformly rotund norm has an equivalent locally uniformly rotund norm.*

Proof. We already proved that the unit sphere S_E is metrizable for the weak topology induced on it, [27]. Let us present here a proof based on the metrization theorem of Arhangelskii, see theorem 9.14 in [7]. We define the symmetric $s(x, y) := 1 - \|(x + y)/2\|$ for every x and y in the unit sphere S_E, then s describes the weak topology on S_E. Indeed, for every $x \in S_E$, every $\mu \in \mathbb{R}$ and every $f \in B_{E^*}$ we have some $\delta > 0$ such that

$$\{y \in S_E : s(x, y) < \delta\} \subset \{y \in S_E : f(y) > f(x) - \mu\}$$

because the norm is weakly uniformly rotund, moreover the lower semicontinuity of the norm tell us that the s-open balls are weakly open sets too. Thus the unit sphere with the weak topology is a symmetrizable with the symmetric s. Further, if $s(x, y_n) \to 0$ and $s(y_n, z_n) \to 0$ then it follows that the sequence $\{z_n : n = 1, 2, ...\}$ goes to x in the weak topology since the sequences $\{(x - y_n) : n = 1, 2, ...\}$ and $\{(y_n - z_n) : n = 1, 2, ...\}$ go to zero by the weakly uniform rotundity condition, and thus $s(x, z_n) \to 0$ too. The conditions of Arhangelskii theorem are fulfilled and consequently the unit sphere $(S_E, \sigma(E, E^*))$ is indeed metrizable. It follows after corollary 2 that the identity map on the unit sphere S_E is σ–continuous from the weak to the norm topology. The fact that open half spaces give basic neighbourhoods at every point in the sphere for the weak topology follows also from the weakly uniform rotundity. Indeed, if $x \in S_E$ and we choose $f_x \in B_{E^*}$ with $f_x(x) = 1$, then $(f_x(x_n)) \to 1$ implies for the sequence $s(x_n, x) \to 0$. Thus the slices:

$$\{\{y \in S_E : f_x(y) > 1 - 1/n\}, n = 1, 2, ...\}$$

are a basis of neighbourhoods at x for the weak topology on the unit sphere. So the identity map on S_E is σ–slicely continuous too and the space E has a **LUR** renorming, see our main theorem in [25]. ■

Of course the same proof works for dual norms and the weak* uniform rotundity condition, see [29]. Banach spaces with weakly uniformly rotund norm as been recently studied by different authors since they connect Asplund property with uniformly Eberlein compacta, see for instance [2, 3, 4, 26], as well as the references in Chapter 12 of [4].

Since there are Banach spaces with Kadec norm and without equivalent **LUR** norm, [15], it is clear that a metrizable unit sphere in the weak topology doest not imply the locally uniformly rotund renormability of a Banach space. The case presented in the former corollary is a very particular one where the proof follows directly from the topological arguments presented in that paper, see[27] for the general result. Nevertheless we can ask here for conditions on the partitions of unity and the associated covers used in our proof to get the **LUR** renormability of our spaces. A natural condition is the following one:

Definition 3 A family $\Lambda = \{F_i : i \in I\}$ of subsets of a normed space E is called *slicely finite* if for every point $x \in \bigcup \Lambda$ there is an open half space H with $x \in H$ and such that H meets a finite number of elements of the family Λ only. We shall say that a family $\Lambda = \{F_i : i \in I\}$ is *σ-slicely finitely decomposable* if we can decompose the elements of the family in countable pieces: $F_i := \bigcup\{F_i^n : n = 1, 2, ...\}$ in such a way that for every positive integer n we will have slicely finite families $\Lambda_n := \{F_i^n : i \in I\}$.

Let us remember the following definitions that has been successfully used in metrization, selection and renorming theories, see [27, 29, 19, 12], and which will be used throughout the rest of the paper.

Definition 4 A family of subsets $\{D_\gamma : \gamma \in \Gamma\}$ in a topological space X is called *discrete* (respectively *isolated*) if for every point $x \in X$ (respectively $x \in \bigcup\{D_\gamma : \gamma \in \Gamma\}$) there is a neighbourhood U of x such that U meets at most one member of the family $\{D_\gamma : \gamma \in \Gamma\}$. When X is a linear topological space and U can be taken to be an open half space then the family is said to be *slicely discrete* (respectively *slicely isolated*).

