WEAK CONTINUITY OF RIEMANN INTEGRABLE FUNCTIONS IN LEBESGUE-BOCHNER SPACES

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Abstract. In general, Banach space-valued Riemann integrable functions defined on $[0,1]$ (equipped with the Lebesgue measure) need not be weakly continuous almost everywhere. A Banach space is said to have the weak Lebesgue property if every Riemann integrable function taking values in it is weakly continuous almost everywhere. In this paper we discuss this property for the Banach space L^1_X of all Bochner integrable functions from $[0,1]$ to the Banach space X. We show that L^1_X has the weak Lebesgue property whenever X has the Radon-Nikodým property and X^* is separable. This generalizes the result by Chonghu Wang and Kang Wan [Rocky Mountain J. Math. 31 (2001), no. 2, 697–703] that $L^1[0,1]$ has the weak Lebesgue property.

1. Introduction

The study of Riemann integration for Banach space-valued functions goes back to the origins of Banach space theory. In a pioneering paper [1] dated on 1927, Graves made clear that Lebesgue's well-known criterion of Riemann integrability for real-valued functions fails in general for vector-valued ones. Indeed, he gave an example of a Riemann integrable function $f : [0,1] \to \ell^\infty([0,1])$ which is not *norm* continuous almost everywhere (a.e. for short) with respect to the Lebesgue measure. In fact, it turns out that this failure is shared by the most familiar infinite-dimensional Banach spaces (a remarkable exception is $\ell^1(\mathbb{N})$). On the contrary, it is not difficult to see that every a.e. norm continuous Banach space-valued function defined on $[0,1]$ is Riemann integrable. For a detailed account on this topic, we refer the reader to the survey [2] by Gordon.

The connection between Riemann integrability and continuity with respect to the *weak topology* was first discussed by Alexiewicz and Orlicz [3]. They showed an example of a weakly continuous function $f : [0,1] \to c_0(\mathbb{N})$ which is not Riemann integrable. The ultimate reason for that relies on the fact that $c_0(\mathbb{N})$ contains weakly convergent sequences which are not norm convergent (i.e. it fails the Schur property): Kadets [4] proved that a Banach space Y has the Schur property if and

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only if every weakly continuous function $f : [0,1] \to Y$ is Riemann integrable. This result has been extended by Chonghu Wang and Zhenhua Yang [5] to arbitrary locally convex topologies on Y weaker than the norm topology. On the other hand, going back to the paper by Alexiewicz and Orlicz, they also constructed a $C[0,1]$-valued Riemann integrable function without points of continuity for the weak topology. A Banach space Y is said to have the *weak Lebesgue property* (*WLP* for short) if every Riemann integrable function $f : [0,1] \to Y$ is weakly continuous a.e. It is not difficult to check that every Banach space with separable dual has the WLP, see [6, 7]. Recently, Chonghu Wang and Kang Wan [7] have shown that the separability of the dual is not a necessary condition to have the WLP by proving that $L^1[0,1]$ has this property.

In this paper we study the WLP for the Banach space L^1_X of all Bochner integrable functions from $[0,1]$ to the Banach space X. Our approach clarifies somehow the role of uniform integrability (relative weak compactness) in the original proof that $L^1[0,1] = L^1_{\mathbb{R}}$ has the WLP and allows us to extend this result to L^1_X under certain assumptions on X. Our main result states that *L^1_X has the WLP whenever X has the Radon-Nikodým property and X^* is separable* (Theorem 2.4). This happens, for instance, if X is separable and reflexive. We finish the paper making clear that *the L^1 space associated to a probability measure fails the WLP whenever it has density character greater than or equal to the continuum* (Proposition 2.10).

