UNIVERSAL BIRKHOFF INTEGRABILITY IN DUAL BANACH SPACES

JOSÉ RODRÍGUEZ

ABSTRACT. We show that some classical results on universal Pettis integrability in dual Banach spaces can be formulated in terms of the Birkhoff integral, thanks to the link between Birkhoff integrability and the Bourgain property.

1. Introduction and preliminaries

It is well known that a bounded function $f : [0,1] \longrightarrow \mathbb{R}$ is *Riemann integrable* if and only if for each $\varepsilon > 0$ there is a finite partition of $[0,1]$ into intervals, say $I_1, \dots, I_n$, such that

$$\sum_{i=1}^{n} \lambda(I_i) \sup(f(I_i)) - \sum_{i=1}^{n} \lambda(I_i) \inf(f(I_i)) \leq \varepsilon$$

(where λ denotes the Lebesgue measure); in this case, the *Riemann integral* of f is the only point in the intersection

$$\bigcap \Big\{ \Big[\sum_{i=1}^{n} \lambda(I_i) \inf(f(I_i)), \sum_{i=1}^{n} \lambda(I_i) \sup(f(I_i))\Big] : \{I_1, \dots, I_n\} \text{ is a finite partition of } [0,1] \text{ into intervals}\Big\}.$$

As Fréchet [11] pointed out, the *Lebesgue integral* can be obtained in a similar fashion by replacing the intervals with arbitrary Lebesgue measurable sets. Inspired by these ideas, Birkhoff [3] proposed the following definition of integral for functions defined on a complete probability space (Ω, Σ, μ) with values in a Banach space $(Y, \|\cdot\|)$.

Definition 1. *Let $f : \Omega \longrightarrow Y$ be a bounded function. We say that f is Birkhoff integrable if for each $\varepsilon > 0$ there is a finite partition of Ω in Σ, say $A_1, \dots, A_n$, such that*

$$\Big\| \sum_{i=1}^{n} \mu(A_i) f(t_i) - \sum_{i=1}^{n} \mu(A_i) f(t_i') \Big\| \leq \varepsilon$$

for arbitrary choices $t_i, t_i' \in A_i$, $1 \leq i \leq n$. In this case, the Birkhoff integral of f is the only point in the intersection

$$\bigcap \Big\{ \overline{\mathrm{co}} \Big\{ \sum_{i=1}^{n} \mu(A_i) f(t_i) : \ t_i \in A_i \textit{ for every } 1 \leq i \leq n \Big\} : \{A_1, \dots, A_n\} \textit{ is a finite partition of } \Omega \textit{ in } \Sigma \Big\}.$$

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In fact, the Birkhoff integral can also be defined for non necessarily bounded functions, namely, by replacing finite partitions by *countable* ones and requiring that the countable Riemann sums obtained in this way are *unconditionally* convergent, see [3]. However, as regards Birkhoff integrability, in this paper we shall deal with bounded functions only.

It is worth mentioning that Birkhoff integrability lies strictly between Bochner and Pettis integrability (see [3], [20] and [21]), and that the Birkhoff integral of any Birkhoff integrable function coincides with its Pettis integral. Moreover, Birkhoff and Pettis integrability are equivalent for functions with values in separable Banach spaces (see [20]).

Unlike the Bochner and Pettis integrals, the Birkhoff integral had hardly been studied until a few years ago, in spite of playing a relevant role in the setting of vector integration, see [5], [12], [26] and [27]. In [5] (joint work with B. Cascales) we have analyzed the Birkhoff integrability of a vector-valued function $f : \Omega \longrightarrow Y$ in terms of the pointwise compact family of real-valued functions

$$Z_f = \{\langle y^*, f\rangle : \ y^* \in B_{Y^*}\} \subset \mathbb{R}^\Omega$$

and certain distinguished subfamilies of Z_f; for instance, when Y is the dual of a Banach space X, one can also look at the subfamily

$$Z_{f,B_X} = \{\langle f, x\rangle : \ x \in B_X\} \subset \mathbb{R}^\Omega.$$

The following characterization (see [5, Corollary 2.5]) will play a fundamental role in the sequel. It is an improvement of a classical result of Riddle and Saab [23], who proved the implication (ii)⇒(i) with 'Pettis' instead of 'Birkhoff'.

