ABSOLUTELY SUMMING OPERATORS AND INTEGRATION OF VECTOR-VALUED FUNCTIONS

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ABSTRACT. Let (Ω, Σ, μ) be a complete probability space and $u : X \longrightarrow Y$ an absolutely summing operator between Banach spaces. We prove that for each Dunford integrable (i.e. scalarly integrable) function $f : \Omega \longrightarrow X$ the composition $u \circ f$ is scalarly equivalent to a Bochner integrable function. Such a composition is shown to be Bochner integrable in several cases, for instance, when f is properly measurable, Birkhoff integrable or McShane integrable, as well as when X is a subspace of an Asplund generated space or a subspace of a weakly Lindelöf space of the form $C(K)$. We also study the continuity of the composition operator $f \mapsto u \circ f$. Some other applications are given.

1. INTRODUCTION

An operator (i.e. linear and continuous map) between Banach spaces is said to be absolutely summing if it takes unconditionally convergent series to absolutely convergent ones. Since absolutely summing operators improve summability properties of sequences, *it is not surprising that they also improve the integrability of vector-valued functions* [12, p. 56]. This fact was first noticed by Diestel [11], who proved that, given a complete probability space (Ω, Σ, μ), if an operator between Banach spaces $u : X \longrightarrow Y$ is absolutely summing then for each *strongly measurable* Pettis integrable function $f : \Omega \longrightarrow X$ the composition $u \circ f$ is Bochner integrable and the linear map

$$(\mathcal{P}_m(\mu, X), \|\cdot\|_P) \longrightarrow (L^1(\mu, Y), \|\cdot\|_1), \qquad f \mapsto u \circ f,$$

is continuous, where $\mathcal{P}_m(\mu, X)$ denotes the space of all strongly measurable Pettis integrable functions from Ω to X and $\|\cdot\|_P$ is the so-called Pettis seminorm (see below for the definitions). Diestel also showed that the converse holds true for atomless μ. Later, Belanger and Dowling [2] proved that the composition of any bounded Pettis integrable function, defined on a *perfect* complete probability space, with an absolutely summing operator is *scalarly equivalent* to a Bochner integrable function. The boundedness assumption has recently been removed by Marraffa [30], who has also obtained the analogue of the aforementioned Diestel's result for *McShane integrable* functions defined on a compact Radon probability space. We also mention that Heiliö [26] studied similar questions in the setting of weak Baire measures in Banach spaces.

The aim of this paper is to go a bit further when studying the composition of an "integrable" vector-valued function and an absolutely summing operator. Our discussion involves non-separable Banach spaces and notions of integral (intermediate between those due to Bochner and Pettis) that have caught the attention of many authors pretty recently, like the Birkhoff, Talagrand and McShane integrals. We next summarize the content of this work.

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Let (Ω, Σ, μ) be a complete probability space and $u : X \longrightarrow Y$ an absolutely summing operator between Banach spaces. Write $\mathcal{D}(\mu, X)$ to denote the space of all Dunford integrable (i.e. scalarly integrable) functions from Ω to X. In Section 2 we prove that for each $f \in \mathcal{D}(\mu, X)$ the composition $u \circ f$ is scalarly equivalent to a Bochner integrable function u_f (Theorem 2.3), even for non-perfect μ. Moreover, the linear map

$$(\mathcal{D}(\mu, X), \|\cdot\|_P) \longrightarrow (L^1(\mu, Y), \|\cdot\|_1), \qquad f \mapsto u_f,$$

is shown to be continuous (Corollary 2.5). Our views also allow us to give an affirmative answer to an open question posed in [26] concerning the image measure $\lambda \circ u^{-1}$ of a weak Baire measure λ on X that is "weakly summable" (Proposition 2.7).

Section 3 is devoted to study conditions on either $f \in \mathcal{D}(\mu, X)$ or X that ensure us that $u \circ f$ is Bochner integrable. Our Lemma 2.2 states that $u \circ f$ is Bochner integrable whenever $u \circ f$ is strongly measurable. Since the last requirement follows automatically provided that $u(X)$ is separable, in Sub-Section 3.1 we pay attention to Banach spaces for which all the absolutely summing operators defined on them have separable range. We show in Theorem 3.3 that this property is shared by a wide class of Banach spaces including, for instance, those that are isomorphic to subspaces of weakly Lindelöf determined spaces of the form $C(K)$ (e.g. weakly countably $\mathcal{K}$-determined spaces), as well as those that are isomorphic to subspaces of Asplund generated spaces (e.g. Asplund spaces). Sub-Section 3.2 deals with the composition of a properly measurable function and an absolutely summing operator. It turns out (Corollary 3.6) that such a composition is Bochner integrable whenever the function is Dunford integrable (e.g. Talagrand integrable). As an application we show that the composition of a Dunford integrable function with values in a subspace of a weakly Lindelöf space of the form $C(K)$ and an absolutely summing operator is always Bochner integrable (Proposition 3.9). We complete Section 3 by establishing the Bochner integrability of the composition of Birkhoff and McShane integrable functions with absolutely summing operators (Sub-Sections 3.3 and 3.4, respectively). Finally, in Section 4 we give two examples making clear that the composition of a Dunford integrable function and an absolutely summing operator is not Bochner integrable in general.

For all unexplained terminology we refer to our standard references [12] (absolutely summing operators), [13] (vector measures), [17, 18] (Banach spaces and related compact spaces) and [36] (Pettis integral). All our vector spaces are assumed to be real. For a complete probability space (Ω, Σ, μ) we denote by $\mathcal{L}^1(\mu)$ the space of all μ-integrable real-valued functions defined on Ω and $L^1(\mu)$ for the corresponding Banach space of equivalence classes with its usual norm $\|\cdot\|_1$. A set $\mathcal{H} \subset \mathcal{L}^1(\mu)$ is *uniformly integrable* iff it is $\|\cdot\|_1$-bounded and for each $\varepsilon > 0$ there is $\delta > 0$ such that $\sup_{h \in \mathcal{H}} \int_E |h| \, d\mu \leq \varepsilon$ whenever $\mu(E) \leq \delta$. Given $A \in \Sigma$, we write μ_A to denote the restriction of μ to the σ-algebra $\{E \subset A : E \in \Sigma\}$.

Now let Z be a Banach space. As usual, B_Z is the closed unit ball of Z and Z^* stands for the topological dual of Z. We denote by w and w^* the weak and weak* topologies on Z and Z^*, respectively. A set $B \subset B_{Z^*}$ is *norming* iff $\|z\| = \sup\{|z^*(z)| : z^* \in B\}$ for every $z \in Z$.

We denote by $L^1(\mu, Z)$ the Banach space of all Bochner integrable functions from Ω to Z (functions that are equal μ-a.e. are identified), endowed with the norm $\|\cdot\|_1$. Recall that a function $f : \Omega \longrightarrow Z$ is *Bochner integrable* iff it is *strongly measurable* (i.e. f is the μ-a.e. limit of a sequence of simple functions) and $\|f\|_1 := \int_\Omega \|f\| \, d\mu < \infty$. A function $f : \Omega \longrightarrow Z$ is *Dunford integrable* (or scalarly integrable) iff the real-valued function $z^* \circ f$ is μ-integrable for every $z^* \in Z^*$. In this case there is a finitely additive vector measure $\nu_f : \Sigma \longrightarrow Z^{**}$ such that $\nu_f(E)(z^*) = \int_E z^* \circ f \, d\mu$ for every $E \in \Sigma$ and $z^* \in Z^*$. Moreover,

$$\|f\|_P := \sup\Big\{ \int_\Omega |z^* \circ f| \, d\mu : \; z^* \in B_{Z^*} \Big\} < \infty,$$

and therefore ν_f is bounded. When ν_f takes its values in Z then f is called *Pettis integrable* (in this case, ν_f is a countably additive vector measure). The set of all Pettis integrable functions $f : \Omega \longrightarrow Z$ for which $\nu_f(\Sigma) = \{\nu_f(E) : E \in \Sigma\}$ is norm relatively compact will be denoted by $\mathcal{P}_c(\mu, Z)$. Two functions $f, g : \Omega \longrightarrow Z$ are said to be *scalarly equivalent* iff for each $z^* \in Z^*$ we have $z^* \circ f = z^* \circ g$ μ-a.e.

Our compact topological spaces are assumed to be Hausdorff. For a compact space K we denote by $C(K)$ the Banach space of all real-valued continuous functions defined on K with the supremum norm. For each $t \in K$ the Dirac delta $\delta_t \in B_{C(K)^*}$ is defined by $\delta_t(f) = f(t)$ for every $f \in C(K)$. Throughout this paper $M^+(K)$ stands for the set of all (completions of) non-negative Radon (i.e. finite and inner regular with respect to compact sets) measures on $\mathrm{Borel}(K)$. Given $\nu \in M^+(K)$, the *support* of ν is

$$\mathrm{supp}(\nu) := K \setminus \bigcup \{U \subset K : \ U \text{ open}, \ \nu(U) = 0\}.$$

It is easy to check that $\nu(K \setminus \mathrm{supp}(\nu)) = 0$ and that $\nu(U \cap \mathrm{supp}(\nu)) > 0$ whenever $U \subset K$ is open and $U \cap \mathrm{supp}(\nu) \neq \emptyset$.