Terminology. All unexplained terminology can be found in our standard references [8, 9]. The density character of a topological space is the minimal cardinality of a dense subset. We work with $[0,1]$ equipped with the Lebesgue measure λ on the σ-algebra $\mathcal{L}$ of all Lebesgue measurable subsets of $[0,1]$. All Banach spaces here are assumed to be real. Let $(Y, \|\cdot\|_Y)$ be a Banach space. Then $B_Y := \{y \in Y : \|y\|_Y \leq 1\}$ and Y^* stands for the topological dual of Y. The weak* topology on Y^* is denoted by w^*. A function $f : [0,1] \to Y$ is said to be *strongly measurable* if it is the a.e. limit of a sequence of simple functions. According to Pettis' measurability theorem (cf. [8, Theorem 2, p. 42]), f is strongly measurable if and only if it is *scalarly measurable* (i.e. for each $y^* \in Y^*$ the composition $y^* \circ f$ is measurable) and there is $E \in \mathcal{L}$ with $\lambda(E) = 1$ such that $f(E)$ is separable. A strongly measurable function $f : [0,1] \to Y$ is *Bochner integrable* if the real-valued function $\|f\|_Y$ given by $t \mapsto \|f(t)\|_Y$ is Lebesgue integrable. After identifying functions which coincide a.e., the linear space L^1_Y of all (equivalence classes of) Bochner integrable functions from $[0,1]$ to Y becomes a Banach space when endowed with the norm

$$\|f\|_{L^1_Y} := \int_0^1 \|f\|_Y \, d\lambda.$$

Recall that Y has the *Radon-Nikodým property* (*RNP* for short) if for each λ-continuous countably additive measure $\nu : \mathcal{L} \to Y$ with bounded variation there is a Bochner integrable function $f : [0,1] \to Y$ such that $\nu(E) = \int_E f \, d\lambda$ for all $E \in \mathcal{L}$. Standard examples of Banach spaces having the RNP are the reflexive ones and the separable duals.

2. The results

Recall that a function f defined on $[0,1]$ and taking values in a Banach space Y is said to be *Riemann integrable*, with integral $y \in Y$, if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\Big\| \sum_{k=1}^{n} \lambda(I_k) f(t_k) - y \Big\|_Y \leq \varepsilon$$

whenever $I_1, \dots, I_n$ is a finite collection of non-overlapping closed subintervals covering $[0,1]$ such that $\max_{1\leq k\leq n} \lambda(I_k) \leq \delta$ and $t_k \in I_k$ for all $1 \leq k \leq n$.

Lemma 2.1. *Let Y be a Banach space with w^*-separable dual. Let $f : [0,1] \to Y$ be a function such that $y^* \circ f$ is Riemann integrable for every $y^* \in Y^*$. The following statements are equivalent:*

- (i) *There is $E \in \mathcal{L}$ with $\lambda(E) = 1$ such that, for each sequence (t_n) in $[0,1]$ converging to a point of E, the set $\{f(t_n) : n \in \mathbb{N}\}$ is relatively weakly compact in Y.*
- (ii) *f is weakly continuous a.e.*

Proof. (ii)⇒(i) is obvious.

(i)⇒(ii) Let $N \subset Y^*$ be a countable w^*-dense set and let $\sigma(Y,N)$ be the topology on Y of pointwise convergence on N, which is Hausdorff because N separates the points of Y. Given $y^* \in N$, the composition $y^* \circ f$ is Riemann integrable and so it is continuous a.e. Since N is countable, we can find $F \in \mathcal{L}$ with $\lambda(F) = 1$ such that f is $\sigma(Y,N)$-continuous at each point of F. We claim that f is weakly continuous at each $t \in E \cap F$. Indeed, let (t_n) be a sequence in $[0,1]$ converging to t. Then $f(t_n) \to f(t)$ in the topology $\sigma(Y,N)$. By the assumption, the set $\{f(t_n) : n \in \mathbb{N}\}$ is relatively weakly compact in Y. Since $\sigma(Y,N)$ is Hausdorff and coarser than the weak topology, both topologies coincide on any weakly compact subset of Y. It follows that $f(t_n) \to f(t)$ weakly and the proof is over. □

Given a Banach space X, we write $L^\infty_{X^*}$ to denote the Banach space of all essentially bounded strongly measurable functions from $[0,1]$ to X^*, equipped with the essential supremum norm (functions which coincide a.e. are identified). It is known (cf. [8, IV.1]) that there is an isometric embedding $L^\infty_{X^*} \hookrightarrow (L^1_X)^*$, where the duality is given by $\langle \varphi, h \rangle = \int_0^1 \langle \varphi(\cdot), h(\cdot) \rangle \, d\lambda$ for all $\varphi \in L^\infty_{X^*}$ and $h \in L^1_X$. This isometry is onto if and only if X^* has the RNP, cf. [8, Theorem 1, p. 98]. In the general case, $B_{L^\infty_{X^*}}$ is still w^*-dense in $B_{(L^1_X)^*}$.