Theorem 1. *Let $f : \Omega \longrightarrow X^*$ be a bounded function. The following conditions are equivalent:*

(i) *f is Birkhoff integrable;*
(ii) *Z_{f,B_X} has the Bourgain property.*

Recall that a family $\mathcal{H} \subset \mathbb{R}^\Omega$ has the *Bourgain property* [23] if for every $\varepsilon > 0$ and every $A \in \Sigma$ with $\mu(A) > 0$ there are $A_1, \dots, A_n \subset A$, $A_i \in \Sigma$ with $\mu(B_i) > 0$, such that for each $h \in \mathcal{H}$ there is some $1 \leq i \leq n$ such that

$$\sup(h(A_i)) - \inf(h(A_i)) \leq \varepsilon.$$

This notion has been widely studied, mostly in connection with the Pettis integral theory (see [17] and [18] for an overview); it is also related to some topological properties of operators from $L^1[0, 1]$ into Banach spaces, see [13, Chapter IV].

The aim of this paper is to show how Theorem 1 can be applied in order to replace Pettis integrability with Birkhoff integrability in some well known results concerning the *universal* Pettis integrability of bounded functions $f : K \longrightarrow X^*$, where K is a compact Hausdorff topological space (as usual, the term 'universal' means 'with respect to each Radon probability measure on K'). Functions that are universally Pettis integrable have been studied widely during many years, see amongst others [1], [2], [14], [15], [22], [23], [24], [31] and [32]. For a detailed survey on this topic, we refer the reader to [18, Section 7].

We next summarize the content of this work.

In Section 2 we discuss the universal integrability of w^*-continuous functions. Our Proposition 1 shows that *a w^*-continuous function $f : K \longrightarrow X^*$ is universally scalarly measurable if and only if it is universally Birkhoff integrable*; the latter is the case if and only if the family Z_{f,B_X} does not contain ℓ^1-sequences (for the supremum norm). As a consequence we infer that *X does not contain subspaces isomorphic to ℓ^1 if and only if the 'identity' function $I : (B_{X^*}, w^*) \longrightarrow X^*$ is universally Birkhoff integrable* (see Theorem 2).

In Section 3 we consider w^*-Luzin measurable functions to obtain some other nice applications of Proposition 1. We prove that *a bounded universally scalarly measurable function $f : K \longrightarrow X^*$ is universally Birkhoff integrable provided that X admits a projectional generator and f takes its values in a w^*-separable subset of X^** (see Corollary 1); remember that both additional hypotheses hold automatically whenever X is separable.

On the other hand, our Theorem 3 states that *if X admits a projectional generator and (B_{X^*}, w^*) has the so-called property (M) then, for a given Radon probability measure μ on K, any bounded universally scalarly measurable function $f : K \longrightarrow X^*$ is w^*-scalarly equivalent (with respect to μ) to some bounded universally Birkhoff integrable function $f_0 : K \longrightarrow X^*$*; when no assumptions are made on X, we just get an f_0 which is Birkhoff integrable with respect to μ (see Theorem 4).

All unexplained notation and terminology can be found in our standard references [9], [10] and [32].

Throughout this paper K is a compact Hausdorff topological space and X is a real Banach space. As usual, $C(K)$ stands for the Banach space of all real-valued continuous functions on K endowed with the supremum norm $\|\cdot\|_\infty$.

Given a Banach space $(Y, \|\cdot\|)$, we write $B_Y = \{y \in Y : \|y\| \leq 1\}$ and we denote by w^* the weak* topology on Y^* (the topological dual of Y). By a 'subspace' of Y we mean a norm closed linear subspace. A bounded sequence (y_n) in Y is called *ℓ^1-sequence* if there exists a constant $\delta > 0$ such that $\delta(\sum_{i=1}^n |a_i|) \leq \|\sum_{i=1}^n a_i y_i\|$ for every $n \in \mathbb{N}$ and every $a_1, \dots, a_n \in \mathbb{R}$.