2. Scalar equivalence to Bochner integrable functions and continuity of the composition operator

It is well known [12, p. 34] that an operator between Banach spaces $u : X \longrightarrow Y$ is absolutely summing if and only if u is 1-*summing*, i.e. there is a constant $C \geq 0$ such that

$$\sum_{i=1}^{n} \|u(x_i)\| \leq C \sup\Big\{\sum_{i=1}^{n} |x^*(x_i)| : \ x^* \in B_{X^*}\Big\}$$

for every finite collection $x_1, \dots, x_n \in X$. In this case, the smallest constant $C \geq 0$ satisfying the inequality above will be denoted by $\pi(u)$. It is easy to check that

$$\sum_{i=1}^{n} \|u(x_i)\| \leq 2\pi(u) \sup\Big\{\Big\|\sum_{i\in S} x_i\Big\| : \ S \subset \{1, \dots, n\}\Big\} \tag{1}$$

for every finite collection $x_1, \dots, x_n \in X$.

As a consequence, the composition of a bounded vector measure with an absolutely summing operator always has bounded variation (see Lemma 2.1 below). Recall that, given a finitely additive vector measure ν defined on an algebra $\mathcal{A}$ (of subsets of a set Ω) with values in a Banach space X, the *total variation* of ν is defined by

$$|\nu|(\Omega) := \sup\Big\{\sum_{i=1}^{n} \|\nu(E_i)\| : \ \{E_1, \dots, E_n\} \text{ is a finite partition of } \Omega \text{ in } \mathcal{A}\Big\} \in [0, \infty].$$

Lemma 2.1. *Let $\mathcal{A}$ be an algebra of subsets of a set Ω and $u : X \longrightarrow Y$ an absolutely summing operator between Banach spaces. If $\nu : \mathcal{A} \longrightarrow X$ is a bounded finitely additive vector measure, then the composition $u \circ \nu$ has bounded variation, i.e. $|u \circ \nu|(\Omega) < \infty$.*

Proof. Given any finite partition of Ω in $\mathcal{A}$, say $\{E_1, \dots, E_n\}$, inequality (1) applies to deduce

$$\begin{aligned}\sum_{i=1}^{n} \|u \circ \nu(E_i)\| &\leq 2\pi(u) \sup\Big\{\Big\|\sum_{i\in S} \nu(E_i)\Big\| : \ S \subset \{1, \dots, n\}\Big\} \\ &\leq 2\pi(u) \sup\{\|\nu(A)\| : \ A \in \mathcal{A}\} < \infty.\end{aligned}$$

Therefore $u \circ \nu$ has bounded variation, as required. $\square$

Let h be a strongly measurable Dunford integrable function defined on a complete probability space (Ω, Σ, μ) with values in a Banach space Z. It follows from Proposition 1 in [31] that $|\nu_h|(\Omega) = \int_\Omega \|h\| \, d\mu$ (maybe infinite). This fact will be used in the proof of the following result.

Lemma 2.2. *Let (Ω, Σ, μ) be a complete probability space, $u : X \longrightarrow Y$ an absolutely summing operator between Banach spaces and $f : \Omega \longrightarrow X$ a Dunford integrable function. Let $g : \Omega \longrightarrow Y$ be scalarly equivalent to $u \circ f$. Then g is Bochner integrable if and only if g is strongly measurable.*

Proof. Assume that g is strongly measurable. Since f is Dunford integrable, the same holds for $u \circ f$, with $\nu_{u \circ f} = u^{**} \circ \nu_f$, where $u^{**} : X^{**} \longrightarrow Y^{**}$ is the second adjoint of u. Hence g is also Dunford integrable and $\nu_g = \nu_{u \circ f} = u^{**} \circ \nu_f$. According to the comment preceding this lemma, $|\nu_g|(\Omega) = \int_\Omega \|g\| \, d\mu$. On the other hand, since u is absolutely summing, u^{**} is absolutely summing too, see [12, Proposition 2.19], hence Lemma 2.1 applied to ν_f and u^{**} ensures that $u^{**} \circ \nu_f = \nu_g$ has bounded variation. Consequently, $\int_\Omega \|g\| \, d\mu < \infty$ and therefore g is Bochner integrable. The proof is over. □

Theorem 2.3. *Let (Ω, Σ, μ) be a complete probability space, $u : X \longrightarrow Y$ an absolutely summing operator between Banach spaces and $f : \Omega \longrightarrow X$ a Dunford integrable function. Then $u \circ f$ is scalarly equivalent to some Bochner integrable function $g : \Omega \longrightarrow Y$.*

Proof. Since u is absolutely summing, u is weakly compact, see [12, Theorem 2.17], and therefore $\overline{u(X)}$ is a weakly compactly generated Banach space in which $u \circ f$ takes its values. Every weakly compactly generated Banach space is weakly Lindelöf (see [18, Chapter 12]) and, therefore, measure-compact in its weak topology. Thus the scalarly measurable function $u \circ f$ is scalarly equivalent to a strongly measurable one $g : \Omega \longrightarrow Y$, see [16, Proposition 5.4]. An appeal to Lemma 2.2 now ensures us that g is Bochner integrable and the proof is complete. □

Remark. As we mentioned in the introduction, the same conclusion was obtained in [2] (in the case of bounded functions) and [30, Proposition 3] for Pettis integrable functions and perfect measures.

We next discuss the continuity of the "composition" operator associated with an absolutely summing operator. Let (Ω, Σ, μ) be a complete probability space and $u : X \longrightarrow Y$ an operator between Banach spaces. Let $\mathcal{M}$ be a linear subspace of $\mathcal{D}(\mu, X)$ such that for each $f \in \mathcal{M}$ the composition $u \circ f$ is scalarly equivalent to a Bochner integrable function $u_f : \Omega \longrightarrow Y$. Then we can consider the map

$$\tilde{u}_{\mathcal{M}} : (\mathcal{M}, \|\cdot\|_P) \longrightarrow (L^1(\mu, Y), \|\cdot\|_1)$$

that sends each $f \in \mathcal{M}$ to the equivalence class of u_f. Observe that $\tilde{u}_{\mathcal{M}}$ does not depend on the particular choice of the u_f's, because two scalarly equivalent strongly measurable functions must coincide μ-a.e., see [13, Corollary 7, p. 48]. For the same reason, $\tilde{u}_{\mathcal{M}}$ is linear.

Lemma 2.4. *With the notations above, the map $\tilde{u}_{\mathcal{M}}$ has closed graph.*

Proof. Fix a sequence (f_n) in $\mathcal{M}$ such that $\lim_n \|f_n\|_P = 0$ and there is a Bochner integrable function $h : \Omega \longrightarrow Y$ with $\lim_n \|u_{f_n} - h\|_1 = 0$. By passing to a further subsequence, we can suppose without loss of generality that (u_{f_n}) converges to h μ-a.e., see [14, Proposition 14, p. 130]. Since h is strongly measurable, in order to check that $h = 0$ μ-a.e. it suffices to show that for each $y^* \in Y^*$ we have $y^* \circ h = 0$ μ-a.e. To this end, fix $y^* \in Y^*$. Since $\lim_n \|f_n\|_P = 0$, we have

$$\lim_n \int_\Omega |y^* \circ u_{f_n}| \, d\mu = \lim_n \int_\Omega |y^* \circ u \circ f_n| \, d\mu = 0$$

and therefore Fatou's lemma yields $\int_\Omega |y^* \circ h| \, d\mu = 0$, hence $y^* \circ h = 0$ μ-a.e. Since $y^* \in Y^*$ is arbitrary, $h = 0$ μ-a.e. Thus $\tilde{u}_{\mathcal{M}}$ has closed graph, as required. □

Although the normed spaces obtained (by identifying scalarly equivalent functions) from $\mathcal{D}(\mu, X)$ and $\mathcal{P}_m(\mu, X)$ are not complete in general, see [27, 38], they are always ultrabornological [10], hence barrelled and therefore every linear map defined on them with values in a Banach space is continuous whenever it has closed graph. (For a detailed account of the theory of barrelled locally convex spaces we refer the reader to [4].) Bearing in mind Theorem 2.3, we can now deduce the following

Corollary 2.5. *Let (Ω, Σ, μ) be a complete probability space and $u : X \longrightarrow Y$ an absolutely summing operator between Banach spaces. Then the linear map*

$$\tilde{u}_{\mathcal{D}(\mu,X)} : (\mathcal{D}(\mu, X), \|\cdot\|_P) \longrightarrow (L^1(\mu, Y), \|\cdot\|_1)$$

is continuous.