Lemma 2.2. *Let X be a Banach space. Let $h, g \in L^1_X$ for which there exist disjoint $A, B \in \mathcal{L}$ such that $\int_A \|h\|_X \, d\lambda > a$ and $\int_B \|g\|_X \, d\lambda > b$. Then either $\|h+g\|_{L^1_X} > a+b$ or $\|h-g\|_{L^1_X} > a+b$.*

Proof. We write $h\chi_A$ (resp. $g\chi_B$) to denote the element of L^1_X which coincides with h (resp. g) on A (resp. B) and vanishes outside A (resp. B). Since $B_{L^\infty_{X^*}}$ is w^*-dense in $B_{(L^1_X)^*}$, we can take $\varphi_h, \varphi_g \in B_{L^\infty_{X^*}}$ such that $\langle \varphi_h, h\chi_A \rangle > a$ and $\langle \varphi_g, g\chi_B \rangle > b$. Clearly, we can assume without loss of generality that φ_h (resp. φ_g)

vanishes outside A (resp. B). Then both $\varphi_h+\varphi_g$ and $\varphi_h-\varphi_g$ belong to $B_{L^\infty_{X^*}}$ and, therefore, we have

$$\begin{aligned}\|h+g\|_{L^1_X}+\|h-g\|_{L^1_X}&\geq\langle\varphi_h+\varphi_g,h+g\rangle+\langle\varphi_h-\varphi_g,h-g\rangle=\\&=2\langle\varphi_h,h\rangle+2\langle\varphi_g,g\rangle=2\langle\varphi_h,h\chi_A\rangle+2\langle\varphi_g,g\chi_B\rangle>2(a+b).\end{aligned}$$

Hence either $\|h+g\|_{L^1_X}>a+b$ or $\|h-g\|_{L^1_X}>a+b$. □

Given a Banach space X, a set $\mathcal{H}\subset L^1_X$ is called *uniformly integrable* if it is bounded and for each $\varepsilon>0$ there is $\delta>0$ such that $\sup_{h\in\mathcal{H}}\int_C\|h\|_X\,d\lambda\leq\varepsilon$ for every $C\in\mathcal{L}$ with $\lambda(C)\leq\delta$.

Lemma 2.3. *Let X be a Banach space with the WLP. Let $f:[0,1]\to L^1_X$ be a Riemann integrable function. Then there is $E\in\mathcal{L}$ with $\lambda(E)=1$ such that, for each sequence (t_n) in $[0,1]$ converging to a point of E, we have:*

(i) *The set $\{f(t_n):n\in\mathbb{N}\}$ is uniformly integrable.*
(ii) *For each $C\in\mathcal{L}$, the set $\{\int_C f(t_n)\,d\lambda:n\in\mathbb{N}\}$ is relatively weakly compact in X.*

Proof. We divide the proof into several steps.

Step 1. Fix $\beta>0$ and let E_β be the set of all $t\in[0,1]$ such that:

For every $\delta>0$ there exist $t'\in[0,1]$ with $|t'-t|<\delta$ and $C\in\mathcal{L}$ with $\lambda(C)<\delta$ such that $\int_C\|f(t)-f(t')\|_X\,d\lambda>\beta$.

We claim that $\lambda^*(E_\beta)=0$ (as usual, λ^* stands for the Lebesgue outer measure). Suppose, if possible, otherwise. Since f is Riemann integrable, there is a finite collection $J_1,\dots,J_m$ of non-overlapping closed subintervals covering $[0,1]$ such that

$$\Big\|\sum_{j=1}^m\lambda(J_j)(f(\xi_j)-f(\xi_j'))\Big\|_{L^1_X}<\beta\lambda^*(E_\beta)$$

for all choices $\xi_j,\xi_j'\in J_j$, $1\leq j\leq m$. Write I_j to denote the interior of J_j, so that the I_j's are pairwise disjoint. For some $I\subset\{1,\dots,m\}$ we have $I_j\cap E_\beta\neq\emptyset$ for every $j\in I$ and $\sum_{j\in I}\lambda^*(I_j\cap E_\beta)\geq\lambda^*(E_\beta)$. Rearranging if necessary, we can assume that $I=\{1,\dots,n\}$ for some $1\leq n\leq m$. Observe that

$$\Big\|\sum_{j=1}^n\lambda(I_j)(f(\xi_j)-f(\xi_j'))\Big\|_{L^1_X}<\beta\lambda^*(E_\beta)\tag{1}$$

for all choices $\xi_j,\xi_j'\in I_j$, $1\leq j\leq n$.