For our purposes it is not necessary to introduce the definition of projectional generator in a Banach space, due to Orihuela and Valdivia [19] (cf. [10, Section 6.1]). We just need to recall Lemma 1 below, which follows from a standard argument used in the construction of projectional resolutions of the identity in non-separable Banach spaces with a projectional generator (cf. [10, Section 6.1]).

Lemma 1. *Suppose that X admits a projectional generator. Let S be a w^*-separable subset of X^*. Then there exist two subspaces $X_0, X_1 \subset X$ such that*

(i) *X_0 is separable;*
(ii) *$X = X_0 \oplus X_1$;*
(iii) *$x^*(x) = 0$ for every $x^* \in S$ and every $x \in X_1$.*

It is well known that every weakly Lindelöf determined (e.g. weakly compactly generated and, more generally, weakly countably $\mathcal{K}$-determined) Banach space admits a projectional generator (cf. [10, Proposition 8.3.1]).

A compact Hausdorff topological space L has the *property (M)* if every Radon probability measure on L has separable support. Remember that the class of weakly Lindelöf determined Banach spaces Y for which (B_{Y^*}, w^*) has the property (M) is bigger than the class of weakly countably $\mathcal{K}$-determined Banach spaces (cf. [10, Chapter 7]).

2. Universal integrability of w^*-continuous functions

Our starting point is the following lemma. The topology on $\mathbb{R}^K$ of pointwise convergence is denoted by $\mathfrak{T}_p(K)$.

Lemma 2. *Let $\mathcal{H} \subset C(K)$ be a uniformly bounded family. The following conditions are equivalent:*

(i) *$\mathcal{H}$ has the Bourgain property with respect to each Radon probability measure on K;*
(ii) *the $\mathfrak{T}_p(K)$-closure of $\mathcal{H}$ is made up of universally measurable functions;*
(iii) *$\mathcal{H}$ does not contain ℓ^1-sequences.*

Proof. The equivalence (ii)⇔(iii) is well known and goes back to Rosenthal [28] and Bourgain, Fremlin and Talagrand [4] (cf. [9, Theorem 3.11] or [32, 14-1-7]). The implication (iii)⇒(i) is a particular case of a result of Musial (cf. [17, Proposition 12.2] or [18, Proposition 4.15]). Finally, (i)⇒(ii) follows from two elementary facts (cf. [23]): (a) the Bourgain property is preserved by taking pointwise closures and (b) every family with the Bourgain property is made up of measurable functions. The proof is complete. □

Proposition 1. *Let $f : K \longrightarrow X^*$ be a w^*-continuous function. The following conditions are equivalent:*

(i) *f is universally Birkhoff integrable;*
(ii) *f is universally scalarly measurable;*
(iii) *$Z_{f,B_X} \subset C(K)$ does not contain ℓ^1-sequences.*

Proof. This is an immediate consequence of Theorem 1 and Lemma 2 applied to the family Z_{f,B_X}, bearing in mind that the closure of Z_{f,B_X} in $(\mathbb{R}^K, \mathfrak{T}_p(K))$ is exactly Z_f (by Goldstine's theorem). □

Haydon [14] (cf. [9, Theorem 6.9]) showed that X does not contain subspaces isomorphic to ℓ^1 if and only if the 'identity' mapping $I : (B_{X^*}, w^*) \longrightarrow X^*$ is universally Pettis integrable (resp. universally scalarly measurable). Now we can replace Pettis integrability with Birkhoff integrability in Haydon's result, as follows.

Theorem 2. *The following conditions are equivalent:*

(i) *X does not contain subspaces isomorphic to ℓ^1;*
(ii) *for each compact Hausdorff topological space L, every w^*-continuous function $f : L \longrightarrow X^*$ is universally Birkhoff integrable;*
(iii) *the 'identity' mapping $I : (B_{X^*}, w^*) \longrightarrow X^*$ is universally Birkhoff integrable;*
(iv) *for each compact Hausdorff topological space L, every w^*-continuous function $f : L \longrightarrow X^*$ is universally scalarly measurable;*
(v) *the 'identity' mapping $I : (B_{X^*}, w^*) \longrightarrow X^*$ is universally scalarly measurable.*