A family of subsets $\mathcal{D}$ is said to be *σ–discrete* (respectively *σ–slicely discrete*, *σ–isolated*, *σ–slicely isolated*) if it can be decomposed into a countable union $\mathcal{D} = \bigcup \mathcal{D}_n$ such that every family $\mathcal{D}_n$ is discrete (respectively slicely discrete, isolated, slicely isolated).

A family of subsets $\{D_\gamma : \gamma \in \Gamma\}$ in a topological space X is called *σ–discretely decomposable* (resp. *σ–isolatedly decomposable*) if $D_\gamma = \bigcup_{n=1}^{\infty} D_\gamma^n$ for every $\gamma \in \Gamma$ and $\left\{D_\gamma^n : \gamma \in \Gamma\right\}$ is discrete (resp. isolated) for each $n \in \mathbb{N}$.

When X is a linear topological space the notion of *σ–slicely discretely decomposable* (resp. *σ–slicely isolatedly decomposable*) means that $\left\{D_\gamma^n : \gamma \in \Gamma\right\}$ is is slicely discrete (resp. slicely isolated) for each $n \in \mathbb{N}$.

Now we can state the following theorem for a radial subset A of a normed space E, i.e. a subset such that for every $x \in E$ there is some positive λ such that $\lambda x \in A$:

Theorem 5 *A normed space $(E, \|\cdot\|)$ with a radial subset A that is assumed to be metrizable for the weak topology is* **LUR** *renormable if and only if every discrete family of subsets in $(A, \sigma(E, E^*))$ is σ-slicely finitely decomposable.*

Proof. A normed space $(E, \|\cdot\|)$ is **LUR** renormable if, and only, if the norm topology has a network $\mathcal{N}$ which is countable union of subfamilies $\{\mathcal{N}_n : n = 1, 2, ...\}$ such that, for every positive integer n, and every $x \in \bigcup\{M : M \in \mathcal{N}_n\}$, there is an open half space H with $x \in H$ and such that H only meets a member of the family $\mathcal{N}_n$; i.e. if $x \in M_0$ then $H \cap M = \emptyset$ for every $M \in \mathcal{N}_n$ with $M \neq M_0$, [27, 29], that is what we call a σ-slicely isolated network. The families $\mathcal{N}_n$ are slicely finite since they are slicely isolated and $\mathcal{N} = \cup\{\mathcal{N}_n : n = 1, 2, ...\}$ is a network for the coarser $\sigma(E, E^*)$ topology on E too. Let us take an isolated family of subsets $\{D_\gamma : \gamma \in \Gamma\}$ for the weak topology in the **LUR** renormable space E , we can define the decomposition $D_\gamma^n := D_\gamma \cap \mathcal{N}_n$ and the family of sets $\left\{D_\gamma^n : \gamma \in \Gamma\right\}$ is slicely isolated, and so slicely finite, since it is the intersection of an slicely isolated family with a fixed set, moreover $D_\gamma = \cup\left\{D_\gamma^n : n = 1, 2, ...\right\}$ for every $\gamma \in \Gamma$ and the family of sets $\{D_\gamma : \gamma \in \Gamma\}$ is σ-slicely finitely decomposable, even more it is σ-slicely isolatedly decomposable. Therefore, in every **LUR** renormable normed space every isolated family of subsets for the weak topology is σ-slicely finitely decomposable. When A is a fixed subset of E and the sets in the family $\{D_\gamma : \gamma \in \Gamma\}$ are subsets of A, then the discreteness of the family in A implies that it is an isolated family, and so it is σ-slicely finitely decomposable whenever E is a **LUR** renormable normed space.

