ON INTEGRATION OF VECTOR FUNCTIONS WITH RESPECT TO VECTOR MEASURES

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Abstract. We study integration of Banach space-valued functions with respect to Banach space-valued measures. The natural extensions to this setting of the Birkhoff and McShane integrals centre our attention. The corresponding generalization of the Birkhoff integral was first considered by Dobrakov under the name S^*-*integral*. Our main result states that S^*-integrability implies McShane integrability in contexts in which the later notion is definable. We also show that a function is measurable and McShane integrable if and only if it is Dobrakov integrable (i.e. Bartle *-integrable).

1. Introduction

The first attempts to establish a theory of integration of vector-valued functions with respect to vector-valued measures go back to the early days of Banach spaces (see [18] for an overview) and, since then, several authors have worked on this topic. Perhaps the most known method is that of Bartle [1], subsequently generalized by Dobrakov (see the survey [22] and the references therein). More recent contributions to this subject are [17, 19, 21, 24].

Most of these theories, including Dobrakov's one, have a common feature: the functions are required to be measurable (in other words, they must be the pointwise limit of a sequence of simple functions). Unfortunately, non measurable vector-valued functions arise naturally and the necessity of integration techniques including such functions becomes evident.

In the particular case of a non-negative measure μ and functions f with values in a Banach space X, the Birkhoff integral [2] and the (generalized) McShane integral [14], which do not require such kind of "strong" measurability, have caught the attention of some authors pretty recently, see [3, 23], [12], [4, 16] and the references given there. Roughly speaking, both integrals

2000 *Mathematics Subject Classification.* Primary: 28B05, 46G10.

Key words and phrases. Bartle *-integral; Dobrakov integral; McShane integral; Birkhoff integral; S^*-integral.

Partially supported by the research grant BFM2002-01719 of MCyT (Spain) and a FPU grant of MEC (Spain).

are defined as limits of sums of the form $\sum_i \mu(A_i)f(t_i)$, where (A_i) is a countable family of pairwise disjoint measurable sets and the t_i's are points of the domain which are related to the A_i's in some way. It is natural to try to extend these integrals to the more general setting of vector-valued measures and our purpose here is to study such generalizations, which are obtained as follows: we will consider a vector measure μ with values in the Banach space $\mathcal{L}(X,Y)$ of all bounded operators from X to another Banach space Y and the sums $\sum_i \mu(A_i)f(t_i)$ will be constructed by replacing the product by scalars $\mathbb{R} \times X \longrightarrow X$ with the natural bilinear map $\mathcal{L}(X,Y) \times X \longrightarrow Y$.

The S^*-integral of Dobrakov [8], derived from Kolmogorov's approach to integration theory [20, 25], is the natural extension of the Birkhoff integral to the case of vector-valued functions and vector-valued measures. Under the assumption that the semivariation of the vector measure is continuous (see below for the definitions), it is known that Dobrakov integrability (i.e. Bartle *-integrability) implies S^*-integrability and that both notions coincide for measurable functions. For the convenience of the reader we have collected the definitions and basic facts (some of them already known) about the S^*-integral and the Dobrakov integral in Section 2.

In Section 3 we develop the theory of the McShane integral with respect to a vector measure. Naturally, throughout this section we work with vector-valued functions defined on topological spaces and we require that the semivariation of the vector measure has a quasi-Radon "control measure". Sub-Section 3.1 contains some preliminary work which paves the way to deal with Sub-Section 3.2, that is devoted to compare in this setting the McShane integral with the Dobrakov and S^* integrals. The main result of this paper, Theorem 3.7, states that *every S^*-integrable function is McShane integrable (and the respective integrals coincide)*. This generalizes partially a result of Fremlin, [12, Proposition 4], regarding the McShane integrability of a Birkhoff integrable function defined on a σ-finite outer regular quasi-Radon measure space. As a consequence of Theorem 3.7 we deduce that *a function is Dobrakov integrable if and only if it is measurable and McShane integrable* (Theorem 3.8).

Notation and terminology. Throughout this paper X and Y are real Banach spaces, (Ω, Σ) is a measurable space and $\mu : \Sigma \longrightarrow \mathcal{L}(X,Y)$ is a countably additive vector measure. $\mathcal{L}(X,Y)$ is the Banach space of all bounded operators from X to Y.

The notion of semivariation defined below differs from the usual one of scalar semivariation of a vector measure ν, [5, p. 2], which will be denoted by $\|\nu\|$. The *semivariation* of μ, [6, p. 513], is the function $\hat{\mu} : \Sigma \longrightarrow [0, \infty]$ defined by $\hat{\mu}(A) = \sup \| \sum_{i=1}^n \mu(A_i)(x_i)\|$, where the supremum is taken over all finite partitions $(A_i)_{i=1}^n$ of A in Σ and all finite collections $(x_i)_{i=1}^n$ in B_X (the closed unit ball of X). $\hat{\mu}$ is always monotone and countably subadditive.

Throughout we will assume that $\hat{\mu}$ is continuous, [7, p. 17]: if $(E_n)_{n=1}^\infty$ is a decreasing sequence in Σ such that $\cap_{n=1}^\infty E_n = \emptyset$, then $\lim_n \hat{\mu}(E_n) = 0$. We emphasize that $\hat{\mu}$ is continuous if and only if there exists a non-negative finite measure λ on Σ such that $\lim_{\lambda(A)\to 0} \hat{\mu}(A) = 0$ and $\lim_{\hat{\mu}(A)\to 0} \lambda(A) = 0$, see [7, Lemma 2]. *Throughout the paper λ will always be such a measure.* From the continuity of $\hat{\mu}$ it follows that $\hat{\mu}(\Omega) < \infty$ (see the remarks after Lemma 2 in [7]). Observe that for each $E \in \Sigma$ the restriction of μ to the σ-algebra $\Sigma_E = \{B \in \Sigma : B \subset E\}$, denoted by μ_E, is countably additive and has continuous semivariation. Moreover, the restriction of λ to Σ_E, denoted by λ_E, fulfills $\lim_{\lambda_E(A)\to 0} \widehat{\mu_E}(A) = 0$ and $\lim_{\widehat{\mu_E}(A)\to 0} \lambda_E(A) = 0$.

There are several cases in which $\hat{\mu}$ is continuous, see for instance [1, 17, 22]. Let us mention two of them:

C1: Integration of X-valued functions with respect to a non-negative finite measure ν on Σ. In such a case we take $Y := X$ and $\mu(E)(x) := \nu(E)x$ for every $E \in \Sigma$ and every $x \in X$. It is obvious that $\hat{\mu} = \nu$ is continuous.

C2: Integration of real-valued functions with respect to a countably additive vector measure $\nu : \Sigma \longrightarrow Y$. In such a case we take $X := \mathbb{R}$ and $\mu(E)(x) := x\nu(E)$ for every $E \in \Sigma$ and every $x \in \mathbb{R}$. Observe that $\hat{\mu} = \|\nu\|$ is continuous, by [5, Corollary 6, p. 14]. The standard integral in this setting is that of Bartle, Dunford and Schwartz, see [10, Section IV.10].

2. Dobrakov and S^* integrals

As said in the introduction, this section contains a brief summary of the definitions and some basic facts on the Dobrakov and S^* integrals.