Proof. (i)⇒(ii) Our proof is by contradiction. Suppose that f is not universally Birkhoff integrable. By Proposition 1, there exist a sequence (x_n) in B_X and a constant $\delta > 0$ such that, for every finite sequence $a_1, \dots, a_m \in \mathbb{R}$, we have

$$\delta\Big(\sum_{n=1}^{m} |a_n|\Big) \leq \Big\|\sum_{n=1}^{m} a_n \langle f, x_n\rangle\Big\|_\infty = \Big\|\langle f, \sum_{n=1}^{m} a_n x_n\rangle\Big\|_\infty \leq \sup_{t\in K} \|f(t)\| \cdot \Big\|\sum_{n=1}^{m} a_n x_n\Big\|.$$

This implies that ℓ^1 embeds into X, a contradiction.

(ii)⇒(iii) and (iv)⇒(v) are obvious, since I is w^*-continuous.

(iii)⇒(v) is a consequence of the scalar measurability of any Birkhoff integrable function defined on a complete probability space.

(v)⇒(i) is part of the aforementioned Haydon's characterization.

(v)⇒(iv) Let f be a w^*-continuous function defined on a compact Hausdorff topological space L with values in X^*. We may and do assume without loss of generality that $f(L) \subset B_{X^*}$. Fix a Radon probability measure μ on L. Since f is Borel(L)-Borel(B_{X^*}, w^*)-measurable, we can consider the image measure μf^{-1} induced on Borel(B_{X^*}, w^*). By the w^*-continuity of f, the completion of μf^{-1}, say ν, is a Radon probability measure on (B_{X^*}, w^*). Thus I is scalarly measurable with respect to ν and, therefore, f is also scalarly measurable with respect to μ (as can be easily seen). The proof is complete. □

Obviously, the proof of (v)⇒(iv) in the previous theorem can be avoided by establishing (i)⇒(iv) with the help of Proposition 1.

A well known result due to Musial, Ryll-Nardzewski, Janicka and Bourgain (cf. [9, Theorem 6.8] or [32, 7-3-8]) states that X does not contain subspaces isomorphic to ℓ^1 if and only if X^* has the so-called *weak Radon-Nikodým property (WRNP)* [16], that is, for each complete probability space (Ω, Σ, μ) and every countably additive and μ-continuous measure $\nu : \Sigma \longrightarrow X^*$, with σ-finite variation, there is a Pettis integrable function $f : \Omega \longrightarrow X^*$ such that ν is the indefinite integral of f. It is remarkable that *in the previous characterization one can replace 'Pettis' with 'Birkhoff'* (see [5, Theorem 3.8]). Theorem 2 now provides another approach to the latter statement. We first need to introduce some terminology.

Given a complete probability space (Ω, Σ, μ) and a lifting ρ on $\mathcal{L}^\infty(\mu)$, a standard way to 'regularize' a bounded w^*-scalarly measurable function $f : \Omega \longrightarrow X^*$ is to consider the associated function $\rho(f) : \Omega \longrightarrow X^*$ given by

$$\langle \rho(f)(t), x\rangle = \rho(\langle f, x\rangle)(t), \quad t \in \Omega,\ x \in X.$$

Plainly $\rho(f)$ is also bounded and w^*-scalarly measurable. It is worth mentioning that $\rho(f)$ is Σ-Borel(X^*, w^*)-measurable and that the completion of the image measure μf^{-1} induced on Borel(X^*, w^*) is a Radon probability measure (these facts are due to Sentilles [29], cf. [9, pp. 67–71]).

Let us turn to the promised 'alternative' proof of Theorem 3.8 in [5]. Consider a complete probability space (Ω, Σ, μ) and a countably additive measure $\nu : \Sigma \longrightarrow X^*$ such that $|\nu|(E) \leq \mu(E)$ for every $E \in \Sigma$, where $|\nu|$ denotes the variation of ν. (Recall that the proof can be reduced to this case.) Fix any lifting ρ on $\mathcal{L}^\infty(\mu)$. It is known (cf. [9, Proposition 6.7]) that there is a bounded w^*-scalarly measurable function $f : \Omega \longrightarrow X^*$ (in fact, $f(\Omega) \subset B_{X^*}$) such that

(a) $f = \rho(f)$, hence f is Σ-Borel(X^*, w^*)-measurable and the completion ϑ of the image measure μf^{-1} induced on Borel(B_{X^*}, w^*) is a Radon probability measure;
(b) $\nu(E)(x) = \int_E \langle f, x\rangle \, d\mu$ for every $E \in \Sigma$.