Proof. Write E to denote the barrelled normed space obtained from $(\mathcal{D}(\mu, X), \|\cdot\|_P)$ by identifying scalarly equivalent functions. Since $\tilde{u}_{\mathcal{D}(\mu,X)}(f) = 0$ whenever f is scalarly null, there is a linear map $T : E \longrightarrow L^1(\mu, Y)$ such that $T \circ I = \tilde{u}_{\mathcal{D}(\mu,X)}$, where $I : \mathcal{D}(\mu, X) \longrightarrow E$ maps each function to its equivalence class. Since $\tilde{u}_{\mathcal{D}(\mu,X)}$ has closed graph (by Lemma 2.4), the same holds for T and therefore T is continuous. Then $\tilde{u}_{\mathcal{D}(\mu,X)}$ is continuous too and the proof is finished. □

In the same manner we obtain

Corollary 2.6. *Let (Ω, Σ, μ) be a complete probability space and $u : X \longrightarrow Y$ an operator between Banach spaces such that $u \circ f$ is Bochner integrable for every $f \in \mathcal{P}_m(\mu, X)$. Then the linear map*

$$\tilde{u}_{\mathcal{P}_m(\mu,X)} : (\mathcal{P}_m(\mu, X), \|\cdot\|_P) \longrightarrow (L^1(\mu, Y), \|\cdot\|_1)$$

is continuous.

Remark. Under the hypotheses of Corollary 2.6 and the additional assumption of the continuity of $\tilde{u}_{\mathcal{P}_m(\mu,X)}$ (which we have shown to be redundant), it was proved in [11] that u must be absolutely summing (with $\pi(u) = \|\tilde{u}_{\mathcal{P}_m(\mu,X)}\|$) provided that μ is atomless.

We end up this section with an application of Theorem 2.3 to Baire measures in Banach spaces. Recall that the Baire σ-algebra of a Banach space X endowed with its weak topology, denoted by $\mathrm{Baire}(X, w)$, is exactly the smallest one for which each $x^* \in X^*$ is measurable, see [16, Theorem 2.3]. As a consequence, if f is a scalarly measurable function defined on a complete probability space (Ω, Σ, μ) with values in X, then f is Σ-$\mathrm{Baire}(X, w)$ measurable and we can consider the *image* probability measure $\mu \circ f^{-1}$ on $\mathrm{Baire}(X, w)$. There are non-trivial relationships between some "smoothness" properties of $\mu \circ f^{-1}$ and properties of f like Pettis integrability and scalar equivalence to strongly measurable functions, see [16, 32, 36] and the references therein.

Heiliö studied in [26] the class of those probability measures on $\mathrm{Baire}(X, w)$ for which the identity map $I_X : X \longrightarrow X$ is Dunford integrable (called by him *weakly summable measures*). Section 8.2 of that paper dealt with the image measure induced by an absolutely summing operator. A weakly summable measure μ is called *absolutely summable* if there is an extension $\tilde{\mu}$ of μ to $\mathrm{Borel}(X, \|\cdot\|)$ such that I_X is Bochner integrable with respect to $\tilde{\mu}$. It was shown in [26, Theorem 8.2.4] that, given an absolutely summing operator between Banach spaces $u : X \longrightarrow Y$ and a probability measure μ on $\mathrm{Baire}(X, w)$ for which $I_X \in \mathcal{P}_c(\mu, X)$, the image measure $\mu \circ u^{-1}$ is absolutely summable. The question of whether the same happens for an arbitrary weakly summable measure μ was left open in [26, 8.2.5]. We next give an affirmative answer to this question.

Proposition 2.7. *Let $u : X \longrightarrow Y$ be an absolutely summing operator between Banach spaces and μ a weakly summable measure on $\mathrm{Baire}(X, w)$. Then $\mu \circ u^{-1}$ is absolutely summable.*

Proof. It is easy to check that $\mu \circ u^{-1}$ is a weakly summable measure. Write Σ to denote the μ-completion of $\mathrm{Baire}(X, w)$ and let $\hat{\mu}$ be the complete extension of μ to Σ. By

Theorem 2.3 applied to I_X the operator u (viewed as a function from the complete probability space $(X, \Sigma, \hat{\mu})$ to Y) is scalarly equivalent to some Bochner integrable function $g : X \longrightarrow Y$. Since g is strongly measurable, g is Σ-$\mathrm{Borel}(Y, \|\cdot\|)$ measurable and there is a separable closed subspace $Y_0 \subset Y$ such that $\hat{\mu}(g^{-1}(Y_0)) = 1$, see [6, Appendix E]. According to Pettis' Measurability Theorem [13, Theorem 2, p. 42], the last condition ensures that the scalarly measurable function I_Y is strongly measurable with respect to $\hat{\mu} \circ g^{-1}$. Since u and g are scalarly equivalent, we have $\hat{\mu} \circ u^{-1} = \hat{\mu} \circ g^{-1}$ on $\mathrm{Baire}(Y, w)$, and therefore $\hat{\mu} \circ g^{-1}$ is an extension of $\mu \circ u^{-1}$ to $\mathrm{Borel}(Y, \|\cdot\|)$ for which I_Y is strongly measurable. On the other hand, since g is Bochner integrable, we have

$$\int_X \|g(x)\| \, d\hat{\mu}(x) < \infty.$$

By a standard change of variable we get

$$\int_Y \|I_Y(y)\| \, d(\hat{\mu} \circ g^{-1})(y) = \int_X \|g(x)\| \, d\hat{\mu}(x) < \infty,$$

hence I_Y is Bochner integrable with respect to $\hat{\mu} \circ g^{-1}$ and the proof is over. $\square$

3. Bochner integrability of the composition

3.1. **Absolutely summing operators with separable range.** As our Lemma 2.2 shows, the composition of a Dunford integrable function f with an absolutely summing operator u is Bochner integrable whenever $u \circ f$ is strongly measurable. By Pettis' Measurability Theorem [13, Theorem 2, p. 42], the last condition holds if the range of u is separable. Thus it is natural to look for Banach spaces X satisfying that each absolutely summing operator defined on X has separable range. In Theorem 3.3 below we show that a wide class of Banach spaces enjoy this property.

For a compact space K and any $\nu \in M^+(K)$, the "identity" operator

$$j_\nu : C(K) \longrightarrow L^1(\nu)$$

(that maps each function to its equivalence class) is absolutely summing, see [12, 2.9], and has dense range, see [6, Proposition 7.4.2]. Moreover, thanks to Pietsch's Factorization Theorem [12, Corollary 2.15], given an absolutely summing summing operator u defined on $C(K)$ with values in another Banach space Y, there exist $\nu \in M^+(K)$ and an operator $v : L^1(\nu) \longrightarrow Y$ such that $u = v \circ j_\nu$. As a consequence, it turns out that $C(K)$ *satisfies that each absolutely summing operator defined on it has separable range if and only if* $L^1(\nu)$ *is separable for every* $\nu \in M^+(K)$.

Following [15], we say that a compact space K *belongs to the class* MS iff each $\nu \in M^+(K)$ is *separable* (i.e. $L^1(\nu)$ is separable). The class MS is closed under subspaces, continuous images and countable products, see [15], and it contains the following compacta:

(a) *Metrizable compacta*, because the L^1 space associated to a non-negative finite measure defined on a countably generated σ-algebra is always separable, see [6, Proposition 3.4.5].

(b) *Corson compacta with property (M)*. Recall that a compact space K has *property (M)* (see [1, Section 3]) iff $\mathrm{supp}(\nu)$ is separable for each $\nu \in M^+(K)$. Thus (a) and the elementary fact that any separable Corson compact space is metrizable (see [18, Exercise 12.56]) imply that MS contains all Corson compacta with property (M). These are exactly those Corson compacta K for which $B_{C(K)^*}$ is also Corson (equivalently, $C(K)$ is weakly Lindelöf or $C(K)$ has property (C) of Corson), see [1, Theorem 3.5]. In particular, all Eberlein compacta and, more generally, all Gul'ko compacta, belong to MS (see Chapter 7 in [17]).

(c) *Rosenthal compacta* (Bourgain, see [39, Theorem 2] for a proof).

(d) *Linearly ordered compacta*, see [15, Theorem 1.0].

(e) *Zero-dimensional compacta K for which $C(K)$ is weakly Lindelöf*, see Lemma 3.5 in [19].
(f) *Radon-Nikodým compacta* (e.g. scattered, see [17, Chapter 1]), as we next show.