For a given simple function $f = \sum_{i=1}^n x_i \chi_{A_i}$, $x_i \in X$, $A_i \in \Sigma$, we write $\int_\Omega f \, d\mu := \sum_{i=1}^n \mu(A_i)(x_i)$. A function $f : \Omega \longrightarrow X$ is called *measurable* if there is a sequence of simple functions converging pointwise to f. A function $f : \Omega \longrightarrow X$ is *Dobrakov integrable* with respect to μ, [6, Definition 2 and Theorem 7], if it is measurable and there is a sequence of simple X-valued functions (f_n) converging to f $\hat{\mu}$-almost everywhere such that for every $E \in \Sigma$ there exists $\lim_n \int_E f_n \, d\mu_E$, for the norm topology of Y; the Dobrakov integral of f with respect to μ is defined by $(D) \int_\Omega f \, d\mu = \lim_n \int_\Omega f_n \, d\mu$.

Within the framework of this paper (that is, under the assumption that the vector measure has continuous semivariation) the differences between the *Bartle bilinear *-integral* and the Dobrakov integral are simply language matters. Indeed, let Z be another real Banach space, $\nu : \Sigma \longrightarrow Z$ a countably additive vector measure and $\phi : X \times Z \longrightarrow Y$ a continuous bilinear map. Then ν has the **-property* with respect to ϕ, [1, Definition 2], if and only if the set function $\mu : \Sigma \longrightarrow \mathcal{L}(X, Y)$ given by $\mu(E)(x) = \phi(x, \nu(E))$ has continuous semivariation. In this case, Theorem 9 in [1] says that a function $f : \Omega \longrightarrow X$

is Bartle *-integrable with respect to ν and ϕ if and only if f is equal $\hat{\mu}$-almost everywhere to a function which is Dobrakov integrable with respect to μ (the respective integrals coincide).

In the particular case C1 (resp. C2) mentioned above, a function is integrable in the sense of Dobrakov if and only if it is measurable and integrable in the sense of Pettis (resp. Bartle, Dunford and Schwartz), see [1, 6].

Given a function $f : \Omega \longrightarrow X$, a countable family $\Gamma = (A_n)$ of pairwise disjoint elements of Σ and a choice $T = (t_n)$ in Γ (i.e., $t_n \in A_n$ for every n), the symbol

$$S(f, \Gamma, T) := \sum_n \mu(A_n)(f(t_n))$$

denotes a formal series. As usual, we say that another countable family Γ' of pairwise disjoint elements of Σ is finer than Γ when each element of Γ' is contained in some element of Γ.

Definition 2.1. *A function $f : \Omega \longrightarrow X$ is S^*-integrable with respect to μ, with S^*-integral $y \in Y$,* [8, Definition 1]*, if for every $\varepsilon > 0$ there is a countable partition Γ_0 of Ω in Σ such that for every countable partition Γ of Ω in Σ finer than Γ_0 and every choice T in Γ*

(i) *the series $S(f, \Gamma, T)$ is unconditionally convergent in Y;*
(ii) $\|S(f, \Gamma, T) - y\| < \varepsilon$.

The vector $y \in Y$ is necessarily unique and will be denoted by $(S^*) \int_\Omega f \, d\mu$.

The set of all functions from Ω to X which are S^*-integrable with respect to μ will be denoted by $S^*(\mu)$. It is easy to check that $S^*(\mu)$ is a linear subspace of X^Ω and that the map from $S^*(\mu)$ to Y given by $f \mapsto (S^*) \int_\Omega f \, d\mu$ is linear.

The basic properties of the S^*-integral and the precise relationship with the Dobrakov integral were studied in [8] (see [9] for a variant of the S^*-integral, called S-integral, which only deals with finite partitions). Theorem 1 in [8] states that *a function $f : \Omega \longrightarrow X$ is Dobrakov integrable with respect to μ if and only if f is measurable and S^*-integrable with respect to μ (in this case, the respective integrals coincide).*

It is worth it to point out that S^*-integrability generalizes Birkhoff integrability. More precisely, Proposition 2.6 in [3] can be read as: if ν is a non-negative finite measure on Σ, then a function $f : \Omega \longrightarrow X$ is Birkhoff integrable with respect to ν if and only if f is S^*-integrable with respect to the set function $\mu : \Sigma \longrightarrow \mathcal{L}(X, X)$ given by $\mu(E)(x) = \nu(E)x$ (in this case, the respective integrals coincide).

We end up the section with two lemmas that will be needed in the proof of Theorem 3.7. We first emphasize that:

(i) if $f : \Omega \longrightarrow X$ is S^*-integrable with respect to μ, then for each $A \in \Sigma$ the restriction $f|_A$ is S^*-integrable with respect to μ_A;

(ii) the set function $\nu_f : \Sigma \longrightarrow Y$ given by

$$\nu_f(A) := (S^*) \int_A f \, d\mu_A$$

is a countably additive vector measure,

see [8, Lemma 1 (1)].

Lemma 2.1. *Let $f \in S^*(\mu)$. Then for each $\varepsilon > 0$ there is a countable partition Γ_0 of Ω in Σ such that for every countable family $\Gamma = (A_n)$ of pairwise disjoint elements of Σ finer than Γ_0 and every choice T in Γ, the series $S(f, \Gamma, T)$ is unconditionally convergent and*

$$\|S(f, \Gamma, T) - \nu_f(\cup_n A_n)\| \leq \varepsilon. \tag{1}$$

Proof. Let Γ_0 be a countable partition of Ω in Σ such that for every countable partition $\tilde{\Gamma}$ of Ω in Σ finer than Γ_0 and every choice $\tilde{T}$ in $\tilde{\Gamma}$

$$\|S(f, \tilde{\Gamma}, \tilde{T}) - \nu_f(\Omega)\| < \varepsilon,$$

the series involved being unconditionally convergent.

Fix an arbitrary countable family $\Gamma = (A_n)$ of pairwise disjoint elements of Σ finer than Γ_0, and take any choice $T = (t_n)$ in Γ. Write $A := \cup_n A_n$ and set $\Gamma' := \{E \setminus A : \ E \in \Gamma_0, \ E \not\subset A\}$. Fix a choice T' in Γ'.

Since $\Gamma \cup \Gamma'$ is a countable partition of Ω in Σ finer than Γ_0 and $T \cup T'$ is a choice in $\Gamma \cup \Gamma'$, the series $S(f, \Gamma \cup \Gamma', T \cup T')$ is unconditionally convergent. Therefore, the subseries $S(f, \Gamma, T)$ is unconditionally convergent.