Assume now that X does not contain subspaces isomorphic to ℓ^1. Then Theorem 2 applies to conclude that the 'identity' mapping $I : B_{X^*} \longrightarrow X^*$ is Birkhoff integrable with respect to ϑ. To see that f is Birkhoff integrable with respect to μ it suffices to bear in mind that, given a finite partition $\{A_1, \dots, A_n\}$ of B_{X^*} in Borel(B_{X^*}, w^*), the collection $\{f^{-1}(A_1), \dots, f^{-1}(A_n)\}$ is a partition of Ω in Σ such that

$$\Big\{\sum_{i=1}^n \mu(f^{-1}(A_i))f(t_i) :\ t_i \in f^{-1}(A_i)\Big\} \subset \Big\{\sum_{i=1}^n \vartheta(A_i)x_i^* :\ x_i^* \in A_i\Big\}.$$

Finally, notice that property (b) ensures that ν is the indefinite integral of f.

3. Applications

We begin by recalling a measurability notion which will play an important role throughout this section.

Definition 2. *Let $f : K \longrightarrow X^*$ be a function and μ a Radon probability measure on K. We say that f is w^*-Luzin measurable (with respect to μ) if for every $\varepsilon > 0$ there is a compact set $F \subset K$ with $\mu(K \setminus F) \leq \varepsilon$ such that the restriction $f|_F$ is w^*-continuous.*

The following lemma isolates a useful sufficient condition for w^*-Luzin measurability.

Lemma 3. *Suppose that X admits a projectional generator. Let μ be a Radon probability measure on K and $f : K \longrightarrow X^*$ a function such that*

(i) *f is w^*-scalarly measurable (with respect to μ);*
(ii) *$f(K) \subset S$ for some w^*-separable set $S \subset X^*$ (notice that this condition holds automatically whenever X is separable).*

Then f is w^*-Luzin measurable (with respect to μ).

Proof. By Lemma 1, there exist a separable subspace $X_0 \subset X$ and a subspace $Y_0 \subset X^*$ which is isomorphic to X_0^* and contains $S \supset f(K)$. Thus we can suppose without loss of generality that X is separable.

Fix a countable dense set $\{x_n\}_{n\in\mathbb{N}} \subset B_X$. Then we have $\|x^*\| = \sup_{n\in\mathbb{N}} |x^*(x_n)|$ for every $x^* \in X^*$, hence the function $t \mapsto \|f(t)\|$ is μ-measurable. Therefore, we can assume further that f is bounded.

Fix $\varepsilon > 0$. Given $n \in \mathbb{N}$, the bounded function $\langle f, x_n\rangle$ is μ-measurable, hence the classical Luzin's theorem (cf. [6, Theorem 7.4.3]) ensures the existence of a compact set $F_n \subset K$ with $\mu(K \setminus F_n) \leq \varepsilon/2^n$ such that the restriction $\langle f, x_n\rangle|_{F_n}$ is continuous. Let us consider the compact set $F = \bigcap_{n\in\mathbb{N}} F_n$. Clearly, we have $\mu(K \setminus F) \leq \varepsilon$ and $\langle f, x_n\rangle|_F$ is continuous for every $n \in \mathbb{N}$. We claim that $f|_F$ is w^*-continuous. Indeed, given $x \in X$, there is a sequence (z_k) in $\operatorname{span}\{x_n\}_{n\in\mathbb{N}}$ converging to x in norm. Since f is bounded, the sequence of continuous functions $(\langle f, z_k\rangle|_F)$ converges uniformly to $\langle f, x\rangle|_F$. It follows that $\langle f, x\rangle|_F$ is continuous too and the proof is finished. □

We now deal with integrability properties of w^*-Luzin measurable functions.