The reverse implication needs a theorem presented in [5]. Indeed, after the assumption on the radial set A it verifies the conclusion of the corollary 2 above. It is possible now to apply theorem 3 in [27] and to get that the norm topology on A has a network $\mathcal{N} = \bigcup\{\mathcal{N}_m : m = 1, 2, ...\}$ with every one of the families $\mathcal{N}_m$ being isolated for the weak topology on A. In a metric space every isolated family is σ-discretely decomposable, see remark 3, and we can assume that our families $\mathcal{N}_m$ are discrete in $(A, \sigma(E, E^*))$. Consequently every family $\mathcal{N}_m$ is going to be σ-slicely finitely decomposable and therefore the norm topology in A admits a network σ-slicely relatively locally finite. The result now follows from theorem 4.1 and corollary 4.5 in [5]. ■

To finish this section let us observe another application of Lebesgue First Theorem giving a Baire one character of weakly continuous maps from a metric space to a Banach space:

Corollary 4 *Let (X, d) be a metric space, E a normed space and $f : X \to (E, weak)$ a continuous map. Then there is a sequence $f_n : X \to (E, \|\cdot\|)$ of continuous functions such that $lim_n f_n(x) = f(x)$ in the weak topology of E*

Proof. It is enough to apply Lebesgue First Theorem to the separately continuous map $\phi : X \times E^* \to \mathbb{R}$ defined as $\phi(x, y) :=< f(x), y >$ in the same way we have done in corollary 1. Indeed, with the same notations for the partitions of unity Ξ_n in the metric space (X, d) we will have here the maps $f_n(x) := \sum \{h_{\alpha,n}(x) f(x_{\alpha,n}) : \alpha \in A\}$ which converge to $f(x)$ in the weak topology of the normed space E for every $x \in X$. ■

Remark 2 Sirivatsa shows in [36] that every map f as above is of the first Baire class to the norm topology, i.e the functions f_n can be constructed in such a way that the sequence $\{f_n(x) : n = 1, 2, ...\}$ converges to $f(x)$ in the norm topology of E for every $x \in E$ too.

3. Slicely–constant approximations of σ–slicely continuous maps

A countable splitting in ε–terms of the continuity property of a given map is the notion of σ-continuity, see definition 1, and the first level of such a splitting is the notion of piecewise locally constant map, see definition 2. All notions introduced so far are connected by the following:

Theorem 6 *Let $(X, \mathcal{T})$ be a topological space and let (Y, ρ) be a metric space. Given a map $\Phi : X \to Y$ the following are equivalent:*

i) Φ is σ–continuous;

ii) if $\{D_\gamma : \gamma \in \Gamma\}$ is a discrete family of subsets in (Y, ρ) then $\left\{\Phi^{-1}(D_\gamma) : \gamma \in \Gamma\right\}$ is σ–isolatedly decomposable in $(X, \mathcal{T})$;

iii) there exists a sequence $\{\Phi_n : X \to Y : n \in \mathbb{N}\}$ of piecewise locally constant functions such that $\lim_{n\to\infty} \Phi_n x = \Phi x$ uniformly on $x \in X$.

The proof of the first part of the theorem is in [29], and has to be completed with the approximation property by maps with the rigidity condition of being locally constant up to a countable partition, (have a look at the proof of our version with slices presented in theorem 7). Our proof uses ideas contained in [19], theorem 5 and corollary 7, see also [13] and[33] for related results.

Remark 3 Given $\Phi : X \to Y$ we say that a family $\mathcal{B}$ of subsets of X is a *function base* for Φ if, whenever V is open in Y, then $\Phi^{-1}(V)$ is union of sets of $\mathcal{B}$. In other words $\mathcal{B}$ is a function base for Φ if, and only if, it is a network of the topology $\left\{\Phi^{-1}(V) : V \text{ is open in } Y\right\}$. The class of maps with σ–isolated function base has been studied by Hansell [11] in connection with the study of Lebesgue–Hausdorff theorem [20, vol 1, p. 393] as well as the descriptive set theory for non separable metric spaces. A map Φ from a topological space $(X, \mathcal{T})$ into the metric space (Y, ρ) is σ–continuous if, and only if, it has a σ–isolated function base, [29]. In connection with measurability properties Hansell [11] studied functions with a σ–isolated function base of $\mathcal{F} \cap \mathcal{G}$ sets. This class of maps is becoming central in the study of measurable selectors of upper semi–continuous multivalued functions when the domain space is not a metric space, [14]. Let Φ a continuous map from a topological space X into $C_p(K)$, where K is a compact space, Hansell has shown that if Φ has a σ–isolated function base, then Φ has also a σ–isolated function base of $\mathcal{F} \cap \mathcal{G}$ sets with respect to the norm topology of $C(K)$, and moreover Φ is a uniform limit of a sequence of piecewise continuous maps for the norm topology, [13]. In a metric space every isolated family is σ–discretely decomposable, [8]. Indeed if $\{D_\gamma :\ \gamma \in \Gamma\}$ is isolated in a metric space (X,d) we define