Let us turn to the proof of (1). There is a sequence $\{\Gamma'_k\}_{k \in \mathbb{N}}$ of countable partitions of $\Omega \setminus A$ in Σ finer than Γ', and a sequence of choices $\{T'_k\}_{k \in \mathbb{N}}$, such that

$$\lim_k \ S(f, \Gamma'_k, T'_k) = \nu_f(\Omega \setminus A). \tag{2}$$

For each $k \in \mathbb{N}$ we define $\Gamma_k := \Gamma \cup \Gamma'_k$, which is a countable partition of Ω in Σ finer than Γ_0, and $T_k := T \cup T'_k$. The choice of Γ_0 implies that $S(f, \Gamma_k, T_k)$ is unconditionally convergent and

$$\|S(f, \Gamma_k, T_k) - \nu_f(\Omega)\| < \varepsilon$$

for every $k \in \mathbb{N}$, which yields

$$\begin{aligned} \|S(f, \Gamma, T) - \nu_f(A)\| &\leq \|S(f, \Gamma_k, T_k) - \nu_f(\Omega)\| + \|S(f, \Gamma'_k, T'_k) - \nu_f(\Omega \setminus A)\| \\ &< \varepsilon + \|S(f, \Gamma'_k, T'_k) - \nu_f(\Omega \setminus A)\| \end{aligned}$$

for every $k \in \mathbb{N}$. Now (1) follows from (2), and the proof is complete. □

Lemma 2.2. *Suppose that Ω is an atom of λ. If $f \in S^*(\mu)$, then there is $E \in \Sigma$ such that $\hat{\mu}(\Omega \setminus E) = 0$ and $\nu_f(\Omega) = \mu(\Omega)(f(\omega))$ for every $\omega \in E$.*

Proof. Since $f \in S^*(\mu)$, for each $m \in \mathbb{N}$ there is a countable partition Γ_m of Ω in Σ such that
$$\|S(f, \Gamma_m, T) - \nu_f(\Omega)\| \leq \frac{1}{m}$$
for every choice T in Γ_m, the series involved being unconditionally convergent. But Ω is an atom of λ, so there is some $E_m \in \Gamma_m$ such that $\lambda(\Omega \setminus E_m) = 0$. The previous inequality can now be read as
$$\sup_{\omega \in E_m} \|\mu(\Omega)(f(\omega)) - \nu_f(\Omega)\| \leq \frac{1}{m}, \quad m \in \mathbb{N},$$
and therefore the set $E := \cap_{m=1}^{\infty} E_m$ fulfills the required properties. □

We should also mention that the Dobrakov and S^*-integrals can be defined in the more general setting of vector measures on δ-rings that are countably additive for the strong operator topology on $\mathcal{L}(X, Y)$, see [6, 8]. The results of [8] quoted above (including Theorem 1) are valid in this context under further assumptions on the functions involved and the semivariation that are fairly close to assume the continuity of the latter.

3. The McShane integral with respect to vector measures

This section is devoted to analyze the McShane integral with respect to a vector measure and its relationship with the Dobrakov and S^* integrals. Our approach here to the McShane integral differs from that of [14] and is inspired by the equivalent formulation given in [11, 13] and [15, Chapter 48]. For all unexplained terminology we refer the reader to [15].

As in the case of non-negative measures, further conditions are needed to set up this method of integration. Throughout this section τ is a topology on Ω with $\tau \subset \Sigma$ and we suppose that $(\Omega, \tau, \Sigma, \lambda)$ is a finite quasi-Radon measure space, in the sense of [15, 411H] (for instance, a finite Radon measure space, see [15, 416A]). Equivalently, $\hat{\mu}$ satisfies the following properties:

(α) for every $E \in \Sigma$ and every $\varepsilon > 0$ there exists a τ-closed set $C \subset E$ such that $\hat{\mu}(E \setminus C) < \varepsilon$;

(β) $\inf_{G \in \mathcal{G}} \hat{\mu}(\cup \mathcal{G} \setminus G) = 0$ for every non-empty upwards directed family $\mathcal{G}$ of τ-open sets;

(γ) if $A \subset E \in \Sigma$ and $\hat{\mu}(E) = 0$, then $A \in \Sigma$.

There are natural examples of topological spaces and vector measures fulfilling the properties above. We next mention some of them.

Example 3.1. Let (Ω, τ) be a compact Hausdorff topological space. Following [5, p. 157], we say that a countably additive vector measure ν defined on the Borel σ-algebra Σ_1 of Ω with values in Y is *regular* if for every $E \in \Sigma_1$ and every $\varepsilon > 0$ there is a compact set $K \subset E$ such that $\|\nu\|(E \setminus K) < \varepsilon$. These measures arise in the representation of weakly compact operators $C(\Omega) \longrightarrow Y$ via the Bartle-Dunford-Schwartz integral, see [5, Chapter VI].

Given such a ν, take $X := \mathbb{R}$ and define $\mu_1 : \Sigma_1 \longrightarrow \mathcal{L}(X,Y)$ by $\mu_1(E)(x) := x\nu(E)$. Then μ_1 is a countably additive vector measure and $\widehat{\mu_1} = \|\nu\|$ is continuous (we are in the conditions of the particular case C2). Fix a non-negative finite measure λ_1 on Σ_1 such that $\lim_{\lambda_1(A)\to 0} \widehat{\mu_1}(A) = 0$ and $\lim_{\widehat{\mu_1}(A)\to 0} \lambda_1(A) = 0$. Write (Ω,Σ,λ) for the completion of $(\Omega,\Sigma_1,\lambda_1)$. It is easy to see that μ_1 can be extended (in a unique way) to a countably additive vector measure $\mu : \Sigma \longrightarrow \mathcal{L}(X,Y)$ such that $\lim_{\lambda(A)\to 0} \hat{\mu}(A) = 0$ and $\lim_{\hat{\mu}(A)\to 0} \lambda(A) = 0$. Moreover, we have $\hat{\mu}|_{\Sigma_1} = \widehat{\mu_1}$. It follows from the regularity of ν that $(\Omega,\tau,\Sigma,\lambda)$ is a finite Radon measure space and therefore $\hat{\mu}$ satisfies (α), (β) and (γ) (so $\widehat{\mu_1}$ fulfills (α) and (β)).

Example 3.2. Let (Ω,τ) be an analytic Hausdorff topological space (e.g. a Polish space). It is well known that the completion of any non-negative finite measure defined on the Borel σ-algebra Σ_1 of Ω is a Radon measure, see [15, 433C]. Consequently, if $\mu_1 : \Sigma_1 \longrightarrow \mathcal{L}(X,Y)$ is a countably additive vector measure with continuous semivariation, then $\widehat{\mu_1}$ satisfies properties (α) and (β), and μ_1 can be extended to a countably additive vector measure with continuous semivariation that also fulfills (γ), as in the previous example.

We will use without explicit mention the fact that properties (α), (β) and (γ) are hereditary: for each $A \in \Sigma$ the set function $\widehat{\mu_A}$ fulfills conditions (α), (β) and (γ) with respect to Σ_A and the induced topology $\tau_A = \{B \cap A : \ B \in \tau\}$, since $(A,\tau_A,\Sigma_A,\lambda_A)$ is a quasi-Radon measure space, see [15, 415B].

To introduce the McShane integral we need some terminology. A *gauge* on (Ω,τ) is a function $\delta : \Omega \longrightarrow \tau$ such that $\omega \in \delta(\omega)$ for every $\omega \in \Omega$. A *partial McShane partition of* Ω is a finite collection $\mathcal{P} = \{(E_i,s_i) : \ 1 \le i \le p\}$ where $(E_i)_{i=1}^p$ are pairwise disjoint elements of Σ and $s_i \in \Omega$ for every $1 \le i \le p$. We write $W_{\mathcal{P}} := \cup_{i=1}^p E_i$. $\mathcal{P}$ is said to be *subordinate to* δ if $E_i \subset \delta(s_i)$ for every $1 \le i \le p$.