Proposition 2. *Let $f : K \longrightarrow X^*$ be a bounded universally scalarly measurable function and μ a Radon probability measure on K. If f is w^*-Luzin measurable (with respect to μ), then f is Birkhoff integrable (with respect to μ).*

Proof. Fix $\varepsilon > 0$ and a compact set $F \subset K$ with $\mu(K \setminus F) \leq \varepsilon$ such that $f|_F$ is w^*-continuous. Since $f|_F$ is universally scalarly measurable, Proposition 1 can be applied to deduce that $f|_F$ is Birkhoff integrable with respect to the restriction of μ to the σ-algebra of the sets of the form $F \cap A$, where A belongs to the domain of μ. As $\varepsilon > 0$ is arbitrary and f is bounded, it follows that f is Birkhoff integrable with respect to μ. □

By putting together Lemma 3 and Proposition 2 we get the following corollary, which improves a result of Riddle, Saab and Uhl [24]. They proved the same statement with 'Pettis' instead of 'Birkhoff' in the particular case of weakly compactly generated Banach spaces.

Corollary 1. *Suppose that X admits a projectional generator. Let $f : K \longrightarrow X^*$ be a bounded function with values in some w^*-separable subset of X^*. Then f is universally Birkhoff integrable if and only if f is universally scalarly measurable.*

Corollary 2. *Suppose that X is separable. Let $f : K \longrightarrow X^*$ be a bounded function. Then f is universally Birkhoff integrable if and only if f is universally scalarly measurable.*

Corollary 2 says, in particular, that universal Pettis integrability and universal Birkhoff integrability are equivalent for bounded functions with values in the dual of separable Banach spaces. Without the separability hypothesis this equivalence is not true in general, see [23] and [27].

In Corollary 1, the assumption 'with values in some w^*-separable subset of X^*' can not be removed even for universal Pettis integrability, as Plebanek has shown in [22] (under the continuum hypothesis, with an example where X is weakly compactly generated). Without that assumption we still have the following

Theorem 3. *Suppose that X admits a projectional generator and that (B_{X^*}, w^*) has the property (M). Let $f : K \longrightarrow X^*$ be a bounded universally scalarly measurable function and μ a Radon probability measure on K. Then there exists a bounded universally Birkhoff integrable function $f_0 : K \longrightarrow X^*$ that is w^*-scalarly equivalent to f (with respect to μ).*

Proof. We may and do assume that $f(K) \subset B_{X^*}$. Fix a lifting ρ on $\mathcal{L}^\infty(\mu)$ and define $g = \rho(f) : K \longrightarrow X^*$. We already know that g is Borel(X^*, w^*)-measurable and that the completion of the image measure μg^{-1} on Borel(B_{X^*}, w^*) is a Radon probability measure. Since (B_{X^*}, w^*) has the property (M), there is a w^*-separable set $S \in$ Borel(B_{X^*}, w^*) such that $\mu(g^{-1}(S)) = 1$.

Lemma 1 now ensures the existence of two subspaces $X_0, X_1 \subset X$ such that X_0 is separable, $X = X_0 \oplus X_1$ and $x^*(x) = 0$ for every $x^* \in S$ and every $x \in X_1$. Hence we can find two subspaces $Y_0, Y_1 \subset X^*$ such that

- $X^* = Y_0 \oplus Y_1$;
- Y_i is isomorphic to X_i^* for $i = 0, 1$;
- $S \subset Y_0$;
- $x^*(x) = 0$ for every $x^* \in Y_0$ (resp. $x^* \in Y_1$) and every $x \in X_1$ (resp. $x \in X_0$).