Since $I_1\cap E_\beta\neq\emptyset$, there exist points $t_1\in I_1\cap E_\beta$ and $t_1'\in I_1$ such that $\int_0^1\|f(t_1)-f(t_1')\|_X\,d\lambda>\beta$, hence $\int_0^1\|\lambda(I_1)(f(t_1)-f(t_1'))\|_X\,d\lambda>\beta\lambda(I_1)$. Fix $1\leq k<n$ and assume that we have already chosen points $t_j,t_j'\in I_j$ for all $1\leq j\leq k$ with the property that

$$\Big\|\sum_{j=1}^k\lambda(I_j)(f(t_j)-f(t_j'))\Big\|_{L^1_X}>\beta\Big(\sum_{j=1}^k\lambda(I_j)\Big).$$

Set $g := \sum_{j=1}^{k} \lambda(I_j)(f(t_j) - f(t'_j)) \in L^1_X$ and

$$\alpha := \|g\|_{L^1_X} - \beta\Big(\sum_{j=1}^{k} \lambda(I_j)\Big) > 0.$$

Choose $\delta > 0$ such that $\int_C \|g\|_X \, d\lambda < \alpha$ whenever $C \in \mathcal{L}$ and $\lambda(C) < \delta$. Since $I_{k+1} \cap E_\beta \neq \emptyset$, there exist $t_{k+1} \in I_{k+1} \cap E_\beta$, $t'_{k+1} \in I_{k+1}$ and $C \in \mathcal{L}$ with $\lambda(C) < \delta$ such that $\int_C \|f(t_{k+1}) - f(t'_{k+1})\|_X \, d\lambda > \beta$, so $h := \lambda(I_{k+1})(f(t_{k+1}) - f(t'_{k+1})) \in L^1_X$ satisfies

$$\int_C \|h\|_X \, d\lambda > \beta\lambda(I_{k+1}). \tag{2}$$

By the choice of δ, we also have $\int_C \|g\|_X \, d\lambda < \int_0^1 \|g\|_X \, d\lambda - \beta(\sum_{j=1}^{k} \lambda(I_j))$, thus

$$\int_{[0,1]\setminus C} \|g\|_X \, d\lambda > \beta\Big(\sum_{j=1}^{k} \lambda(I_j)\Big). \tag{3}$$

Bearing in mind inequalities (2) and (3), an appeal to Lemma 2.2 (interchanging the role of t_{k+1} and t'_{k+1} if necessary) ensures that

$$\Big\|\sum_{j=1}^{k+1} \lambda(I_j)(f(t_j) - f(t'_j))\Big\|_{L^1_X} > \beta\Big(\sum_{j=1}^{k+1} \lambda(I_j)\Big).$$

Continuing the process in this way, we are able to find $t_j, t'_j \in I_j$ for all $1 \leq j \leq n$ such that

$$\Big\|\sum_{j=1}^{n} \lambda(I_j)(f(t_j) - f(t'_j))\Big\|_{L^1_X} > \beta\Big(\sum_{j=1}^{n} \lambda(I_j)\Big) \geq \beta\Big(\sum_{j=1}^{n} \lambda^*(I_j \cap E_\beta)\Big) \geq \beta\lambda^*(E_\beta),$$

which contradicts (1). This shows that $\lambda^*(E_\beta) = 0$.