For every gauge δ on (Ω,τ) and every $\eta > 0$ the set $\Pi_{\delta,\eta}$, made up of all partial McShane partitions $\mathcal{P}$ of Ω subordinate to δ such that $\hat{\mu}(\Omega \setminus W_{\mathcal{P}}) \le \eta$, is not empty (the arguments in [14, 1B(d)] can be applied since $(\Omega,\tau,\Sigma,\lambda)$ is a finite quasi-Radon measure space). It is clear that the family

$$\mathcal{B} = \{\Pi_{\delta,\eta} : \ \delta \text{ is a gauge on } (\Omega,\tau), \ \eta > 0\}$$

is a filter base on the set Π of all partial McShane partitions of Ω. Let us denote by $\mathcal{F}$ the filter on Π generated by $\mathcal{B}$.

From now on, given a function $f : \Omega \longrightarrow X$ and a partial McShane partition $\mathcal{P} = \{(E_i,s_i) : \ 1 \le i \le p\}$ of Ω, we write

$$f(\mathcal{P}) := \sum_{i=1}^{p} \mu(E_i)(f(s_i)).$$

Definition 3.1. *Let $f : \Omega \longrightarrow X$ be a function. We say that f is McShane integrable with respect to μ if there exists $\lim_{\mathcal{P}\to\mathcal{F}} f(\mathcal{P}) = y$ for the norm topology of Y, i.e., for every $\varepsilon > 0$ the set $\{\mathcal{P} \in \Pi : \|f(\mathcal{P}) - y\| < \varepsilon\}$ belongs to $\mathcal{F}$. The vector $y \in Y$ is called the McShane integral of f and will be denoted by $(M)\int_\Omega f\, d\mu$.*

The set of all functions from Ω to X which are McShane integrable with respect to μ, denoted by $M(\mu)$, is a linear subspace of X^Ω and the map from $M(\mu)$ to Y given by $f \mapsto (M)\int_\Omega f\, d\mu$ is linear.

The McShane integral of Banach-valued functions defined on finite quasi-Radon measure spaces, in the sense of [14], turns out to be a particular case of the McShane integral with respect to a vector measure, as we have defined it. More precisely, if $(\Omega, \tau, \Sigma, \nu)$ is a finite quasi-Radon measure space, then a function $f : \Omega \longrightarrow X$ is McShane integrable according to [14, 1A] if and only if f is McShane integrable with respect to the set function $\mu : \Sigma \longrightarrow \mathcal{L}(X, X)$ given by $\mu(E)(x) = \nu(E)x$ (in this case, the respective integrals coincide), see [13, Proposition 3].

3.1. **Preliminary results.** In this sub-section we establish the basics of the theory of the McShane integral with respect to vector measures.

Given a gauge δ on (Ω, τ) and $E \in \Sigma$, we will denote by δ_E the gauge on (E, τ_E) defined by $\delta_E(\omega) := \delta(\omega) \cap E$ for every $\omega \in E$.

Lemma 3.1. *Let $f \in M(\mu)$. Then for each $A \in \Sigma$ the restriction $f|_A$ is McShane integrable with respect to μ_A. Its McShane integral will be denoted by $\zeta_f(A)$.*

Proof. By the completeness of X, it suffices to show that for every $\varepsilon > 0$ there exist a gauge δ_A on (A, τ_A) and $\eta > 0$ such that $\|f(\mathcal{P}_1) - f(\mathcal{P}_2)\| < \varepsilon$ whenever $\mathcal{P}_1$ and $\mathcal{P}_2$ are partial McShane partitions of A subordinate to δ_A such that $\hat{\mu}(A \setminus W_{\mathcal{P}_i}) \le \eta$ for $i = 1, 2$.

Since $f \in M(\mu)$, there exist a gauge δ on (Ω, τ) and $\eta > 0$ such that

$$\|f(\mathcal{P}) - f(\mathcal{P}')\| < \varepsilon \tag{3}$$

for every $\mathcal{P}, \mathcal{P}' \in \Pi_{\delta,2\eta}$. Fix a partial McShane partition $\mathcal{P}_0$ of $\Omega \setminus A$ subordinate to $\delta_{\Omega\setminus A}$ such that $\hat{\mu}(\Omega \setminus (A \cup W_{\mathcal{P}_0})) \le \eta$.

Now let $\mathcal{P}_1$ and $\mathcal{P}_2$ be partial McShane partitions of A subordinate to δ_A such that $\hat{\mu}(A \setminus W_{\mathcal{P}_i}) \le \eta$ for $i = 1, 2$. Then $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_0$ and $\mathcal{P}' = \mathcal{P}_2 \cup \mathcal{P}_0$ belong to $\Pi_{\delta,2\eta}$, and (3) applies to get

$$\|f(\mathcal{P}_1) - f(\mathcal{P}_2)\| = \|f(\mathcal{P}) - f(\mathcal{P}')\| < \varepsilon,$$

as required. □

Lemma 3.2. *Let $f \in M(\mu)$. Then*

(i) $\lim_{\hat{\mu}(E)\to 0} \zeta_f(E) = 0$;

(ii) *$\zeta_f : \Sigma \longrightarrow Y$ is a countably additive vector measure.*

Proof. It is easy to check that ζ_f is a finitely additive vector measure. In view of this, (ii) follows directly from (i) and the continuity of $\hat{\mu}$.

In order to prove (i) fix $\varepsilon > 0$. Since $f \in M(\mu)$, there are $\eta > 0$ and a gauge δ on (Ω, τ) such that $\|f(\mathcal{P}) - \zeta_f(\Omega)\| < \varepsilon$ whenever $\mathcal{P} \in \Pi_{\delta,\eta}$. Fix $E \in \Sigma$ such that $\hat{\mu}(E) \leq \frac{\eta}{2}$. We claim that $\|\zeta_f(E)\| < 3\varepsilon$.

Indeed, take a partial McShane partition $\mathcal{P}_1$ of $\Omega \setminus E$ subordinate to $\delta_{\Omega \setminus E}$ such that $\hat{\mu}(\Omega \setminus (E \cup W_{\mathcal{P}_1})) \leq \frac{\eta}{2}$, and fix another partial McShane partition $\mathcal{P}_2$ of E subordinate to δ_E such that $\|f(\mathcal{P}_2) - \zeta_f(E)\| < \varepsilon$. Since $\mathcal{P}_1$ and $\mathcal{P}_1 \cup \mathcal{P}_2$ are in $\Pi_{\delta,\eta}$, we have

$$\begin{aligned}\|\zeta_f(E)\| \leq \|\zeta_f(E) - f(\mathcal{P}_2)\| + \|f(\mathcal{P}_1 \cup \mathcal{P}_2) - \zeta_f(\Omega)\| \\ + \|f(\mathcal{P}_1) - \zeta_f(\Omega)\| < 3\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, (i) holds and the proof is complete. □

The following version of the *Henstock-Saks lemma* will be needed in the proof of Proposition 3.4.

Lemma 3.3. *Let $f \in M(\mu)$. Then for each $\varepsilon > 0$ there is a gauge δ on (Ω, τ) such that*

$$\Big\|\sum_{i=1}^{p} \mu(E_i)(f(s_i)) - \zeta_f(\cup_{i=1}^{p} E_i)\Big\| < \varepsilon$$

for every partial McShane partition $\{(E_i, s_i) : 1 \leq i \leq p\}$ of Ω subordinate to δ.