Lemma 3.1. *Let K be a Radon-Nikodým compact space. Then K belongs to MS.*

Proof. We denote by $\mathfrak{T}$ the original topology on K. Since K is a Radon-Nikodým compact, there is a lower semicontinuous metric d on K, whose corresponding topology is finer than $\mathfrak{T}$, such that K is *fragmented* by d (i.e. for every $\varepsilon > 0$ and every non-empty set $H \subset K$ there is a non-empty relatively open subset of H with d-diameter less than ε), see [7, Section I.5]. Fix $\nu \in M^+(K)$ and $n \in \mathbb{N}$. By [28, Theorem 4.1] there is a d-compact set $F_n \subset K$ such that $\nu(K \setminus F_n) \leq 1/n$. Since the topology induced by d is finer than $\mathfrak{T}$, the set F_n is compact and metrizable when endowed with the restriction of $\mathfrak{T}$. It follows from (a) above that $L^1(\nu_{F_n})$ is separable. Hence $E_n := \{h\chi_{F_n} : h \in L^1(\nu)\}$ is a separable subset of $L^1(\nu)$ (where χ_A is the characteristic function of a set $A \subset K$). Consequently, $\bigcup_{n=1}^{\infty} E_n$ is separable. Since $\lim_n \nu(K \setminus F_n) = 0$, the set $\bigcup_{n=1}^{\infty} E_n$ is dense in $L^1(\nu)$ and therefore $L^1(\nu)$ is separable. The proof is finished. $\square$

Under some additional set-theoretic assumptions we can say more about the class MS. Solving a long-standing problem posed by Haydon, Fremlin showed in [22] that under Martin's Axiom (MA) and the negation of the Continuum Hypothesis (CH), any compact space not belonging to MS can be mapped continuously onto $[0,1]^{\omega_1}$ (the converse holds without further assumptions). It follows that all compact spaces with countable tightness (e.g. Corson compacta or, more generally, angelic compacta, as well as compact spaces K for which $C(K)$ has property (C), see [19]) belong to MS whenever both MA and ¬CH are assumed. On the other hand, under CH there are examples of Corson compacta not belonging to MS, see [33, §5] (the Kunen-Haydon-Talagrand space) and [1, Section 3]. For more information on separable Radon measures on compact spaces and related questions we refer the reader to [1, 15, 19, 22, 29, 34] and the references therein.

We now introduce the Banach space counterpart of the class MS. We say that a Banach space *belongs to the class $\mathcal{MS}$* iff it is isomorphic to a subspace of $C(L)$, where L is a compact space belonging to MS. Bearing in mind that any Banach space X is isometric to a subspace of $C(B_{X^*})$, it follows from (b) above that all weakly countably $\mathcal{K}$-determined (e.g. weakly compactly generated) Banach spaces belong to $\mathcal{MS}$ (they have Gul'ko compact dual unit ball, see [17, Chapter 7]). On the other hand, taking into account (f) and the fact that B_{X^*} is a Radon-Nikodým compactum whenever X is an Asplund generated Banach space (i.e. there exist an Asplund Banach space Z and an operator from Z to X with dense range), see [17, Chapter 1], we conclude that all Asplund generated spaces (and their subspaces) belong to $\mathcal{MS}$. Moreover, under MA and ¬CH, the class $\mathcal{MS}$ contains all Banach spaces with angelic dual unit ball, as well as all $C(K)$ spaces with property (C) (and their subspaces).

The following easy observation will be used in the proofs of Theorem 3.3 and Proposition 3.9.

Lemma 3.2. *Let L be a compact space and X a closed subspace of $C(L)$. Then there is a continuous mapping from L onto a w^*-compact norming set $K \subset B_{X^*}$.*

Proof. Notice that $D := \{\delta_t : t \in L\}$ is a w^*-compact subset of $B_{C(L)^*}$ that is homeomorphic to L. The restriction mapping $r : B_{C(L)^*} \longrightarrow B_{X^*}$ is w^*-w^*-continuous and maps D onto a w^*-compact norming set $K := \{\delta_t|_X : t \in L\} \subset B_{X^*}$. Hence there is a continuous map from L onto K and the proof is over. $\square$

From now on, given a Banach space X and a w^*-compact norming set $K \subset B_{X^*}$, we denote by i_K the natural isometry from X into $C(K)$ given by $i_K(x)(x^*) := x^*(x)$.

Theorem 3.3. *Let X be a Banach space.*

(i) *If X belongs to $\mathcal{MS}$, then each absolutely summing operator defined on X has separable range.*
(ii) *Assume that $X = C(L)$ for some compact space L. If each absolutely summing operator defined on X has separable range, then L belongs to MS.*

Proof. Part (ii) has already been established at the beginning of this section. For the proof of (i) we can suppose without loss of generality that X is a closed subspace of $C(L)$, where L is a compact space belonging to MS. According to Lemma 3.2, there is a continuous mapping from L onto a w^*-compact norming set $K \subset B_{X^*}$. As we already mentioned, MS is closed under continuous images, hence K belongs to MS. Now let $u : X \longrightarrow Y$ be an absolutely summing operator between Banach spaces. Pietsch's Factorization Theorem, see [12, Theorem 2.13], ensures us the existence of $\nu \in M^+(K)$, a closed subspace $Z \subset L^1(\nu)$ and an operator $v : Z \longrightarrow Y$ such that $j_\nu(i_K(X)) \subset Z$ and $u = v \circ j_\nu \circ i_K$. Since $L^1(\nu)$ is separable, the same holds for Z and therefore $u(X)$ is separable. The proof is complete. $\square$

Corollary 3.4. *Let (Ω, Σ, μ) be a complete probability space and $u : X \longrightarrow Y$ an absolutely summing operator between Banach spaces. If X belongs to $\mathcal{MS}$, then $u \circ f$ is Bochner integrable for every $f \in \mathcal{D}(\mu, X)$.*

Remark. Our Corollary 3.4 improves the *if* part of [30, Corollary 7], where an analogous result is proved for Pettis integrable functions in the particular case of a Radon probability measure μ on a compact space Ω and a superreflexive (hence reflexive) Banach space X.

3.2. **Properly measurable functions.** We next study the composition of properly measurable functions with absolutely summing operators. Properly measurable vector-valued functions and stable families of real-valued measurable functions were thoroughly studied in [36], mostly in connection with the Pettis integral. Recall that a family $\mathcal{H}$ of real-valued functions defined on a complete probability space (Ω, Σ, μ) is *stable* iff for each $A \in \Sigma$ with $\mu(A) > 0$ and each pair of real numbers $\alpha < \beta$ there exist $k, l \in \mathbb{N}$ such that

$$\mu^*_{k+l}\Big(\bigcup_{h \in \mathcal{H}} \big(\{h < \alpha\}^k \times \{h > \beta\}^l\big) \cap A^{k+l}\Big) < \mu(A)^{k+l},$$

where μ_{k+l} is the product of $k + l$ copies of μ. In particular, $\mathcal{H}$ is made up of measurable functions. A well-known result of Talagrand [36, Theorem 9-5-2] states that *if $\mathcal{H} \subset \mathbb{R}^\Omega$ is stable, then the identity map $(\mathcal{H}, \mathfrak{T}_p) \longrightarrow (\mathcal{H}, \mathfrak{T}_m)$ is continuous*, where $\mathfrak{T}_p$ is the topology of pointwise convergence and $\mathfrak{T}_m$ is the topology of convergence in measure. Under Axiom L (a weakening of MA, see [36, p. 14] for the definition), every pointwise relatively compact sequence of real-valued measurable functions defined on a perfect complete probability space is stable, see [36, Section 9.3].

Recall that a function f defined on Ω with values in a Banach space X is *properly measurable* iff the family

$$Z_f = \{x^* \circ f : \ x^* \in B_{X^*}\}$$

is stable. In view of the above, such a function belongs to $\mathcal{P}_c(\mu, X)$ whenever Z_f is a uniformly integrable subset of $\mathcal{L}^1(\mu)$, see [36, Theorem 6-1-2].

Inspired by some ideas in [2], in Theorem 3.5 below we apply a result of Talagrand (Theorem 10-2-1 in [36]) linking stability and joint measurability in order to study the Bochner integrability of the composition of vector-valued functions that are "almost" properly measurable with absolutely summing operators.

Theorem 3.5. *Let (Ω, Σ, μ) be a complete probability space, X a Banach space and $f : \Omega \longrightarrow X$ a Pettis integrable function. Let us consider the following statements:*

(i) *there is a w^*-compact norming set $K \subset B_{X^*}$ such that for each $\nu \in M^+(K)$ the family $\{x^* \circ f : \ x^* \in \mathrm{supp}(\nu)\}$ is stable;*

(ii) *there is a w^*-compact norming set $K \subset B_{X^*}$ such that for each $\nu \in M^+(K)$ the function*

$$f_K : \Omega \times K \longrightarrow \mathbb{R}, \quad f_K(\omega, x^*) := (x^* \circ f)(\omega),$$

is $\mu \times \nu$-measurable;

(iii) *for each absolutely summing operator u defined on X the composition $u \circ f$ is Bochner integrable.*

Then (i)⇒(ii)⇒(iii). Moreover, under Axiom L all the statements are equivalent provided that μ is perfect (in this case, (i) and (ii) hold for any w^*-compact norming set $K \subset B_{X^*}$).