$$D_p^\gamma := \{x \in D_\gamma :\ B_d(x, 1/p) \cap D_\beta = \emptyset \text{ for all } \beta \in \Gamma \setminus \{\gamma\}\}$$

and $D_\gamma = \bigcup_{p=1}^{\infty} D_p^\gamma$. Thanks to the triangle inequality we have now that every family $\left\{D_p^\gamma :\ \gamma \in \Gamma\right\}$ is discrete. Indeed if $x \in X$ then the ball $B_d(x, 1/2p)$ meets at most one member of the family $\left\{D_p^\gamma :\ \gamma \in \Gamma\right\}$. Consequently a map $\Phi : X \to Y$ between metric spaces is σ–continuous if, and only if, it has a σ–discrete function base. This class of maps has been called σ–discrete maps, and they play an important role in the study of non separable descriptive topology, [11]. For instance when X is a metric space and Y is a normed space a map $\Phi : X \to Y$ in the first Borel class; i.e. $\Phi^{-1}(U)$ is a F_σ–set for every open set U of Y, will be the pointwise limit of a sequence of continuous functions if, and only if, the map Φ is σ–discrete [9], [34]. When X is a complete metric space Hansell [8] showed that any Borel measurable map $\Phi : X \to Y$ is a σ–discrete map. The same assertion for an arbitrary metric space is independent of the usual axioms of the set theory, [10].

In our applications we shall be mainly interested in a particular class of σ–continuous maps where the continuity property is required for a fixed subbasis of the topology. Indeed we shall work in a locally convex topological vector space X and we will play with the subbasis of the weak topology made up with all the open half spaces in X:

Definition 5 Let A be a subset of a locally convex topological vector space X, let Φ be a map from A into a metric space (Y, ρ). We say that Φ is *slicely continuous at* $x \in A$ if for every $\varepsilon > 0$ there exists an open half space H of X containing x with osc $(\Phi_{\restriction_{H\cap A}}) = \text{diam } \Phi(H \cap A) < \varepsilon$. We say that Φ is *σ–slicely continuous on* A if for every $\varepsilon > 0$ we can write

$$A = \bigcup_{n\in\mathbb{N}} A_{n,\varepsilon} \tag{1}$$

in such a way that for every $x \in A_{n,\varepsilon}$ there exists an open half space H of X containing x with osc $\left(\Phi_{\restriction_{H\cap A_{n,\varepsilon}}}\right) = diam\ \Phi(H \cap A_{n,\varepsilon}) < \varepsilon$.

As before, the easiest way to produce σ–slicely continuous maps is by means of the locally constant maps, although using open half spaces as neighbourhoods in that case:

Definition 6 Let A be a subset of a locally convex topological vector space space X, let Φ be a map from A into a set Y. We say that Φ is *slice-locally constant* if for every $x \in A$ there exists an open half space H of X containing x with $\Phi_{\restriction_{H\cap A}}$ a constant map.

The map Φ is said to be *piecewise slicely constant* if A can be expressed as the union of a sequence of sets $\{A_n :\ n \in \mathbb{N}\}$ such that the restrictions $\Phi_{\restriction_{A_n}}$ are slice-locally constant for all $n \in \mathbb{N}$.

The following theorem is a central result of the present paper:

Theorem 7 *Let A be a subset of a locally convex linear topological space and let (Y, ρ) be a metric space. For a map $\Phi : A \to Y$ the following assertions are equivalent:*

i) Φ is σ–slicely continuous.

ii) If $\{D_\gamma :\ \gamma \in \Gamma\}$ is a discrete family of subsets of (Y, ρ) then $\left\{\Phi^{-1}(D_\gamma) :\ \gamma \in \Gamma\right\}$ is σ–slicely isolatedly decomposable in A.

iii) There exists a sequence $\Phi_n : A \to Y$, $n \in \mathbb{N}$ of piecewise slicely constant functions such that $\lim_{n\to\infty} \Phi_n x = \Phi x$ uniformly on $x \in A$.