Proof. Fix $\eta > 0$ and a gauge δ on (Ω, τ) such that

$$\|f(\mathcal{P}') - \zeta_f(\Omega)\| < \frac{\varepsilon}{2} \quad \text{for every } \mathcal{P}' \in \Pi_{\delta,\eta}.$$

Take an arbitrary partial McShane partition $\mathcal{P} = \{(E_i, s_i) : 1 \leq i \leq p\}$ of Ω subordinate to δ. Since $f|_{\Omega \setminus W_{\mathcal{P}}} \in M(\mu_{\Omega \setminus W_{\mathcal{P}}})$, there is a partial McShane partition $\mathcal{P}_0$ of $\Omega \setminus W_{\mathcal{P}}$ subordinate to $\delta_{\Omega \setminus W_{\mathcal{P}}}$ such that $\hat{\mu}(\Omega \setminus (W_{\mathcal{P}} \cup W_{\mathcal{P}_0})) \leq \eta$ and

$$\|f(\mathcal{P}_0) - \zeta_f(\Omega \setminus W_{\mathcal{P}})\| < \frac{\varepsilon}{2}.$$

Since $\mathcal{P} \cup \mathcal{P}_0 \in \Pi_{\delta,\eta}$ and ζ_f is finitely additive, we have

$$\begin{aligned}\Big\|\sum_{i=1}^{p} \mu(E_i)(f(s_i)) - \zeta_f(\cup_{i=1}^{p} E_i)\Big\| \leq \|f(\mathcal{P} \cup \mathcal{P}_0) - \zeta_f(\Omega)\| \\ + \|f(\mathcal{P}_0) - \zeta_f(\Omega \setminus W_{\mathcal{P}})\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

and the proof is complete. □

For a given function $f : \Omega \longrightarrow X$ and $A \in \Sigma$, we denote by $f\chi_A$ the function from Ω to X defined by $f\chi_A(\omega) = f(\omega)$ if $\omega \in A$, $f\chi_A(\omega) = 0$ if $\omega \in \Omega \setminus A$.

Proposition 3.4. *Let $f : \Omega \longrightarrow X$ be a function and $A \in \Sigma$ such that $f|_A$ is McShane integrable with respect to μ_A. Then $f\chi_A \in M(\mu)$ and*

$$(M)\int_\Omega f\chi_A \, d\mu = (M)\int_A f \, d\mu_A.$$

Proof. Fix $\varepsilon > 0$. By property (α) of $\hat{\mu}$, for every $m \in \mathbb{N}$ we can choose an open set $G_m \supset A$ such that

$$\hat{\mu}(G_m \setminus A) < \frac{\varepsilon}{2^m \cdot m}. \tag{4}$$

Since $f|_A$ is McShane integrable with respect to μ_A, Lemma 3.3 applies to get a gauge δ' on (A, τ_A) such that

$$\|f(\mathcal{P}') - \zeta_{f|_A}(W_{\mathcal{P}'})\| \leq \varepsilon \tag{5}$$

for every partial McShane partition $\mathcal{P}'$ of A subordinate to δ'. On the other hand, by Lemma 3.2 (i) there is $\eta > 0$ such that

$$\|\zeta_{f|_A}(E)\| \leq \varepsilon \tag{6}$$

whenever $\hat{\mu}(E) \leq \eta$, $E \in \Sigma_A$. Fix a closed set $K \subset A$ such that $\hat{\mu}(A \setminus K) \leq \frac{\eta}{2}$ (use again property (α)).

Let δ be a gauge on (Ω, τ) such that

- $\delta(\omega) \cap A = \delta'(\omega)$ and $\delta(\omega) \subset G_m$ if $\omega \in A$ and $m - 1 \leq \|f(\omega)\| < m$;
- $\delta(\omega) = \Omega \setminus K$ if $\omega \in \Omega \setminus A$.

We claim that

$$\Big\|\sum_{i=1}^{p} \mu(E_i)(f\chi_A(s_i)) - (M)\int_A f \, d\mu_A\Big\| \leq 3\varepsilon \tag{7}$$

for every $\mathcal{P} = \{(E_i, s_i) : 1 \leq i \leq p\} \in \Pi_{\delta, \frac{\eta}{2}}$. Indeed, since the collection $\{(E_i \cap A, s_i) : s_i \in A\}$ is a partial McShane partition of A subordinate to δ', it follows from (5) that

$$\Big\|\sum_{s_i \in A} \mu(E_i \cap A)(f(s_i)) - \zeta_{f|_A}\big(\cup_{s_i \in A}(E_i \cap A)\big)\Big\| \leq \varepsilon. \tag{8}$$

Note that $A \cap (\cup_{s_i \notin A} E_i) \subset A \setminus K$, by the choice of δ. Therefore

$$\begin{aligned}\hat{\mu}\big(A \setminus \cup_{s_i \in A} E_i\big) &\leq \hat{\mu}\big(A \cap (\cup_{s_i \notin A} E_i)\big) + \hat{\mu}(\Omega \setminus W_{\mathcal{P}}) \\ &\leq \hat{\mu}(A \setminus K) + \hat{\mu}(\Omega \setminus W_{\mathcal{P}}) \leq \frac{\eta}{2} + \frac{\eta}{2} = \eta,\end{aligned}$$

and (6) yields $\|\zeta_{f|_A}(A \setminus \cup_{s_i \in A} E_i)\| \leq \varepsilon$. This inequality and (8) imply

$$\Big\|\sum_{s_i \in A} \mu(E_i \cap A)(f(s_i)) - \zeta_{f|_A}(A)\Big\| \leq 2\varepsilon. \tag{9}$$

On the other hand, set $I_m = \{1 \le i \le p : \; s_i \in A, \; m-1 \le \|f(s_i)\| < m\}$ for every $m \in \mathbb{N}$. Since $\cup_{i \in I_m} E_i \subset G_m$ for every $m \in \mathbb{N}$ (by the choice of δ), (4) yields

$$\Big\| \sum_{s_i \in A} \mu(E_i \setminus A)(f(s_i)) \Big\| \le \sum_{m=1}^{\infty} \Big\| \sum_{i \in I_m} \mu(E_i \setminus A)(f(s_i)) \Big\|$$
$$\le \sum_{m=1}^{\infty} \hat{\mu}(G_m \setminus A) \cdot m \le \sum_{m=1}^{\infty} \frac{\varepsilon}{2^m} = \varepsilon,$$

which combined with (9) implies

$$\Big\| \sum_{i=1}^{p} \mu(E_i)(f\chi_A(s_i)) - \zeta_{f|_A}(A) \Big\| = \Big\| \sum_{s_i \in A} \mu(E_i)(f(s_i)) - \zeta_{f|_A}(A) \Big\| \le 3\varepsilon.$$

Hence (7) holds. Since $\varepsilon > 0$ is arbitrary, $f\chi_A$ is McShane integrable with respect to μ and $(M)\int_\Omega f\chi_A \, d\mu = (M)\int_A f \, d\mu_A$. □

Corollary 3.5. *Let $f : \Omega \longrightarrow X$ be a simple function, $f = \sum_{i=1}^n x_i \chi_{A_i}$. Then f is McShane integrable with respect to μ and $(M)\int_\Omega f \, d\mu = \sum_{i=1}^n \mu(A_i)(x_i)$.*

Proof. It suffices to consider the case $f = x\chi_A$, which follows from Proposition 3.4 and the fact that constant functions are McShane integrable. □

Corollary 3.6. *Let $f, g : \Omega \longrightarrow X$ be two functions which are equal $\hat{\mu}$-almost everywhere. Then $f \in M(\mu)$ if and only if $g \in M(\mu)$. In this case,*

$$(M)\int_\Omega f \, d\mu = (M)\int_\Omega g \, d\mu.$$

Proof. It suffices to check that $h := f - g$ is McShane integrable with respect to μ and $(M)\int_\Omega h \, d\mu = 0$. Fix $A \in \Sigma$ such that $h(\omega) = 0$ for every $\omega \in A$ and $\hat{\mu}(\Omega \setminus A) = 0$. Since $h|_{\Omega \setminus A}$ is McShane integrable with respect to $\mu_{\Omega \setminus A}$, with integral 0, an appeal to Proposition 3.4 ensures us that $h\chi_{\Omega \setminus A} = h \in M(\mu)$ and $(M)\int_\Omega h \, d\mu = 0$, as required. □

3.2. **Relationship with the Dobrakov and S^* integrals.** In this subsection we discuss the relationship between the different integrals considered in this paper.