Step 2. Let (C_n) be a sequence in $\mathcal{L}$ such that

$$\inf_{n\in\mathbb{N}} \lambda(C \triangle C_n) = 0 \quad \text{for every } C \in \mathcal{L}. \tag{4}$$

Fix $n \in \mathbb{N}$ and consider the linear continuous mapping $T_n : L^1_X \to X$ given by $T_n(h) := \int_{C_n} h \, d\lambda$. Since the composition $T_n \circ f : [0,1] \to X$ is Riemann integrable and X has the WLP, there is $F_n \in \mathcal{L}$ with $\lambda(F_n) = 1$ such that $T_n \circ f$ is weakly continuous at each point of F_n. Then $F := \bigcap_{n\in\mathbb{N}} F_n$ is measurable, $\lambda(F) = 1$ and, for each $n \in \mathbb{N}$, the function $T_n \circ f$ is weakly continuous at each point of F.

Step 3. Define $E := F \setminus \bigcup_{n\in\mathbb{N}} E_{1/n}$. Then E is measurable with $\lambda(E) = 1$. We will check that E satisfies the required property. Fix a sequence (t_n) in $[0,1]$ converging to a point $t \in E$. On the one hand, f is bounded (since it is Riemann integrable) and so $\{f(t_n) : n \in \mathbb{N}\}$ is bounded in L^1_X. On the other hand, fix $m \in \mathbb{N}$. Since $t \notin E_{1/m}$, there is $\delta > 0$ such that $\int_C \|f(t) - f(t')\|_X \, d\lambda \leq 1/m$ for every $t' \in [0,1]$ with $|t' - t| < \delta$ and every $C \in \mathcal{L}$ with $\lambda(C) < \delta$. Since $t_n \to t$ as $n \to \infty$, we may and do assume that $|t_n - t| < \delta$ for every $n \in \mathbb{N}$. Moreover, we can also assume that $\int_C \|f(t)\|_X \, d\lambda \leq 1/m$ for every $C \in \mathcal{L}$ with $\lambda(C) < \delta$. Hence $\sup_{n\in\mathbb{N}} \int_C \|f(t_n)\|_X \, d\lambda \leq 2/m$ for every $C \in \mathcal{L}$ with $\lambda(C) < \delta$. As $m \in \mathbb{N}$ is arbitrary, it follows that $\{f(t_n) : n \in \mathbb{N}\}$ is uniformly integrable.

Fix $C \in \mathcal{L}$ and $\varepsilon > 0$. Since the set $\{f(t_n) : n \in \mathbb{N}\}$ is uniformly integrable, we can find $\delta > 0$ such that $\sup_{n\in\mathbb{N}} \int_D \|f(t_n)\|_X \, d\lambda \le \varepsilon/2$ for every $D \in \mathcal{L}$ with $\lambda(D) \le \delta$. By (4), there is $m \in \mathbb{N}$ such that $\lambda(C \triangle C_m) \le \delta$. Since $T_m \circ f$ is weakly continuous at $t \in F$, we have

$$\int_{C_m} f(t_n) \, d\lambda \to \int_{C_m} f(t) \, d\lambda \quad \text{weakly in } X \text{ as } n \to \infty.$$

In particular, the set $K := \{\int_{C_m} f(t_n) \, d\lambda : n \in \mathbb{N}\}$ is relatively weakly compact. Moreover, we have $\lambda(C_m \setminus C) \le \delta$ and $\lambda(C \setminus C_m) \le \delta$, hence

$$\Big\| \int_C f(t_n) \, d\lambda - \int_{C_m} f(t_n) \, d\lambda \Big\|_X \le \int_{C \setminus C_m} \|f(t_n)\|_X \, d\lambda + \int_{C_m \setminus C} \|f(t_n)\|_X \, d\lambda \le \varepsilon$$

for every $n \in \mathbb{N}$. Thus $\{\int_C f(t_n) \, d\lambda : n \in \mathbb{N}\} \subset K + \varepsilon B_X$. An appeal to Grothendieck's test for weak compactness (cf. [10, Lemma 2, p. 227]) ensures that $\{\int_C f(t_n) \, d\lambda : n \in \mathbb{N}\}$ is relatively weakly compact in X, as required. □