Proof. i)$\Longrightarrow$ii). If $\{D_\gamma : \gamma \in \Gamma\}$ is a discrete family of subsets of (Y, ρ) we can firstly decompose every set D_γ in the following way

$$D_{\gamma,p} := \{x \in D_\gamma : B_\rho(x, 1/p) \cap D_\beta = \emptyset \text{ for all } \beta \neq \gamma, \ \beta \in \Gamma\}$$

and we have $D_\gamma = \bigcup_{p=1}^{\infty} D_{\gamma,p}$. Now each family $\{D_{\gamma,p} : \gamma \in \Gamma\}$ is $1/p$–discrete. Since Φ is σ–slicely continuous, for every positive integer p we decompose the domain $A = \bigcup_{n=1}^{\infty} A_{n,1/p}$ in such a way that for every $x \in A_{n,1/p}$ there exists an open half space H with $x \in H$ and such that

$$\rho - \text{diam}\left(\Phi\left(A_{n,1/p} \cap H\right)\right) < 1/p. \tag{2}$$

Then, if we consider for n and p fixed the family

$$\left\{\Phi^{-1}\left(D_{\gamma,p}\right) \cap A_{n,1/p} : \gamma \in \Gamma\right\}$$

it now follows that it is an slicely isolated family. Indeed, let us take $x \in \Phi^{-1}\left(D_{\gamma,p}\right) \cap A_{n,1/p}$, if H is the open half space containing x which satisfies (2), and according to the fact that the sets $D_{\gamma,p}$ form a $1/p$–discrete family in (Y, ρ) for the fixed p and $\gamma \in \Gamma$, we have

$$H \cap \Phi^{-1}\left(D_{\beta,p}\right) \cap A_{n,1/p} = \emptyset, \text{ for all } \beta \neq \gamma, \ \beta \in \Gamma.$$

Since $\Phi^{-1}\left(D_\gamma\right) = \bigcup_{n,p=1}^{\infty} \Phi^{-1}\left(D_{\gamma,p}\right) \cap A_{n,1/p}$, we have shown that $\left\{\Phi^{-1}\left(D_\gamma\right) : \gamma \in \Gamma\right\}$ is σ–slicely isolatedly decomposable in A.

ii)$\Longrightarrow$iii) Let us fix the $\epsilon > 0$ and we will construct a piecewise slicely constant function $\Phi_\varepsilon : X \to Y$ such that $\rho\left(\Phi_\varepsilon x, \Phi x\right) < \varepsilon$ for all $x \in X$. Every open cover of the metric space (Y, ρ) has a σ–discrete open refinement by the Stone's theorem on the paracompactness of metric spaces. Taking an open cover by sets of diameter less than ϵ we find a refinement $\mathcal{B}$ of it such that $\mathcal{B} = \bigcup_{m=1}^{\infty} \mathcal{B}_m$ and every $\mathcal{B}_m$ is a discrete family of open sets with diameter less than ε. Using that each family $\Phi^{-1}\left(\mathcal{B}_m\right)$ will be σ–slicely isolatedly decomposable we will build up the piecewise slicely constant function $\Phi_\varepsilon : A \to Y$ we are looking for. Indeed, let us begin with the construction on the set

$$A_1 := \bigcup \left\{\Phi^{-1}(B) : B \in \mathcal{B}_1\right\}$$

and an induction process will follow to complete the construction. We can write $\Phi^{-1}(B) = \bigcup_{m=1}^{\infty} A_m^{1,B}$ for every $B \in \mathcal{B}_1$ where every family $\left\{A_m^{1,B} : B \in \mathcal{B}_1\right\}$ is slicely isolated for every positive integer m. Choose $y_B \in B$ for every $B \in \mathcal{B}_1$ and let us define

$$\Phi_\varepsilon^1 : \bigcup \left\{A_1^{1,B} : B \in \mathcal{B}_1\right\} \to Y$$

by $\Phi_\varepsilon^1 x = y_B$ if $x \in A_1^{1,B}$. Then Φ_ε^1 is a slice-locally constant function. Set

$$A_{1,1} := \bigcup \left\{A_1^{1,B} : B \in \mathcal{B}_1\right\}$$

and $A_{1,2} := \bigcup \left\{A_2^{1,B} : B \in \mathcal{B}_1\right\} \setminus A_{1,1}$. Again we can define a slice-locally constant function