Theorem 3.7. *If $f : \Omega \longrightarrow X$ is S^*-integrable with respect to μ, then f is McShane integrable with respect to μ and*

$$(S^*)\int_\Omega f \, d\mu = (M)\int_\Omega f \, d\mu.$$

The proof will be divided into four cases.

Case 1. Suppose that Ω is an atom of λ.

By Lemma 2.2 there is $E \in \Sigma$ such that $\hat{\mu}(\Omega \setminus E) = 0$ and

$$\nu_f(\Omega) = \mu(\Omega)(f(\omega)) \quad \text{for every } \omega \in E.$$

Therefore $f(\mathcal{P}) = \nu_f(\Omega)$ for every partial McShane partition $\mathcal{P}$ of E such that $\hat{\mu}(E \setminus W_{\mathcal{P}}) < \hat{\mu}(E)$ (keep in mind that E is an atom of λ). It follows that $f|_E$ is McShane integrable with respect to μ_E, with McShane integral $\nu_f(\Omega)$. An appeal to Proposition 3.4 ensures us that $f\chi_E \in M(\mu)$ and $(M)\int_\Omega f\chi_E \, d\mu = \nu_f(\Omega)$. Since f and $f\chi_E$ are equal $\hat{\mu}$-almost everywhere, it follows that f is McShane integrable with respect to μ and $(M)\int_\Omega f \, d\mu = \nu_f(\Omega)$ (by Corollary 3.6). □

Case 2. Suppose that there is a countable partition (A_n) of Ω made up of atoms of λ.

Given $E \in \Sigma$ and $n \in \mathbb{N}$, the restriction $f|_{A_n \cap E}$ is S^*-integrable with respect to $\mu_{A_n \cap E}$. It is obvious that $f|_{A_n \cap E} \in M(\mu_{A_n \cap E})$, with McShane integral $\nu_f(A_n \cap E) = 0$, when $\hat{\mu}(A_n \cap E) = 0$. If, on the contrary, $A_n \cap E$ is an atom of λ, *Case 1* applies to deduce that $f|_{A_n \cap E} \in M(\mu_{A_n \cap E})$, with McShane integral $\nu_f(A_n \cap E)$. We conclude from Proposition 3.4 that $f\chi_{A_n \cap E} \in M(\mu)$ and

$$(M)\int_\Omega f\chi_{A_n \cap E} \, d\mu = (M)\int_{A_n \cap E} f \, d\mu_{A_n \cap E} = \nu_f(A_n \cap E).$$

In particular, $f\chi_{A_n} \in M(\mu)$ and

$$\zeta_{f\chi_{A_n}}(E) = \zeta_{f\chi_{A_n}}(A_n \cap E) = (M)\int_{A_n \cap E} f \, d\mu_{A_n \cap E} = \nu_f(A_n \cap E) \tag{10}$$

for every $n \in \mathbb{N}$ and every $E \in \Sigma$.

Fix $\varepsilon > 0$. By Lemma 3.3, for each n there is a gauge δ_n on (Ω, τ) such that

$$\|f\chi_{A_n}(\mathcal{P}') - \zeta_{f\chi_{A_n}}(W_{\mathcal{P}'})\| \leq \frac{\varepsilon}{2^n} \tag{11}$$

for every partial McShane partition $\mathcal{P}'$ of Ω subordinate to δ_n. On the other hand, since ν_f is countably additive, there is $N_0 \in \mathbb{N}$ such that

$$\Big\|\sum_{n \in F} \nu_f(A_n)\Big\| \leq \varepsilon \tag{12}$$

for every finite set $F \subset \mathbb{N}$ satisfying $F \cap \{1, \ldots, N_0\} = \emptyset$.

By property (α) of $\hat{\mu}$, for every $1 \leq n \leq N_0$ we can choose a closed set $F_n \subset A_n$ such that $\hat{\mu}(A_n \setminus F_n) = 0$ (keep in mind that A_n is an atom of λ). Define a gauge δ on (Ω, τ) by

$$\delta(\omega) := \delta_n(\omega) \setminus \bigcup_{\substack{m=1 \\ m \neq n}}^{N_0} F_m \quad \text{whenever } \omega \in A_n.$$

Fix $0 < \eta < \min\{\hat{\mu}(A_n) : 1 \le n \le N_0\}$. We claim that

$$\|f(\mathcal{P}) - \nu_f(\Omega)\| \le 2\varepsilon \quad \text{for every } \mathcal{P} \in \Pi_{\delta,\eta}. \tag{13}$$

To prove this fix $\mathcal{P} = \{(E_i, s_i) : 1 \le i \le p\} \in \Pi_{\delta,\eta}$. Given $1 \le n \le N_0$, the definition of δ implies that

$$A_n \setminus \bigcup_{s_i \in A_n} E_i \subset (A_n \setminus F_n) \cup (\Omega \setminus \cup_{i=1}^{p} E_i),$$

hence $\hat{\mu}(A_n \setminus \cup_{s_i \in A_n} E_i) \le \eta < \hat{\mu}(A_n)$ and, since A_n is an atom of λ, we conclude that $\hat{\mu}(A_n \setminus \cup_{s_i \in A_n} E_i) = 0$. Therefore

$$\nu_f\Big(A_n \setminus \bigcup_{s_i \in A_n} E_i\Big) = 0 \quad \text{for every } 1 \le n \le N_0.$$

On the other hand, for each $n > N_0$ we have

$$\nu_f\Big(A_n \setminus \bigcup_{s_i \in A_n} E_i\Big) \in \{0, \nu_f(A_n)\},$$

because A_n is an atom of λ. It follows from (12) that

$$\Big\|\sum_n \nu_f\Big(A_n \setminus \bigcup_{s_i \in A_n} E_i\Big)\Big\| \le \varepsilon. \tag{14}$$

Note that for each $n \in \mathbb{N}$ the collection $\{(E_i, s_i) : s_i \in A_n\}$ is a partial McShane partition of Ω subordinate to δ_n, hence

$$\Big\| \sum_{s_i \in A_n} \mu(E_i)(f\chi_{A_n}(s_i)) - \nu_f\Big(\Big(\bigcup_{s_i \in A_n} E_i\Big) \cap A_n\Big)\Big\| \le \frac{\varepsilon}{2^n},$$

by (11) and (10). Combining this inequality with (14) we get

$$\begin{aligned}
\|f(\mathcal{P}) - \nu_f(\Omega)\| &= \Big\|\sum_n \sum_{s_i \in A_n} \mu(E_i)(f\chi_{A_n}(s_i)) - \sum_n \nu_f(A_n)\Big\| \\
&\le \sum_n \Big\| \sum_{s_i \in A_n} \mu(E_i)(f\chi_{A_n}(s_i)) - \nu_f\Big(\Big(\bigcup_{s_i \in A_n} E_i\Big) \cap A_n\Big)\Big\| \\
&\qquad + \Big\|\sum_n \nu_f\Big(A_n \setminus \bigcup_{s_i \in A_n} E_i\Big)\Big\| \le \sum_n \frac{\varepsilon}{2^n} + \varepsilon = 2\varepsilon.
\end{aligned}$$

Therefore, (13) holds. Since $\varepsilon > 0$ is arbitrary, f is McShane integrable with respect to μ and $(M)\int_\Omega f \, d\mu = \nu_f(\Omega)$. □

Case 3. Suppose that λ is atomless.