Write $f = f_0 + f_1$, where each $f_i : K \longrightarrow Y_i \subset X^*$. Since f is bounded and universally scalarly measurable, the same holds true for both f_0 and f_1, as can be easily seen. Bearing in mind the separability of X_0, Corollary 2 can be applied to conclude that f_0 is universally Birkhoff integrable. In order to finish the proof we only have to check that f and f_0 are w^*-scalarly equivalent (with respect to μ). Fix $x \in X$ and write $x = x_0 + x_1$, where $x_0 \in X_0$ and $x_1 \in X_1$. Since f and g are w^*-scalarly equivalent, we have $\langle f, x\rangle = \langle g, x\rangle$ μ-a.e. and $\langle f_0, x\rangle = \langle f, x_0\rangle = \langle g, x_0\rangle$ μ-a.e. On the other hand, recall that we have $\mu(g^{-1}(S)) = 1$, hence $\langle g, x_0\rangle = \langle g, x\rangle$ μ-a.e. It follows that $\langle f, x\rangle = \langle f_0, x\rangle$ μ-a.e., as required. □

The previous theorem with 'Pettis' instead of 'Birkhoff' was proved in the particular case of weakly compactly generated Banach spaces by Stefánsson [31]. His proof does not make use of liftings and relies on the following fact: *(+) Let (Ω, Σ, μ) be a complete probability space, X a weakly compactly generated Banach space and $f : \Omega \longrightarrow X^*$ a w^*-scalarly integrable function. Then Z_{f,B_X} is $\|\cdot\|_1$-separable.* This result was proved by Stefánsson [30] using the factorization theorem of Davis, Figiel, Johnson and Pelczynski [7]. A different method now allows us to extend (+) to the class of Banach spaces considered in Theorem 3.

Lemma 4. *Suppose that X admits a projectional generator and that (B_{X^*}, w^*) has the property (M). Then every Radon probability measure on B_{X^*} has w^*-metrizable support.*

Proof. Fix a Radon probability measure ν on B_{X^*} and let S be its support. Since (B_{X^*}, w^*) has the property (M), S is w^*-separable. By Lemma 1, there exist two subspaces $X_0, X_1 \subset X$ such that X_0 is separable, $X = X_0 \oplus X_1$ and $x^*(x) = 0$ for every $x^* \in S$ and every $x \in X_1$. Notice that the mapping $\pi : S \longrightarrow B_{X_0^*}$, $\pi(x^*) = x^*|_{X_0}$, is injective and w^*-w^*-continuous, hence π establishes a w^*-w^*-homeomorphism between the compact space S and its image $\pi(S)$. Since $B_{X_0^*}$ is w^*-metrizable, we conclude that S is w^*-metrizable. □

Proposition 3. *Suppose that X admits a projectional generator and that (B_{X^*}, w^*) has the property (M). Let (Ω, Σ, μ) be a complete probability space and $f : \Omega \longrightarrow X^*$ a w^*-scalarly integrable function. Then Z_{f,B_X} is $\|\cdot\|_1$-separable.*

Proof. We begin with the following:

Particular case.- Suppose that there is a constant $M > 0$ such that for each $x \in B_X$ we have $|\langle f, x\rangle| \leq M$ μ-a.e. Then we can consider the operator (i.e. linear and continuous map) $\phi : X \longrightarrow L^\infty(\mu)$ that maps each $x \in X$ to the equivalence class of $\langle f, x\rangle$. Recall that the 'identity' operator $\xi : L^\infty(\mu) \longrightarrow L^1(\mu)$ is *absolutely summing* (cf. [8, 2.9]), that is, it takes unconditionally convergent series to absolutely convergent ones. Hence the same holds for the composition $\xi \circ \phi$. On the other hand, by Lemma 4, each Radon probability measure ν on B_{X^*} has w^*-metrizable support and, in particular, the space $L^1(\nu)$ is separable (cf. [6, Proposition 3.4.5]). Therefore, by Pietch's factorization theorem (cf. [8, 2.13]), every absolutely summing operator defined on X has separable range, see [25]. In particular, $\xi \circ \phi$ has separable range, as required.