Proof. Assume that (i) holds, fix $\nu \in M^+(K)$ and write $F := \mathrm{supp}(\nu)$. The function $f_K|_{\Omega\times F} : \Omega \times F \longrightarrow \mathbb{R}$ is measurable in the first variable and continuous in the second one. Since, in addition, the family

$$\{f_K|_{\Omega\times F}(\cdot, x^*) = x^* \circ f : \ x^* \in F\}$$

is stable, Theorem 10-2-1 in [36] applies to conclude that $f_K|_{\Omega\times F}$ is $\mu \times \nu_F$-measurable. Therefore f_K is $\mu \times \nu$-measurable. This proves the implication (i)⇒(ii).

Let us turn to the proof of (ii)⇒(iii). Fix a w^*-compact norming set $K \subset B_{X^*}$ satisfying the conditions in (ii) and consider an absolutely summing operator u defined on X with values in another Banach space Y. By Lemma 2.2, in order to check that $u \circ f$ is Bochner integrable it suffices to check that $u \circ f$ is strongly measurable.

By Pietsch's Factorization Theorem there exist $\nu \in M^+(K)$, a closed subspace $Z \subset L^1(\nu)$ and an operator $v : Z \longrightarrow Y$ such that $j_\nu(i_K(X)) \subset Z$ and $u = v \circ j_\nu \circ i_K$. Write $F := \mathrm{supp}(\nu)$ and consider the restriction operator $R : C(K) \longrightarrow C(F)$ and a linear isometry $I : L^1(\nu_F) \longrightarrow L^1(\nu)$ such that $j_\nu = I \circ j_{\nu_F} \circ R$. The function $g := R \circ i_K \circ f : \Omega \longrightarrow C(F)$ is Pettis integrable. Since F is the support of ν, Rosenthal's theorem (see [36, Theorem 12-1-5]) ensures that every weakly compact subset of $C(F)$ is separable. In particular, $\nu_g(\Sigma)$ is separable (the range of any countably additive vector measure defined on a σ-algebra is relatively weakly compact, see [13, Corollary 7, p. 14]). Then there is a sequence of simple functions $s_n : \Omega \longrightarrow C(F)$ such that

(α) $\{h \circ s_n : \ h \in B_{C(F)^*}, \ n \in \mathbb{N}\}$ is uniformly integrable,

(β) for each $h \in C(F)^*$ we have $\lim_n h \circ s_n = h \circ g$ μ-a.e.,

see [36, Theorem 5-3-2]. Define $g_n = g - s_n$ for every $n \in \mathbb{N}$ and notice that the family $\mathcal{F} := \{\delta_{x^*} \circ g_n : \ x^* \in F, \ n \in \mathbb{N}\}$ is uniformly integrable, by (α) and the fact that Z_g is uniformly integrable (because g is Pettis integrable, see [36, Theorem 4-2-2]).

Since f_K is $\mu \times \nu$-measurable, the restriction $f_K|_{\Omega\times F}$ is $\mu \times \nu_F$-measurable. On the other hand, given $n \in \mathbb{N}$, it is easy to see that the function

$$\Omega \times F \longrightarrow \mathbb{R}, \quad (\omega, x^*) \mapsto (\delta_{x^*} \circ s_n)(\omega),$$

is $\mu \times \nu_F$-measurable. Therefore the same holds for the function

$$\Omega \times F \longrightarrow \mathbb{R}, \quad (\omega, x^*) \mapsto (\delta_{x^*} \circ g_n)(\omega) = f_K(\omega, x^*) - (\delta_{x^*} \circ s_n)(\omega).$$

Since the family $\mathcal{F}$ is $\|\cdot\|_1$-bounded, we have

$$\int_F \Big(\int_\Omega |(\delta_{x^*} \circ g_n)(\omega)| \, d\mu(\omega) \Big) \, d\nu(x^*) < \infty,$$

and therefore we can apply Fubini's theorem obtaining

$$\int_F \Big(\int_\Omega |(\delta_{x^*} \circ g_n)(\omega)| \, d\mu(\omega) \Big) \, d\nu(x^*) = \int_\Omega \Big(\int_F |(\delta_{x^*} \circ g_n)(\omega)| \, d\nu(x^*) \Big) \, d\mu(\omega) \tag{2}$$

for every $n \in \mathbb{N}$. Define $G_n \in \mathcal{L}^1(\nu_F)$ by $G_n(x^*) = \int_\Omega |(\delta_{x^*} \circ g_n)(\omega)| \, d\mu(\omega)$. Since $\mathcal{F}$ is uniformly integrable and for each $x^* \in F$ we have $\lim_n \delta_{x^*} \circ g_n = 0$ μ-a.e. (by (β)), Vitali's convergence theorem implies that (G_n) converges pointwise to 0, and thus Lebesgue's dominated convergence theorem applied to the uniformly bounded sequence (G_n) yields $\lim_n \|G_n\|_1 = 0$. Now (2) applies to conclude that

$$\lim_n \int_\Omega \Big(\int_F |(\delta_{x^*} \circ g_n)(\omega)| \, d\nu(x^*) \Big) \, d\mu(\omega) = 0.$$

Therefore the sequence (H_n) in $\mathcal{L}^1(\mu)$ defined by $H_n(\omega) = \int_F |(\delta_{x^*} \circ g_n)(\omega)| \, d\nu(x^*)$ converges to 0 in the norm $\|\cdot\|_1$. Thus there is a subsequence (H_{n_k}) converging to 0 μ-a.e.

Define the operator $Q := I \circ j_{\nu_F} : C(F) \longrightarrow L^1(\nu)$ and observe that

$$\|(Q \circ g_{n_k})(\omega)\| = \int_F |(\delta_{x^*} \circ g_{n_k})(\omega)| \, d\nu(x^*) = H_{n_k}(\omega) \quad \text{for every } \omega \in \Omega \text{ and } k \in \mathbb{N}.$$

Thus $\lim_k \|Q \circ s_{n_k} - Q \circ g\| = 0$ μ-a.e. and therefore $Q \circ g$ is strongly measurable. To finish the proof of (ii)⇒(iii) notice that $Q \circ g$ takes its values in Z and that $v \circ Q \circ g = u \circ f$, thus $u \circ f$ is also strongly measurable, as required.

Finally, assume that f satisfies (iii), and fix *any* w^*-compact norming set $K \subset B_{X^*}$ and $\nu \in M^+(K)$. Let us define $F := \mathrm{supp}(\nu)$ and consider the restriction operator $R : C(K) \longrightarrow C(F)$. The composition $j_{\nu_F} \circ R \circ i_K$ is absolutely summing, hence $g := j_{\nu_F} \circ R \circ i_K \circ f$ is strongly measurable, i.e. g is $\Sigma - \mathrm{Borel}(L^1(\nu_F), \|\cdot\|_1)$-measurable and essentially separably valued (see [6, Appendix E]). If we assume, in addition, that μ is perfect and Axiom L holds, then the criterion in [36, Theorem 10-2-4] can be applied to $f_K|_{\Omega \times F}$ to deduce that $\{x^* \circ f : x^* \in F\}$ is stable. The proof is over. □

Corollary 3.6. *Let (Ω, Σ, μ) be a complete probability space, $u : X \longrightarrow Y$ an absolutely summing operator between Banach spaces and $f : \Omega \longrightarrow X$ a function.*

(i) *If f is properly measurable, then $u \circ f$ is strongly measurable.*
(ii) *If f is properly measurable and Dunford integrable, then $u \circ f$ is Bochner integrable.*

Proof. The proof of (i) is as follows. Since f is scalarly measurable, there is a non-negative measurable function h on Ω such that for each $x^* \in B_{X^*}$ we have $|x^* \circ f| \leq h$ μ-a.e. (see e.g. [32, Proposition 3.1]). Fix $n \in \mathbb{N}$ and define $A_n := \{\omega \in \Omega : n-1 \leq h(\omega) < n\} \in \Sigma$. Then the family of restrictions $Z_{f|_{A_n}}$ is a stable uniformly integrable subset of $\mathcal{L}^1(\mu_{A_n})$, hence $f|_{A_n}$ is Pettis integrable, by [36, Theorem 6-1-2], and an appeal to Theorem 3.5 ensures that $u \circ f|_{A_n}$ is strongly measurable. Since $n \in \mathbb{N}$ is arbitrary, it follows that $u \circ f$ is strongly measurable, as required. Part (ii) now follows immediately from (i) and Lemma 2.2. □

Recall that a function f defined on a complete probability space (Ω, Σ, μ) with values in a Banach space X is *Talagrand integrable* [24] iff f satisfies the *law of large numbers*, that is, there exists $\lim_n (1/n) \sum_{i=1}^n f(\omega_i)$ (in norm) for almost every $(\omega_i)_{i \in \mathbb{N}} \in \Omega^{\mathbb{N}}$, where $\Omega^{\mathbb{N}}$ is given its product probability. Equivalently, f is properly measurable and $\|f\|$ has an integrable majorant, see [37]. Every Talagrand integrable function is Pettis integrable.