For each $\omega \in \Omega$ we have $\lambda^*(\{\omega\}) = 0$ (here λ^* stands for the outer measure induced by λ) and, since $(\Omega, \Sigma, \lambda)$ is complete, $\{\omega\} \in \Sigma$ and $\hat{\mu}(\{\omega\}) = 0$.

Fix $\varepsilon > 0$. By Lemma 2.1 there exists a countable partition $\Gamma_0 = (A_n)$ of Ω in Σ such that

$$\|S(f, \Gamma, T) - \nu_f(\cup_m B_m)\| < \varepsilon \tag{15}$$

for every finite collection $\Gamma = (B_m)$ of pairwise disjoint elements of Σ finer than Γ_0 and any choice T in Γ.

Since ν_f is countably additive and $\nu_f(E) = 0$ whenever $\lambda(E) = 0$, we have $\lim_{\lambda(E)\to 0} \nu_f(E) = 0$, see [5, Theorem 1, p. 10], and we can choose $\eta > 0$ such that

$$\|\nu_f(E)\| \leq \varepsilon \quad \text{for every } E \in \Sigma \text{ with } \hat{\mu}(E) \leq \eta. \tag{16}$$

Fix $N_0 \in \mathbb{N}$ large enough such that

$$\hat{\mu}(\cup_{n>N_0} A_n) \leq \frac{\eta}{3}. \tag{17}$$

Property (α) of $\hat{\mu}$ ensures us that for each $n, m \in \mathbb{N}$ there exist a closed set $K_n \subset A_n$ and an open set $G_{n,m} \supset A_n$ such that

$$\hat{\mu}(A_n \setminus K_n) \leq \frac{\eta}{2^n \cdot 3} \tag{18}$$

and

$$\hat{\mu}(G_{n,m} \setminus A_n) \leq \frac{\varepsilon}{2^{n+m} \cdot m}. \tag{19}$$

Let us consider the gauge δ on (Ω, τ) defined by

$$\delta(\omega) := G_{n,m} \setminus \bigcup_{\substack{i=1 \\ i \neq n}}^{N_0} K_i$$

whenever $\omega \in A_n$ and $m - 1 \leq \|f(\omega)\| < m$. We will prove that

$$\Big\| \sum_{i=1}^{p} \mu(E_i)(f(s_i)) - \nu_f(\Omega) \Big\| \leq 3\varepsilon \tag{20}$$

for every $\mathcal{P} = \{(E_i, s_i) : \ 1 \leq i \leq p\} \in \Pi_{\delta, \frac{\eta}{3}}$. To this end, observe that we can suppose that $s_i \neq s_j$ for $i \neq j$. Choose $N \geq N_0$ such that $s_1, \dots, s_p \in \cup_{n=1}^{N} A_n$. Define $F := \Omega \setminus \{s_i : \ 1 \leq i \leq p\}$ and write $I_n = \{1 \leq i \leq p : \ s_i \in A_n\}$ for each $1 \leq n \leq N$ (some I_n may be empty). Let us define

$$E_{i,n} := (E_i \cap A_n \cap F) \cup \{s_i\}$$

for every $1 \leq n \leq N$ and every $i \in I_n$. Since

$$\Gamma = \{E_{i,n} : \ 1 \leq n \leq N, \ i \in I_n\}$$

is a finite collection of pairwise disjoint elements of Σ finer than Γ_0, (15) applies to get

$$\Big\| \sum_{n=1}^{N} \sum_{i \in I_n} \big(\mu(E_{i,n})(f(s_i)) - \nu_f(E_{i,n})\big) \Big\| \leq \varepsilon. \tag{21}$$

Since $\lambda(\{\omega\}) = 0$ for every $\omega \in \Omega$, we have $\lambda((E_i \cap A_n)\triangle E_{i,n}) = 0$ for every $1 \leq n \leq N$ and every $i \in I_n$, hence

$$\nu_f\Big(\bigcup_{n=1}^{N} \bigcup_{i\in I_n} E_{i,n}\Big) = \nu_f\Big(\bigcup_{n=1}^{N} \bigcup_{i\in I_n} (E_i \cap A_n)\Big).$$

This equality, the fact that $\mu(E_{i,n}) = \mu(E_i \cap A_n)$ (for all $1 \leq n \leq N$ and $i \in I_n$) and (21) imply

$$\Big\|\sum_{n=1}^{N}\sum_{i\in I_n} \mu(E_i \cap A_n)(f(s_i)) - \nu_f\big(\cup_{n=1}^{N}(P_n \cap A_n)\big)\Big\| \leq \varepsilon, \tag{22}$$

where $P_n := \cup_{i\in I_n} E_i$ for every $1 \leq n \leq N$.

We will now show that

$$A_n \setminus P_n \subset (A_n \setminus K_n) \cup (\cup_{s>N_0} A_s) \cup (\Omega \setminus \cup_{i=1}^{p} E_i) \quad \text{for every } 1 \leq n \leq N. \tag{23}$$

It suffices to check that $(A_n \setminus P_n) \cap (\cup_{i=1}^{p} E_i) \subset A_n \setminus K_n$ for every $1 \leq n \leq N_0$. To this end, take $1 \leq i \leq p$ and suppose that $(A_n \setminus P_n) \cap E_i \neq \emptyset$. Then there is some $k \neq n$ such that $s_i \in A_k$ and, therefore, we have

$$E_i \cap K_n \subset \delta(s_i) \cap K_n \subset \Big(\Omega \setminus \bigcup_{\substack{i=1\\ i\neq k}}^{N_0} K_i\Big) \cap K_n = \emptyset,$$

hence $(A_n \setminus P_n) \cap E_i \subset A_n \setminus K_n$, as required. This completes the proof of (23).