The classical Dunford's theorem (cf. [8, Theorem 15, p. 76]) states that the relatively weakly compact subsets of $L^1[0,1]$ are precisely those which are uniformly integrable. However, in general this characterization is not valid when dealing with spaces of Bochner integrable functions. The problem of characterizing the relatively weakly compact subsets of L^1_X for an arbitrary Banach space X has attracted the attention of several authors over the years; see [11], [8, IV.2], [12] and the references therein. *When both X and X^* have the RNP, a set $\mathcal{H} \subset L^1_X$ is relatively weakly compact if and only if it is uniformly integrable and, for each $C \in \mathcal{L}$, the set $\{\int_C h \, d\lambda : h \in \mathcal{H}\}$ is relatively weakly compact in X*, cf. [8, Theorem 1, p. 101]. This characterization is used to prove our main result:

Theorem 2.4. *Let X be a Banach space with the RNP such that X^* is separable. Then L^1_X has the WLP.*

Proof. The space L^1_X is separable, because simple functions are dense in it, X is separable and there is a sequence in $\mathcal{L}$ satisfying condition (4) above. Hence $(L^1_X)^*$ is w^*-separable. On the other hand, the separability of X^* ensures that it has the RNP (cf. [8, Theorem 1, p. 79]) and also that X has the WLP (as we have mentioned in the introduction). The conclusion now follows from Lemmas 2.1 and 2.3, taking into account the comments preceding the theorem. □

Corollary 2.5. *Let X be a separable reflexive Banach space. Then L^1_X has the WLP.*

As a particular case we get the aforementioned result of Chonghu Wang and Kang Wan [7]:

Corollary 2.6. *$L^1[0,1]$ has the WLP.*

Remark 2.7. Theorem 2.4 and its corollaries are still true for the Banach space $L^1_X(\mu)$ of all Bochner integrable functions from a probability space (Ω, Σ, μ) to a Banach space X provided that $L^1(\mu) = L^1_{\mathbb{R}}(\mu)$ *is separable* (or, equivalently, there is a sequence (C_n) in Σ such that $\inf_{n\in\mathbb{N}} \mu(C \triangle C_n) = 0$ for every $C \in \Sigma$).

It is worth pointing out here that if $L^1(\mu)$ is separable and μ *is atomless*, then $L^1(\mu)$ is isometrically isomorphic to $L^1[0,1]$, cf. [13, Theorem 9 and its Corollary, p. 128]; bearing in mind the identification of $L^1_X(\mu)$ with the projective tensor product of $L^1(\mu)$ and X (cf. [8, Example 10, p. 228]), it follows that $L^1_X(\mu)$ is isometrically isomorphic to L^1_X.

We finish the paper with some remarks on the WLP for *non-separable* Banach spaces. Our starting point is the following observation.

Remark 2.8. *Let Y be a Banach space and $f : [0,1] \to Y$ a function which is weakly continuous a.e. Then f is strongly measurable.*

Proof. There is $E \in \mathcal{L}$ with $\lambda(E) = 1$ such that f is weakly continuous at each point of E. In particular, the restriction $f|_E : E \to Y$ is weakly continuous. Since E is separable, the set $f(E) \subset Y$ is weakly separable and so it is norm separable. On the other hand, for each $y^* \in Y^*$, the composition $y^* \circ f$ is continuous a.e. and, in particular, it is measurable. □

As a consequence of the previous remark, a Banach space Y fails the WLP whenever there is a Riemann integrable function $f : [0,1] \to Y$ which is not strongly measurable. It is known that this happens in the following cases (as usual, $\mathfrak{c}$ stands for the cardinality of $\mathbb{R}$):

- $Y = c_0(\mathfrak{c})$ (see [14, proof of Theorem 2.5]).
- Y has density character $\mathfrak{c}$ and admits an equivalent uniformly convex norm (see [15, proof of Lemma 3.2]).

Recall that the canonical norm of $L^p(\mu)$ for $1 < p < \infty$ (where μ is any non-negative measure) is always uniformly convex. In fact, in the particular case of the Hilbert space $\ell^2(\mathfrak{c})$ it is not difficult to give a concrete example of a Riemann integrable function $f : [0,1] \to \ell^2(\mathfrak{c})$ which is not strongly measurable:

Example 2.9. *Define $f : [0,1] \to \ell^2(\mathfrak{c})$ by $f(t) := e_{\varphi(t)}$, where $(e_\alpha)_{\alpha<\mathfrak{c}}$ is the canonical basis of $\ell^2(\mathfrak{c})$ and $\varphi : [0,1] \to \mathfrak{c}$ is any bijection. Then f is Riemann integrable but not strongly measurable.*