$$\Phi_\varepsilon^2 : A_{1,2} \to Y$$

by $\Phi_{\varepsilon}^{2}x = y_B$ if $x \in A_2^{1,B}$ because the intersection of a slicely isolated family of sets with a fixed set is a slicely isolated family too. Inductively we define a partition of A_1

$$A_1 = \bigcup_{n=1}^{\infty} A_{1,n}$$

and slice-locally constant functions $\Phi_{\varepsilon}^{n} : A_{1,n} \to Y$ with the property that $\rho\left(\Phi_{\varepsilon}^{n}x, \Phi x\right) < \varepsilon$, $\forall x \in A_{1,n}$. Finally we have $\Phi_1 : A_1 \to Y$ defined by $\Phi_1 x = \Phi_{\varepsilon}^{n}x$ if $x \in A_{1,n}$ which is piecewise slicely constant and verifies $\rho\left(\Phi_1 x, \Phi x\right) < \varepsilon$, for every $x \in A_1$. Inductively we define piecewise slicely constant functions $\Phi_n : A_n \to Y$ where

$$A_2 := \bigcup \left\{\Phi^{-1}(B) : \ B \in \mathcal{B}_2\right\} \setminus A_1, \ldots, A_n := \bigcup \left\{\Phi^{-1}(B) : \ B \in \mathcal{B}_n\right\} \setminus A_{n-1} \ldots$$

verifying $\rho\left(\Phi_n x, \Phi x\right) < \varepsilon$, for every $x \in A_n$. Finally $\Phi_{\varepsilon}x = \Phi_n x$ if $x \in A_n$. Then Φ_{ε} is a piecewise slicely constant function $\Phi_{\varepsilon} : A \to Y$ verifying

$$\rho\left(\Phi_{\varepsilon}x, \Phi x\right) < \varepsilon, \quad x \in A.$$

iii)$\Longrightarrow$i) Given $\varepsilon > 0$ there exists an integer q such that $\rho\left(\Phi_q x, \Phi x\right) < \varepsilon/2$, for every $x \in A$. Since Φ_q is piecewise slicely constant there is a sequence of subsets $A_1, A_2, \ldots, A_n, \ldots$ with $A = \bigcup_{n=1}^{\infty} A_n$ and $\Phi_{q\restriction A_n}$ slice-locally constant for every $n \in \mathbb{N}$. Then for every $x \in A_n$ there is an open half space H containing x such that $\Phi_{q\restriction A_n \cap H}$ is constant. Consequently osc $\left(\Phi_{\restriction A_n \cap H}\right) < \varepsilon$ and the proof is over. ■

4. Applications to locally uniformly rotund renormings

The following result contains theorem 3 from the introduction. We can see the aproximation and rigidity conditions characterizing locally uniformly rotund renormability:

Theorem 8 *Let $(E, \|\cdot\|)$ be a normed space with a norming subspace $F \subset E^*$, then the following conditions are equivalent:*

i) The normed space E admits an equivalent $\sigma(E,F)$-lower semicontinuous **LUR** *norm on it.*

ii) The identity map on E is a uniform limit of a sequence $\{I_n : E \to E, n = 1,2,...\}$ of piecewise slicely constant maps for $\sigma(E,F)$.

iii) There is a sequence of piecewise slicely constant maps $\{I_n : E \to E, n = 1,2,...\}$ for $\sigma(E,F)$ such that $x \in \overline{\{I_n(x) : n = 1,2,...\}}^{\sigma(E,E^*)} \forall x \in E$.

iv) There is a sequence of piecewise slicely constant maps $\{I_n : E \to E, n = 1,2,...\}$ for $\sigma(E,F)$ such that $x \in \overline{span\{I_n(x) : n = 1,2,...\}}^{\|\cdot\|} \forall x \in E$.