As a consequence of (23) we have (recall that $N \geq N_0$)

$$\hat{\mu}\Big(\Big(\bigcup_{n=1}^{N}(A_n \setminus P_n)\Big) \cup \Big(\bigcup_{n>N} A_n\Big)\Big) \leq \sum_{n=1}^{N} \hat{\mu}(A_n \setminus K_n)$$
$$+ \hat{\mu}(\cup_{s>N_0} A_s) + \hat{\mu}(\Omega \setminus \cup_{i=1}^{p} E_i) \leq \sum_{n=1}^{N} \frac{\eta}{2^n \cdot 3} + \frac{\eta}{3} + \frac{\eta}{3} \leq \eta,$$

by (18), (17) and the fact that $\mathcal{P} \in \Pi_{\delta,\frac{\eta}{3}}$. An appeal to (16) yields

$$\Big\|\nu_f\Big(\Big(\bigcup_{n=1}^{N}(A_n \setminus P_n)\Big) \cup \Big(\bigcup_{n>N} A_n\Big)\Big)\Big\| \leq \varepsilon,$$

which combined with (22) allows us to conclude that

$$\Big\|\sum_{n=1}^{N}\sum_{i\in I_n} \mu(E_i \cap A_n)(f(s_i)) - \nu_f(\Omega)\Big\| \leq 2\varepsilon. \tag{24}$$

By the definition of δ we have $E_i \setminus A_n \subset G_{n,m} \setminus A_n$ whenever $i \in I_n$ and $m-1 \leq \|f(s_i)\| < m$. Therefore

$$\Big\|\sum_{n=1}^{N}\sum_{i\in I_n}\mu(E_i\setminus A_n)(f(s_i))\Big\| \leq \sum_{n=1}^{N}\sum_{m=1}^{\infty}\Big\|\sum_{\substack{i\in I_n\\ m-1\leq\|f(s_i)\|<m}}\mu(E_i\setminus A_n)(f(s_i))\Big\|$$
$$\leq \sum_{n=1}^{N}\sum_{m=1}^{\infty}\hat{\mu}(G_{n,m}\setminus A_n)\cdot m \leq \sum_{n=1}^{N}\sum_{m=1}^{\infty}\frac{\varepsilon}{2^{n+m}} \leq \varepsilon,$$

by (19). This inequality and (24) imply

$$\Big\|\sum_{i=1}^{p}\mu(E_i)(f(s_i)) - \nu_f(\Omega)\Big\| = \Big\|\sum_{n=1}^{N}\sum_{i\in I_n}\mu(E_i)(f(s_i)) - \nu_f(\Omega)\Big\| \leq 3\varepsilon.$$

Therefore, (20) holds. Since $\varepsilon > 0$ is arbitrary, f is McShane integrable with respect to μ and $(M)\int_\Omega f\, d\mu = \nu_f(\Omega)$. □

General case.

Since λ is finite, there is a countable family (A_n) of pairwise disjoint atoms of λ such that $\lambda_{\Omega\setminus A}$ is atomless, where $A := \cup_n A_n$.

Since $f|_A$ (resp. $f|_{\Omega\setminus A}$) is S^*-integrable with respect to μ_A (resp. $\mu_{\Omega\setminus A}$), *Case 2* (resp. *Case 3*) implies that $f|_A$ (resp. $f|_{\Omega\setminus A}$) is McShane integrable with respect to μ_A (resp. $\mu_{\Omega\setminus A}$) and

$$(M)\int_A f\, d\mu_A = \nu_f(A) \quad \Big(\text{resp. } (M)\int_{\Omega\setminus A} f\, d\mu_{\Omega\setminus A} = \nu_f(\Omega\setminus A)\Big).$$

We conclude from Proposition 3.4 that $f\chi_A \in M(\mu)$ (resp. $f\chi_{\Omega\setminus A} \in M(\mu)$) and $(M)\int_\Omega f\chi_A\, d\mu = \nu_f(A)$ (resp. $(M)\int_\Omega f\chi_{\Omega\setminus A}\, d\mu = \nu_f(\Omega\setminus A)$).

It follows that $f = f\chi_A + f\chi_{\Omega\setminus A}$ is McShane integrable with respect to μ and $(M)\int_\Omega f\, d\mu = \nu_f(A) + \nu_f(\Omega\setminus A) = \nu_f(\Omega)$. The proof of Theorem 3.7 is complete. □

Proposition 4 in [12] states that *every Birkhoff integrable function defined on a σ-finite outer regular quasi-Radon measure space is McShane integrable in the sense of* [14] *(and the respective integrals coincide)*. The converse does not hold in general, although it is true if the closed unit ball of the dual of the range space is weak*-separable, [12, Theorem 10]. Examples of McShane integrable functions defined on $[0,1]$ (with the Lebesgue measure) which are not Birkhoff integrable can be found in [12, Example 8] and [23, Corollaries 2.4 and 2.6]. Observe that the finite measure case of the aforementioned Fremlin's result is included in our Theorem 3.7.

We end up the paper by pointing out the precise relationship between McShane and Dobrakov integrability.

Theorem 3.8. *Let $f : \Omega \longrightarrow X$ be a function. The following conditions are equivalent:*

(i) *f is Dobrakov integrable with respect to μ;*
(ii) *f is measurable and McShane integrable with respect to μ.*

In this case, $(D)\int_\Omega f\ d\mu = (M)\int_\Omega f\ d\mu$.

Proof. As pointed out in Section 2, (i) is equivalent to

(i') *f is measurable and S^*-integrable with respect to μ;*

and, in this case, $(D)\int_\Omega f\ d\mu = (S^*)\int_\Omega f\ d\mu$. It follows from Theorem 3.7 that (i) implies (ii) and that $(D)\int_\Omega f\ d\mu = (M)\int_\Omega f\ d\mu$.

The proof of (ii)⇒(i) is as follows. Since f is measurable, there is a countably valued function $g : \Omega \longrightarrow X$, $g = \sum_{n=1}^\infty x_n \chi_{A_n}$, $x_n \in X$, $A_n \in \Sigma$ pairwise disjoint, such that $\|f - g\| \leq 1$ $\hat{\mu}$-almost everywhere, see [5, Corollary 3, p. 42]. Hence $f - g$ is Dobrakov integrable with respect to μ, by [6, Theorem 5], and, in view of (i)⇒(ii), McShane integrable with respect to μ. Define $g_n = \sum_{k=1}^n x_k \chi_{A_k}$ for every $n \in \mathbb{N}$. Clearly $(g_n)_{n=1}^\infty$ converges to g pointwise and, since $g = f - (f - g)$ is McShane integrable with respect to μ, Lemma 3.2 (ii) implies that for each $E \in \Sigma$ there exists the limit

$$\lim_n \int_E g_n\ d\mu_E = \lim_n \sum_{k=1}^n \mu(E \cap A_k)(x_k) = \lim_n \sum_{k=1}^n \zeta_g(E \cap A_k).$$

Therefore, g is Dobrakov integrable with respect to μ and the same is true for $f = g + (f - g)$. The proof is finished. □

Finally, Corollary 3.6 allows us to translate the previous theorem into the language of the Bartle bilinear *-integral.

Corollary 3.9. *Let Z be a real Banach space, $\nu : \Sigma \longrightarrow Z$ a countably additive vector measure and $\phi : X \times Z \longrightarrow Y$ a continuous bilinear map. Suppose that ν has the *-property with respect to ϕ and that the semivariation of the set function $\mu : \Sigma \longrightarrow \mathcal{L}(X, Y)$ given by $\mu(E)(x) = \phi(x, \nu(E))$ fulfills properties (α), (β) and (γ). Let $f : \Omega \longrightarrow X$ be a function. The following conditions are equivalent:*

(i) *f is Bartle *-integrable with respect to ν and ϕ;*
(ii) *f is McShane integrable with respect to μ and there is a sequence of simple functions converging to f $\hat{\mu}$-almost everywhere.*

In this case the respective integrals coincide.

Acknowledgements.
The author wishes to express his gratitude to David Fremlin and Gabriel Vera for several helpful comments and suggestions that improved the content of this paper. The author also thanks the referee for valuable remarks leading to the final version.

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