General case.- Since Z_{f,B_X} is a pointwise bounded family of measurable functions, there is a non-negative measurable function h on Ω such that for each $x \in B_X$ we have $|\langle f, x\rangle| \leq h$ μ-a.e. (cf. [17, Proposition 3.1]). Fix $n \in \mathbb{N}$ and define $A_n = \{t \in \Omega : n - 1 \leq h(t) < n\} \in \Sigma$. In view of the *Particular case*, the family of restrictions $\{\langle f, x\rangle|_{A_n} : x \in B_X\}$ is $\|\cdot\|_1$-separable. Since $n \in \mathbb{N}$ is arbitrary and $\Omega = \bigcup_{n=1}^\infty A_n$, we conclude that Z_{f,B_X} is $\|\cdot\|_1$-separable. The proof is complete. □

Given a bounded universally scalarly measurable function $f : K \longrightarrow X^*$ and a Radon probability measure μ on K, Bator [2] proved that there exists a bounded function $g : K \longrightarrow X^*$ such that f and g are w^*-scalarly equivalent and Z_{g,B_X} has the Bourgain property (with respect to μ); in view of Theorem 1, the function g is Birkhoff integrable (with respect to μ). The same happens with $\rho(f)$ for any lifting ρ on $\mathcal{L}^\infty(\mu)$, as we show in Theorem 4 below.

Lemma 5. *Let (Ω, Σ, μ) be a complete probability space and ρ a lifting on $\mathcal{L}^\infty(\mu)$. Let $\mathcal{H} \subset \mathcal{L}^\infty(\mu)$ be a family with the Bourgain property. Then $\{\rho(h) : h \in \mathcal{H}\}$ has the Bourgain property too.*

Proof. Given $E \in \Sigma$, we define $E^\rho = E \cap \{t \in \Omega : \rho(\chi_E)(t) = 1\} \in \Sigma$. Fix $\varepsilon > 0$ and $A \in \Sigma$ with $\mu(A) > 0$. Since $\mathcal{H}$ has the Bourgain property, there are $A_1, \dots, A_n \in \Sigma$, $A_i \subset A$ with $\mu(A_i) > 0$, such that for each $h \in \mathcal{H}$ there is some $1 \leq i_h \leq n$ such that $\sup(h(A_{i_h})) - \inf(h(A_{i_h})) \leq \varepsilon$. Notice that $A_i^\rho \subset A$ and that $\mu(A_i^\rho) = \mu(A_i) > 0$ for every $1 \leq i \leq n$.

Fix $h \in \mathcal{H}$ and set $M_h = \sup(h(A_{i_h}))$ and $m_h = \inf(h(A_{i_h}))$. Since $m_h\chi_{A_{i_h}} \leq h\chi_{A_{i_h}} \leq M_h\chi_{A_{i_h}}$, we have

$$m_h\rho(\chi_{A_{i_h}}) = \rho(m_h\chi_{A_{i_h}}) \leq \rho(h\chi_{A_{i_h}}) = \rho(h)\rho(\chi_{A_{i_h}}) \leq \rho(M_h\chi_{A_{i_h}}) = M_h\rho(\chi_{A_{i_h}}).$$

Therefore, $m_h \leq \rho(h)(t) \leq M_h$ for every $t \in A_{i_h}^\rho$ and, consequently, we obtain $\sup(\rho(h)(A_{i_h})) - \inf(\rho(h)(A_{i_h})) \leq M_h - m_h \leq \varepsilon$. It follows that $\{\rho(h) : h \in \mathcal{H}\}$ has the Bourgain property. The proof is over. □

Theorem 4. *Let $f : K \longrightarrow X^*$ be a bounded universally scalarly measurable function. Let μ be a Radon probability measure on K and ρ a lifting on $\mathcal{L}^\infty(\mu)$. Then $\rho(f)$ is Birkhoff integrable (with respect to μ).*

Proof. By the aforementioned result of Bator, there exists a bounded function $g : K \longrightarrow X^*$ such that Z_{g,B_X} has the Bourgain property and g is w^*-scalarly equivalent to f (with respect to μ). The latter condition ensures that $\rho(f) = \rho(g)$. On the other hand, since Z_{g,B_X} has the Bourgain property, the same holds for the family $\{\rho(\langle g, x\rangle) : x \in B_X\} = Z_{\rho(f),B_X}$ (Lemma 5). An appeal to Theorem 1 establishes that $\rho(f)$ is Birkhoff integrable (with respect to μ). □

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Departamento de Matemáticas, Universidad de Murcia, 30100 Espinardo, Murcia, Spain

E-mail address: joserr@um.es