Corollary 3.7. *Let (Ω, Σ, μ) be a complete probability space and $u : X \longrightarrow Y$ an absolutely summing operator between Banach spaces. If $f : \Omega \longrightarrow X$ is a Talagrand integrable function, then $u \circ f$ is Bochner integrable.*

As we have already mentioned, under Axiom L, every pointwise relatively compact sequence of real-valued measurable functions defined a perfect complete probability space is stable. Since stability is preserved by taking pointwise closures, we get the following

Corollary 3.8 (Axiom L). *Let (Ω, Σ, μ) be a perfect complete probability space and let $u : X \longrightarrow Y$ be an absolutely summing operator between Banach spaces, where B_{X^*} is w^*-separable. Then $u \circ f$ is Bochner integrable for every $f \in \mathcal{D}(\mu, X)$.*

For the proof of the following proposition it is useful to recall that property (M) is preserved by continuous mappings. Indeed, let $\phi : L \longrightarrow K$ be a continuous surjection between compact spaces, where L has property (M), and fix $\nu \in M^+(K)$. Then there exists $\nu' \in M^+(L)$ such that $\nu'(\phi^{-1}(B)) = \nu(B)$ for every $B \in \mathrm{Borel}(K)$, see e.g. [36, 1-2-5]. Since $\mathrm{supp}(\nu')$ is separable, the same holds true for $\phi(\mathrm{supp}(\nu')) = \mathrm{supp}(\nu)$. It follows that K has property (M).

Proposition 3.9. *Let (Ω, Σ, μ) be a complete probability space and $u : X \longrightarrow Y$ an absolutely summing operator between Banach spaces, where X is isomorphic to a subspace of a weakly Lindelöf space of the form $C(L)$. Then $u \circ f$ is Bochner integrable for every $f \in \mathcal{D}(\mu, X)$.*

Proof. Obviously we can suppose without loss of generality that X is a subspace of $C(L)$. Now Lemma 3.2 can be applied to find a w^*-compact norming set $K \subset B_{X^*}$ and a continuous map from L onto K. Since $C(L)$ is weakly Lindelöf, it has property (C). It follows from a result of Pol (see [18, Exercise 12.31]) that L has property (M), and therefore the same also holds for its continuous image K (see the comments preceding this proposition).

Fix $f \in \mathcal{D}(\mu, X)$. Since $C(L)$ is weakly Lindelöf, X is weakly Lindelöf too and therefore X is measure-compact in its weak topology. Hence the scalarly measurable function f is scalarly equivalent to a strongly measurable one $h : \Omega \longrightarrow X$, see [16, Proposition 5.4]. By Lemma 2.2 the composition $u \circ h$ is Bochner integrable, so in order to finish the proof it suffices to check that $u \circ (f - h)$ is Bochner integrable. Define $h' := f - h$ and observe that each countable subset of $Z_{h'}$ is stable (because h' is scalarly null). Since stability is preserved by taking pointwise closures and K has property (M), h' fulfills condition (i) in Theorem 3.5 and therefore $u \circ h'$ is Bochner integrable. The proof is over. □

3.3. **Birkhoff integrable functions.** The notion of integrability introduced by Garrett Birkhoff in [3] (that lies strictly between Bochner and Pettis integrability) has been widely studied pretty recently in [5, 20, 35]. Recall that a function $f : \Omega \longrightarrow X$ defined on a complete probability space (Ω, Σ, μ) with values in a Banach space X is *Birkhoff integrable* iff for every $\varepsilon > 0$ there is a countable partition (A_n) of Ω in Σ such that

$$\Big\|\sum_n \mu(A_n) f(t_n) - \sum_n \mu(A_n) f(t'_n)\Big\| \leq \varepsilon$$

for arbitrary choices $t_n, t'_n \in A_n$, the series involved being unconditionally convergent. It was proved in [20] that every Birkhoff integrable function is properly measurable. Thus, according to Corollary 3.6, the composition of such a function with an absolutely summing operator is Bochner integrable. We provide in Theorem 3.11 a simpler proof of this fact that is based on the following well-known lemma (whose proof we include here for the sake of completeness).

Lemma 3.10. *Let $f : \Omega \longrightarrow X$ be a function defined on a complete probability space (Ω, Σ, μ) with values in a Banach space X. The following conditions are equivalent:*

(i) *f is strongly measurable;*
(ii) *for each $\varepsilon > 0$ and each $E \in \Sigma$ with $\mu(E) > 0$ there is $B \subset E$, $B \in \Sigma$ with $\mu(B) > 0$, such that $\sup_{t,t' \in B} \|f(t) - f(t')\| \leq \varepsilon$.*

Proof. It is known (see [13, Corollary 3, p. 42]) that (i) is equivalent to

(iii) *for each $\varepsilon > 0$ there exist a countable partition (E_n) of Ω in Σ and a sequence (x_n) in X such that the function $g : \Omega \longrightarrow X$ given by $g = \sum_n x_n \chi_{E_n}$ satisfies $\|f - g\| \leq \varepsilon$ μ-a.e.*

Therefore, it suffices to check that (ii) and (iii) are equivalent. The implication (iii)⇒(ii) is straightforward. Conversely, assume that (ii) holds and fix $\varepsilon > 0$. Let $\mathcal{S}_\varepsilon$ be the (non-empty) set of all countable families (E_n) made up of pairwise disjoint elements of Σ with

$\mu(E_n) > 0$ such that $\sup_{t,t' \in E_n} \|f(t) - f(t')\| \leq \varepsilon$ for every n. It is easy to see that $\mathcal{S}_\varepsilon$, with the order given by the inclusion, satisfies that every totally ordered subset has an upper bound (use that any family of pairwise disjoint elements of Σ with positive measure must be countable). An appeal to Zorn's lemma yields a maximal element $(A_n) \in \mathcal{S}_\varepsilon$. By maximality and (ii) we have $\mu(\Omega \setminus \cup_n A_n) = 0$. Define $E_1 := \Omega \setminus \cup_n A_n$ and $E_{n+1} := A_n$ for every n. Fix $t_n \in E_n$ and define $x_n := f(t_n)$ for every n. Now $g = \sum_n x_n \chi_{E_n}$ satisfies the requirement in (iii) and the proof is finished. $\square$

Theorem 3.11. *Let (Ω, Σ, μ) be a complete probability space and $u : X \longrightarrow Y$ an absolutely summing operator between Banach spaces. If $f : \Omega \longrightarrow X$ is a Birkhoff integrable function, then $u \circ f$ is Bochner integrable.*

Proof. By Lemma 2.2, in order to prove that $u \circ f$ is Bochner integrable it suffices to check that $u \circ f$ is strongly measurable. To this end, we will apply the characterization isolated in Lemma 3.10. Fix $\varepsilon > 0$ and $A \in \Sigma$ with $\mu(A) > 0$. Since the restriction $f|_A$ is Birkhoff integrable, there is a countable partition (A_n) of A in Σ such that

$$2\pi(u)\Big\|\sum_{n=1}^{m}\big(\mu(A_n)f(t_n) - \mu(A_n)f(t_n')\big)\Big\| < \varepsilon$$

for arbitrary choices $t_n, t_n' \in A_n$ and every $m \in \mathbb{N}$. Then inequality (1) on page 3 yields

$$\sum_{n=1}^{m} \mu(A_n)\|u \circ f(t_n) - u \circ f(t_n')\| < \varepsilon$$

for arbitrary choices $t_n, t_n' \in A_n$ and every $m \in \mathbb{N}$. It follows that there is some A_n with $\mu(A_n) > 0$ for which $\sup_{t,t' \in A_n} \|u \circ f(t) - u \circ f(t')\| \leq \varepsilon$. An appeal to Lemma 3.10 ensures that $u \circ f$ is strongly measurable and the proof is over. $\square$

Remark. The same result was obtained in [30, Corollary 8] in the particular case of a compact Radon probability space.

3.4. **McShane integrable functions.** In this subsection we prove that the composition of a McShane integrable function with an absolutely summing operator is always Bochner integrable (Theorem 3.13). The McShane integral of vector-valued functions has caught the attention of many authors in recent years, see [9, 20, 21, 24] and the references therein, and to recall its definition we need some terminology.

Let $(\Omega, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon probability space [23, Chapter 41], i.e. (Ω, Σ, μ) is a complete probability space and $\mathfrak{T} \subset \Sigma$ is a topology on Ω such that:

(i) $\mu(E) = \sup\{\mu(C) : C \subset E,\ C \text{ closed}\}$ for every $E \in \Sigma$;
(ii) $\mu(\bigcup \mathcal{G}) = \sup\{\mu(G) : G \in \mathcal{G}\}$ for every (non-empty) upwards directed family $\mathcal{G} \subset \mathfrak{T}$.