Proof. A normed space $(E, \|\cdot\|)$ has an equivalent $\sigma(E,F)$-lower semicontinuous norm if, and only if, the identity map from $(E, \sigma(E,F))$ into $(E, \|\cdot\|)$ is σ-slicely continuous, [25, 29, 32].This fact together with our theorem 7 is enough to prove the equivalence between i) and ii). From the properties described in [29] for the σ-slicely continuous maps it follows the result for the coarser convergence topologies. Indeed, if we assume only the condition iii), that is x is a cluster point of the sequence $\{I_n(x) : n = 1,2,...\}$ in the weak topology of E for every $x \in E$, then the Hahn Banach theorem tell us that condition iv) is satisfied:

$$x \in \overline{span\{I_n(x) : n = 1,2,...\}}^{\|\cdot\|} \forall x \in E.$$

Finally, if the condition iv) is fulfilled, then the set of all rational convex combinations of maps from $\{I_n : E \to E, n = 1,2,...\}$ forms a countable family of maps that we call $\{J_n : E \to E, n = 1,2,...\}$ and they are σ–slicely continuous too for the $\sigma(E, F)$-topology by our results in [29, chapter 4]. For every $x \in E$ we will have that $x \in \overline{\{J_n(x), n = 1, 2, ...\}}^{\|\cdot\|}$, it now follows that the identity map in E is σ–slicely continuous for the $\sigma(E, F)$-topology, so the space E is going to have an equivalent $\sigma(E, F)$-lower semicontinuous **LUR** norm and we arrive to i) and the proof is over. ■

Let us remark here that the corresponding result for descriptive spaces is in [27], the result presented here for **LUR** renormings needs our chapter 4 in [29]. Let us apply the former result to the space $c_0(\Gamma)$ since it is a characteristic example in the theory:

Example 1 *In the space* $c_0(\Gamma)$ *we describe the sequence of piecewise slicely constant maps approximating the identity.*

Proof. Given an $\varepsilon > 0$ and $x \in c_0(\Gamma)$, we set $L^\varepsilon(|x|) := \{\gamma : |x(\gamma)| \geq \varepsilon\}$. Let us fix l, $p \in \mathbb{N}$, and $q \in \mathbb{Q}$ such that we have:

$$\#L^\varepsilon(|x|) = l \tag{3}$$

$$\varepsilon - \sup\{|x(\gamma)| : \gamma \notin L^\varepsilon(|x|)\} > p^{-1}, \tag{4}$$

and

$$0 \leq q - \sum \left\{x^2(\gamma) : \gamma \in L^\varepsilon(|x|)\right\} < (p^{-1})^2/3 \tag{5}$$

First we consider the cardinality of the finite set $L^\varepsilon(|x|)$, then we consider the jump of x after ϵ, (4); both are the integers of the*rigidity* condition, and finally we adjust the width of the required open half space with (5). It exists because we can adjust the parameter q here in order to have an almost maximum for the $\sum \left\{x^2(\gamma) : \gamma \in L^\varepsilon(|x|)\right\}$ in our piece. Indeed, let us denote by $X_{l,p,q}$ the set of all $y \in c_0(\Gamma)$ which satisfy the same conditions as x, i.e. the conditions (3),(4)and (5) with y instead of x. It is clear that

$$c_0(\Gamma) = \bigcup \{X_{l,p,q} : l, p \in \mathbb{N}; q \in \mathbb{Q}\}$$

Let us write $L^\varepsilon(|x|) = \{\gamma_i\}_1^\ell$, and let f_x be the linear functional on $c_0(\Gamma)$ defined by the formula

$$f_x(u) := \sum_1^\ell x(\gamma_i)\, u(\gamma_i), \quad u \in c_0(\Gamma).$$

Denote with H_x the open half space of $c_0(\Gamma)$ given by

$$\left\{u \in c_0(\Gamma) : f_x(u) > f_x(x) - (p^{-1})^2/3\right\}$$

Pick $y \in X_{l,p,q} \cap H_x$. From (5) we get

$$\sum_1^\ell (x(\gamma_i) - y(\gamma_i))^2 = 2f_x(x-y) + \sum_1^\ell \left(y^2(\gamma_i) - x^2(\gamma_i)\right) < \tag{6}$$
$$2f_x(x-y) + \sum \left\{y^2(\gamma) : \gamma \in L^\varepsilon(|y|)\right\} - \sum \left\{x^2(\gamma) : \gamma \in L^\varepsilon(|x|)\right\} < (p^{-1})^2.$$

From (3) and (4) we get that

$$L^\varepsilon(|x|) = L^\varepsilon(|y|).$$

Thus the map $\Theta_\varepsilon : c_0(\Gamma) \to 2^\Gamma$ defined by $\Theta_\varepsilon(x) = L^\varepsilon(|x|)$ is piecewise slicely constant.