Proof. Fix $\varepsilon > 0$. Let $I_1, \dots, I_n$ be a finite collection of non-overlapping closed subintervals covering $[0,1]$ such that $\max_{1\le k\le n} \lambda(I_k) \le \varepsilon$ and take $t_k \in I_k$ for all $1 \le k \le n$. Observe that for each ordinal $\alpha < \mathfrak{c}$ there are at most two different $k, k' \in \{1, \dots, n\}$ for which $\varphi(t_k) = \varphi(t_{k'}) = e_\alpha$. Therefore, we can write

$$\sum_{k=1}^{n} \lambda(I_k) f(t_k) = \sum_{k=1}^{n} \lambda(I_k) e_{\varphi(t_k)} = \sum_{\alpha\in A} c_\alpha e_\alpha,$$

where $A \subset \mathfrak{c}$ is finite, $c_\alpha \in [0, 2\varepsilon]$ for all $\alpha \in A$ and $\sum_{\alpha\in A} c_\alpha = 1$. Then

$$\Big\|\sum_{k=1}^{n} \lambda(I_k) f(t_k)\Big\|_{\ell^2(\mathfrak{c})} = \Big(\sum_{\alpha\in A} c_\alpha^2\Big)^{1/2} \le \Big(2\varepsilon\Big(\sum_{\alpha\in A} c_\alpha\Big)\Big)^{1/2} = \sqrt{2\varepsilon}.$$

As $\varepsilon > 0$ is arbitrary, f is Riemann integrable, with integral 0. On the other hand, f is not strongly measurable, because $f(C)$ is non-separable for every uncountable set $C \subset [0,1]$ (observe that $\|f(t) - f(t')\|_{\ell^2(\mathfrak{c})} = \sqrt{2}$ for every $t \neq t'$). □

Clearly, the WLP is inherited by closed subspaces and it is preserved by isomorphisms. In view of the previous example and the comments preceding it, a Banach space containing a closed subspace isomorphic to $\ell^2(\mathfrak{c})$ fails automatically the WLP. This happens for "large" L^1 spaces and we arrive at the following result.

Proposition 2.10. *Let μ be a probability measure. If the density character of $L^1(\mu)$ is greater than or equal to $\mathfrak{c}$, then $L^1(\mu)$ fails the WLP.*

Proof. It suffices to show that $L^1(\mu)$ contains a closed subspace isomorphic to $\ell^2(\mathfrak{c})$. Given an infinite cardinal κ, we write λ_κ to denote the usual product probability measure on $\{0,1\}^\kappa$. As a consequence of Maharam's theorem, the space $L^1(\mu)$ is isometrically isomorphic to the ℓ^1-sum

$$\Big(\ell^1(\Gamma) \oplus \Big(\bigoplus_{i\in I} L^1(\lambda_{\kappa_i})\Big)_1\Big)_1$$

where Γ and I are countable sets, each κ_i is an infinite cardinal and $\kappa_i \neq \kappa_{i'}$ whenever $i \neq i'$, cf. [13, Theorem 9, p. 127]. The density character of each $L^1(\lambda_{\kappa_i})$ is exactly κ_i, cf. [16, §254]. Since the union of countably many sets of cardinality strictly less than $\mathfrak{c}$ also has cardinality strictly less than $\mathfrak{c}$ (cf. [17, Corollary 5.14]) and the density character of $L^1(\mu)$ is greater than or equal to $\mathfrak{c}$, there is $j \in I$ such that $\kappa_j \geq \mathfrak{c}$. Finally, since $\ell^2(\kappa_{\mathrm{j}})$ is isomorphic to a closed subspace of $L^1(\lambda_{\kappa_j})$ (cf. [13, Theorem 12, p. 128]), it follows that $L^1(\mu)$ contains a closed subspace isomorphic to $\ell^2(\mathfrak{c})$. □

Combining Corollary 2.6 (in its slightly more general version pointed out in Remark 2.7) with Proposition 2.10 we get:

Corollary 2.11 (Under the Continuum Hypothesis)**.** *Let μ be a probability measure. Then $L^1(\mu)$ has the WLP if and only if it is separable.*

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