(For instance, every Radon probability space is quasi-Radon, see [23, 416A].) A *generalized McShane partition* of Ω is a sequence $\{(E_i, s_i)\}_{i \in \mathbb{N}}$ where $\{E_i\}_{i \in \mathbb{N}}$ is a family of pairwise disjoint measurable sets such that $\mu(\Omega \setminus \bigcup_{i \in \mathbb{N}} E_i) = 0$ and $s_i \in \Omega$ for every $i \in \mathbb{N}$. A *partial McShane partition* of Ω is a countable collection $\{(E_i, s_i)\}_{i \in I}$ where $\{E_i\}_{i \in I}$ is a family of pairwise disjoint measurable sets and $s_i \in \Omega$ for every $i \in I$. A *gauge* on Ω is a function $\delta : \Omega \longrightarrow \mathfrak{T}$ such that $\omega \in \delta(\omega)$ for every $\omega \in \Omega$, and a partial McShane partition $\{(E_i, s_i)\}_{i \in I}$ of Ω is *subordinate* to δ iff $E_i \subset \delta(s_i)$ for every $i \in I$. Recall that for every gauge δ on Ω there is a generalized McShane partition of Ω subordinate to δ, see [21, 1B(d)].

A function f defined on Ω with values in a Banach space X is *McShane integrable*, with McShane integral $x \in X$, see [21, 1A], iff for every $\varepsilon > 0$ there is a gauge δ on Ω such that

$$\limsup_n \Big\|\sum_{i=1}^{n} \mu(E_i) f(s_i) - x\Big\| \leq \varepsilon$$

for every generalized McShane partition $\{(E_i, s_i)\}_{i\in\mathbb{N}}$ of Ω subordinate to δ. In this context, every Birkhoff integrable function is McShane integrable, see [20, Proposition 3], and every McShane integrable function is Pettis integrable, see [21, 1Q] (and the respective integrals coincide).

The following stronger notion (which appears naturally in the proof of Theorem 3.13) was studied in [8]. A function $f : \Omega \longrightarrow X$ is *variationally McShane integrable* iff it is Pettis integrable and for every $\varepsilon > 0$ there is a gauge δ on Ω such that

$$\sum_{i=1}^{\infty} \|\mu(E_i)f(s_i) - \nu_f(E_i)\| \leq \varepsilon$$

for every generalized McShane partition $\{(E_i, s_i)\}_{i\in\mathbb{N}}$ of Ω subordinate to δ. For our purposes here, the key fact is that *every variationally McShane integrable function is strongly measurable*, see [8, Lemma 3]. The proof of this result given in [8] relies on the norm relative compactness of the range of the indefinite integral of any McShane integrable function f, which Fremlin deduced in [21, 3E] from the fact that each countable subset of Z_f is stable (see [21, 3C]). In Proposition 3.12 below we give a short and more elementary proof of Fremlin's result that does not involve techniques of stable families of measurable functions.

Proposition 3.12. *Let $(\Omega, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon probability space, X a Banach space and $f : \Omega \longrightarrow X$ a McShane integrable function. Then $\nu_f(\Sigma)$ is norm relatively compact.*

Proof. Notice that if $g = \sum_{i=1}^n x_i\chi_{A_i}$ is a simple function, then

$$\nu_g(\Sigma) = \Big\{\sum_{i=1}^{n} \mu(A_i \cap A)x_i : \ A \in \Sigma\Big\}$$

is totally bounded. Thus in order to prove the proposition it suffices to check the following:

(+) for every $\varepsilon > 0$ there is a simple function $g : \Omega \longrightarrow X$ such that

$$\sup_{E\in\Sigma} \|\nu_f(E) - \nu_g(E)\| \leq \varepsilon.$$

To prove (+) fix $\varepsilon > 0$. By the Henstock-Saks lemma [21, 2B] there is a gauge δ on Ω such that

$$\Big\|\sum_{i=1}^{p} \mu(F_i)f(t_i) - \nu_f\Big(\bigcup_{i=1}^{p} F_i\Big)\Big\| \leq \frac{\varepsilon}{2} \tag{3}$$

for every partial McShane partition $\{(F_i, t_i) : 1 \leq i \leq p\}$ of Ω subordinate to δ. On the other hand, since ν_f is absolutely continuous with respect to μ, see [36, 4-2-2], there is $\eta > 0$ such that

$$\|\nu_f(A)\| \leq \frac{\varepsilon}{2} \quad \text{for every } A \in \Sigma \text{ with } \mu(A) \leq \eta. \tag{4}$$

Fix a generalized McShane partition $\{(E_i, s_i)\}_{i\in\mathbb{N}}$ of Ω subordinate to δ. Choose $n \in \mathbb{N}$ large enough such that $\mu(\Omega \setminus \bigcup_{i=1}^n E_i) \leq \eta$ and define $g := \sum_{i=1}^n f(s_i)\chi_{E_i}$. We claim that

$$\sup_{E\in\Sigma} \|\nu_f(E) - \nu_g(E)\| \leq \varepsilon. \tag{5}$$

Indeed, given $E \in \Sigma$, the collection $\{(E_i \cap E, s_i) : 1 \leq i \leq n\}$ is a partial McShane partition of Ω subordinate to δ and (3) implies

$$\Big\|\sum_{i=1}^{n} \mu(E_i \cap E)f(s_i) - \nu_f\Big(E \cap \Big(\bigcup_{i=1}^{n} E_i\Big)\Big)\Big\| \leq \frac{\varepsilon}{2},$$

which, in view of (4) and the choice of n, yields

$$\Big\|\sum_{i=1}^{n}\mu(E_i\cap E)f(s_i)-\nu_f(E)\Big\|\leq\Big\|\nu_f\Big(E\setminus\Big(\bigcup_{i=1}^{n}E_i\Big)\Big)\Big\|$$
$$+\Big\|\sum_{i=1}^{n}\mu(E_i\cap E)f(s_i)-\nu_f\Big(E\cap\Big(\bigcup_{i=1}^{n}E_i\Big)\Big)\Big\|\leq\varepsilon.$$

From the equality $\sum_{i=1}^{n}\mu(E_i\cap E)f(s_i)=\nu_g(E)$ we obtain $\|\nu_f(E)-\nu_g(E)\|\leq\varepsilon$. Since $E\in\Sigma$ is arbitrary, (5) holds and the proof is complete. □

Theorem 3.13. *Let $(\Omega,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon probability space and $u:X\longrightarrow Y$ an absolutely summing operator between Banach spaces. If $f:\Omega\longrightarrow X$ is a McShane integrable function, then $u\circ f$ is Bochner integrable.*

Proof. We claim that $u\circ f$ is variationally McShane integrable. Indeed, on the one hand, $u\circ f$ is McShane integrable and so it is Pettis integrable. On the other hand, given $\varepsilon>0$, the Henstock-Saks lemma [21, 2B] ensures the existence of a gauge δ on Ω such that

$$\Big\|\sum_{i=1}^{q}\big(\mu(F_i)f(t_i)-\nu_f(F_i)\big)\Big\|\leq\varepsilon$$

for every partial McShane partition $\{(F_i,t_i):1\leq i\leq q\}$ of Ω subordinate to δ. It follows from inequality (1) on page 3 that

$$\sum_{i=1}^{p}\Big\|\mu(E_i)(u\circ f)(s_i)-\nu_{u\circ f}(E_i)\Big\|=\sum_{i=1}^{p}\Big\|u\big(\mu(E_i)f(s_i)-\nu_f(E_i)\big)\Big\|\leq 2\pi(u)\varepsilon$$

for every partial McShane partition $\{(E_i,s_i):1\leq i\leq p\}$ of Ω subordinate to δ. Since $\varepsilon>0$ is arbitrary, $u\circ f$ is variationally McShane integrable and, in particular, strongly measurable. An appeal to Lemma 2.2 establishes that $u\circ f$ is Bochner integrable and the proof finishes. □

Remark. Our Theorem 3.13 generalizes the *if* part of [30, Theorem 5], where an analogous result is proved for compact Radon probability spaces.

Combining Theorem 3.13 with Theorem 3.5 we can deduce the following (partial) extension of [21, 3C]:

Corollary 3.14 (Axiom L)**.** *Let $(\Omega,\mathfrak{T},\Sigma,\mu)$ be a perfect quasi-Radon probability space, X a Banach space and $f:\Omega\longrightarrow X$ a McShane integrable function. Then for each $\nu\in M^+(B_{X^*})$ the family $\{x^*\circ f:x^*\in\operatorname{supp}(\nu)\}$ is stable.*

The results in [9] on the coincidence of Pettis and McShane integrability (that remain valid for functions defined on arbitrary quasi-Radon probability spaces) allow us to obtain the following proposition.