Given the partition $\Pi = \{t_0 < t_1 < ... < t_s\}$ of the interval $[-n, n]$ with $t_{i+1} - t_i = \delta$, let us define $f_\delta^n(t) := \sum_{-n}^{n-1} t_i.1_{[t_i,t_{i+1})}$. For a given $f : \mathbb{R} \to \mathbb{R}$ with finite range we define the map

$$\Xi_f : c_0(\Gamma) \to c_0(\Gamma)$$

by $\Xi_f(x)(\gamma) := f(x(\gamma))$. The map $\Xi_{f_\delta^n}$ is piecewise slicely constant. Indeed, let us take $\epsilon > 0$ with 0 and ϵ in the same subinterval of the partition Π, and let us consider for that ϵ the former pieces $X_{l,p,q}$ where Θ_ε is slice-locally constant. If we think in all possible distributions of all possible values of elements $x \in X_{l,p,q}$ in the partition Π, we see that we only have a finite number of cases when we restrict ourselves to the values $x(\gamma)$ for $\gamma \in L^\varepsilon(|x|)$. If we decompose the set $X_{l,p,q}$ according with all such possibilities, it follows that the map $\Xi_{f_\delta^n}$ is piecewise slicely locally constant on $X_{l,p,q}$ and thus on all $c_0(\Gamma)$. We can now take the sequence of functions $\left\{f_{\delta_n}^n : n = 1, 2, 3, ...\right\}$ where $\delta_n = 1/2^n$ and we have $lim_n f_{\delta_n}^n(t) = t$ for every t in $\mathbb{R}$. So the sequence of functions $\left\{\Xi_{f_{\delta_n}^n} : n = 1, 2, ...\right\}$ approximate the identity map uniformly on bounded sets of $c_0(\Gamma)$. ∎

In a similar way we may construct an approximating sequence

$$\{I_n : E \to E, n = 1, 2, ...\}$$

of piecewise slicely constant maps in most of the Banach space E with a **LUR** renorming, for instance in the following case:

Example 2 *In a Banach space $(E, \|\cdot\|)$ with a strong Markushevich basis $\{(x_\gamma, f_\gamma) : gamma \in \Gamma\}$, i.e.*

$$x \in \overline{span\,\{x_\gamma : f_\gamma(x) \neq 0\}}^{\|\cdot\|} \forall x \in E,$$

we can describe a sequence of piecewise slicely constant maps $\{I_n : E \to E, n = 1, 2, ...\}$ such that $x \in \overline{\{I_n(x) : n = 1, 2, ...\}}^{\|\cdot\|} \forall x \in E$.

Proof. Indeed, for every $x \in E$ we can assume without lose of generality that $(f_\gamma(x))_{\gamma\in\Gamma}$ is a vector of $c_0(\Gamma)$. With the same notations as above we can take now the functions $\Theta_\varepsilon : E \to E$ defined by $\Theta_\varepsilon(x) := \sum \{x_i : |f_i(x)| \geq \varepsilon\}$. These maps are piecewise slicely constant, indeed it is enough to consider the sets $E_{l,p,q} := \{x \in E : (f_\gamma(x))_{\gamma\in\Gamma} \in X_{l,p,q}\}$, and for $x \in E_{l,p,q}$ to take the functional

$$f_x(u) := \sum_1^\ell f_{\gamma_j}(x) f_{\gamma_j}(u) \quad u \in E.$$

and we operate as above denoting with $\{\gamma_j : j = 1, 2, ...l\}$ the finite set of indexes for which $|f_\gamma(x)| \geq \varepsilon$. Since the basis is strong we have that the $\mathbb{Q}$ linear span of the functions Θ_ε for all rational positive numbers ϵ is a sequence of maps

$$\{I_n : E \to E, n = 1, 2, ...\}$$

approximating the identity on E as required. The fact that they are piecewise slicely constant maps follows from our study in Chapter 4 of [29]. ∎

Let us mention to finish that the main renorming result in the recent paper [21] is formulated in terms of our piecewise slicely constant maps.

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