Proposition 3.15. *Let $(\Omega,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon probability space and $u:X\longrightarrow Y$ an absolutely summing operator between Banach spaces. Then $u\circ f$ is McShane integrable for every $f\in\mathcal{D}(\mu,X)$.*

Proof. Fix a Dunford integrable function $f:\Omega\longrightarrow X$. Since u is absolutely summing, u is also 2-summing (see [12, Theorem 2.8]) and therefore [12, Corollary 2.16] applies to ensure the existence of $\nu\in M^+(B_{X^*})$ and an operator $v:L^2(\nu)\longrightarrow Y$ such that $u=v\circ j\circ i_{B_{X^*}}$, where $j:C(B_{X^*})\longrightarrow L^2(\nu)$ is the "identity" operator (that maps each function to its equivalence class). Since $j\circ i_{B_{X^*}}\circ f$ is Dunford integrable and $L^2(\nu)$ is reflexive, $j\circ i_{B_{X^*}}\circ f$ is Pettis integrable. Using the fact that Pettis and McShane integrability coincide for functions with values in superreflexive spaces [9] we infer that $j\circ i_{B_{X^*}}\circ f$ is McShane integrable. Therefore $u\circ f$ is McShane integrable and the proof is over. □

4. EXAMPLES

We end up the paper with two examples showing that the composition of a Dunford integrable function with an absolutely summing operator is not always Bochner integrable.

The first one uses the Pettis integrable function whose indefinite integral does not have norm relatively compact range that Fremlin and Talagrand constructed in [25].

Example 4.1. *There exist a complete probability space $(\Omega, \mathcal{A}, \theta)$, a Pettis integrable function $f : \Omega \longrightarrow \ell^\infty$ and an absolutely summing operator u defined on ℓ^∞ with values in another Banach space Y such that $u \circ f$ is not Bochner integrable.*

Proof. The cardinality of a set S will be denoted by $|S|$. In the sequel we identify $\mathcal{P}(\mathbb{N})$ (the set of all subsets of $\mathbb{N}$) with $\{0,1\}^{\mathbb{N}}$ in the standard way

$$a \subset \mathbb{N} \longleftrightarrow \chi_a \in \{0,1\}^{\mathbb{N}} \text{ (the characteristic function of } a).$$

Let us denote by $(\{0,1\}^{\mathbb{N}}, \Sigma, \mu)$ the complete probability space obtained after completing the usual product probability measure on $\{0,1\}^{\mathbb{N}}$. From now on the term *measurable* will refer to this measure space. Recall that a free filter $F \subset \{0,1\}^{\mathbb{N}}$ is non-measurable if and only if $\mu^*(F) = 1$, see [36, Proposition 13-1-1].

Let us recall the definition of the so-called *Talagrand's measure space* $(\{0,1\}^{\mathbb{N}}, \overline{\Sigma}, \bar{\mu})$ (see sections 13-1 and 13-2 in [36]). The σ-algebra $\overline{\Sigma}$ (that contains Σ) is made up of all the sets $A \subset \{0,1\}^{\mathbb{N}}$ for which there exist $B \in \Sigma$ and a non-measurable free filter $F \subset \{0,1\}^{\mathbb{N}}$ such that $A \cap F = B \cap F$. The measure $\bar{\mu}$ is a (complete) extension of μ defined on $\overline{\Sigma}$ by saying that $\bar{\mu}(A) = \mu(B)$ whenever $A \in \overline{\Sigma}$, $B \in \Sigma$ and there is a non-measurable free filter $F \subset \{0,1\}^{\mathbb{N}}$ such that $A \cap F = B \cap F$.

The completion of the product probability space $(\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}, \overline{\Sigma} \otimes \overline{\Sigma}, \bar{\mu} \times \bar{\mu})$ will be denoted by $(\Omega, \mathcal{A}, \theta)$. It was shown in [25, 2C] (alternatively see [36, Theorem 4-2-5]) that the function

$$f : \Omega \longrightarrow \ell^\infty, \quad f(a,b) := \chi_a - \chi_b,$$

is Pettis integrable with respect to θ (and that $\nu_f(\mathcal{A})$ is not norm relatively compact).

Fix any measurable finitely additive functional $\lambda : \mathcal{P}(\mathbb{N}) \longrightarrow [0,1]$ with $\lambda(\mathbb{N}) = 1$ and vanishing on all finite sets. For instance, we can take

$$\lambda(a) = \lim_n \frac{|\{m \in a : \ m \leq n\}|}{n},$$

see [23, 464J(b)]. Denote by $\beta\mathbb{N}$ the Stone-Cech compactification of $\mathbb{N}$ (with the discrete topology) and let ν be the unique element of $M^+(\beta\mathbb{N})$ such that $\int_{\beta\mathbb{N}} \chi_a \, d\nu = \lambda(a)$ for every $a \subset \mathbb{N}$ (from now on we identify ℓ^∞ and $C(\beta\mathbb{N})$). Let us consider the absolutely summing operator $u = j_\nu : \ell^\infty \longrightarrow L^1(\nu)$. We will check that *the composition $u \circ f$ is not strongly measurable with respect to θ*.

To prove this, fix $A \in \overline{\Sigma} \otimes \overline{\Sigma}$ with $(\bar{\mu} \times \bar{\mu})(A) = 1$. Then there is $d \subset \mathbb{N}$ such that $\bar{\mu}(A^d) = 1$, where $A^d = \{a \subset \mathbb{N} : (a,d) \in A\}$. From the definition of $\bar{\mu}$ it follows that there exist $B \in \Sigma$ and a non-measurable free filter $F \subset \{0,1\}^{\mathbb{N}}$ such that $A^d \cap F = B \cap F$ and $\mu(B) = \bar{\mu}(A^d) = 1$. Since F is non-measurable, we have $\mu^*(F) = 1$ and therefore $\mu^*(B \cap F) = 1$. On the other hand, since λ is a measurable bounded finitely additive functional vanishing on finite sets, for each $b \subset \mathbb{N}$ we have

$$\lambda(a \cap b) = \frac{\lambda(b)}{2} \quad \text{for } \mu\text{-almost all } a \subset \mathbb{N}, \tag{6}$$

see [25, 1J]. The fact that $\mu^*(B \cap F) = 1$ and (6) can now be used to construct by transfinite induction a set $\{a_\alpha : \alpha < \omega_1\} \subset B \cap F = A^d \cap F$ such that

$$\lambda\Big(\bigcap_{\alpha \in I} a_\alpha\Big) = \frac{1}{2^{|I|}} \tag{7}$$

for every non-empty finite set $I \subset \omega_1$. Finally, for each pair $\alpha, \beta < \omega_1$, $\alpha \neq \beta$, we have $(a_\alpha, d), (a_\beta, d) \in A$, and (7) yields

$$\|(u \circ f)(a_\alpha, d) - (u \circ f)(a_\beta, d)\| = \int_{\beta\mathbb{N}} |f(a_\alpha, d) - f(a_\beta, d)| \, d\nu$$
$$= \int_{\beta\mathbb{N}} |\chi_{a_\alpha} - \chi_{a_\beta}| \, d\nu = \int_{\beta\mathbb{N}} \chi_{(a_\alpha \triangle a_\beta)} \, d\nu = \lambda(a_\alpha \triangle a_\beta) = \frac{1}{2}.$$

Hence $(u \circ f)(A)$ is not separable. Since A was arbitrarily chosen among all the elements of $\overline{\Sigma} \otimes \overline{\Sigma}$ of measure 1, we infer that $u \circ f$ is not strongly measurable with respect to θ. The proof is finished. $\square$

Recall that a cardinal κ is *of measure zero* if there is no probability measure μ on $\mathcal{P}(\kappa)$ (the set of all subsets of κ) such that $\mu(\{\alpha\}) = 0$ for every $\alpha < \kappa$. It is consistent with ZFC that every cardinal is of measure zero. For a detailed account on measure zero cardinals we refer the reader to [23, §438] and the references therein.

Example 4.2. *Assume that there is a cardinal κ that is not of measure zero. Then there exist a complete probability space (Ω, Σ, μ), a Dunford integrable function $f : \Omega \longrightarrow \ell^1(\kappa)$ and an absolutely summing operator $u : \ell^1(\kappa) \longrightarrow \ell^2(\kappa)$ such that $u \circ f$ is not Bochner integrable.*

Proof. There is a probability measure μ on $\mathcal{P}(\kappa)$ such that $\mu(\{\alpha\}) = 0$ for every $\alpha < \kappa$. Define $f : \kappa \longrightarrow \ell^1(\kappa)$ by $f(\alpha) := e_\alpha$, where $e_\alpha(\beta) = \delta_{\alpha,\beta}$ (the Dirac delta) for every $\alpha, \beta < \kappa$. Clearly f is bounded and scalarly measurable, hence Dunford integrable. On the other hand, it is well known that the "identity" operator $u : \ell^1(\kappa) \longrightarrow \ell^2(\kappa)$ is absolutely summing, see [12, Theorem 3.4]. Finally, observe that $\|(u \circ f)(\alpha) - (u \circ f)(\beta)\| = \sqrt{2}$ whenever $\alpha, \beta < \kappa$, $\alpha \neq \beta$. Hence for each $A \subset \kappa$ with $\mu(A) > 0$ the set $(u \circ f)(A)$ is not separable (because A is uncountable) and therefore $u \circ f$ is not strongly measurable. The proof is finished. $\square$

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