ALUR DUAL RENORMINGS OF BANACH SPACES

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ABSTRACT. We give a covering type characterization for the class of dual Banach spaces with an equivalent ALUR dual norm.

Let K be a closed convex subset of a real Banach space $(X, \|\cdot\|)$, and $\epsilon > 0$. An element $x \in K$ is called an:

(1) ϵ-very strong extreme point of K if there exists a positive real number δ such that $\int_0^1 \|f(t) - x\| dt < \epsilon$ whenever f is a Bochner integrable function from $[0, 1]$ into K such that $\| \int_0^1 f(t)dt - x\| < \delta$. In this case we will say that x is an (ϵ, δ)-very strong extreme point of K.

(2) ϵ-denting point of K if $x \notin \overline{\mathrm{conv}(K \setminus B(x; \epsilon))}^{\|\cdot\|}$, or equivalently, if there is an open half space $H \subset X$ such that $x \in H$ and $\|\cdot\| - diam(H \cap K) < \epsilon$ (recall that an open half space of X is a set of the form $f^{-1}(a, \infty)$, with $f \in X^*$ and $a \in \mathbb{R}$).

The element x is called a very strong extreme (resp. denting) point of K if it is an ϵ-very strong extreme (resp. ϵ-denting) point of K, for every $\epsilon > 0$. Recall also that x is a point of continuity of K if the identity map $id : (K, weak) \longrightarrow (K, \|\cdot\|)$ is continuous at x. The above notions were characterized in the following

Proposition 1 (Lin-Lin-Troyanski, [4])**.** *Let x be an element in a bounded closed convex set K of a Banach space. The following are equivalent:*

(1) *x is a very strong extreme point of K.*
(2) *x is a denting point of K.*
(3) *x is an extreme point and a point of continuity of K.*

We say that the space X (or the norm of X) is average locally uniformly rotund (ALUR for short) if every point of the unit sphere of X is a very strong extreme point of the unit ball. This property was introduced in [9].

The space X is rotund if the points of the unit sphere are extreme points of the unit ball. The space X is said to have the Kadec property when the weak and the norm topologies coincide on the sphere, this is equivalent to say that the points of the unit sphere are points of continuity of the unit ball. Thus, from Proposition 1 it

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follows that a Banach space is ALUR if, and only if, it is rotund and has the Kadec property.

On the other hand, the space X has the G property (introduced in [2]) if all the points on the unit sphere are denting points of the unit ball. Proposition 1 also gives the equivalence between ALUR and G property.

Recall also that X is locally uniformly rotund (LUR for short) if for every point x and every sequence $(x_n)_n$ in the unit sphere of X such that $\lim_n \|\frac{x_n+x}{2}\| = 1$ we have $\lim_n \|x_n - x\| = 0$.

It is easy to see that a LUR norm is rotund and has the Kadec property; in particular, every LUR Banach space is ALUR. In [9] it is shown that the converse is true up to renormings, i.e., that every ALUR space admits an equivalent LUR norm. This result suggests the following question:

If a dual Banach space is ALUR, does it necessarily admit an equivalent LUR dual norm?

This problem has a negative answer. In [8] it was shown that the dual of the James Tree space JT has an equivalent dual norm with the Kadec property, and since JT is separable, the space JT^* admits also an equivalent rotund dual norm (see e.g. [1, p. 48]). The sum of these two norms is a dual norm which shares both properties, and consequently, JT^* is ALUR dual renormable. On the other hand, it is known (see e.g. [1, p. 43 and 51]) that the dual of a separable Banach space X is separable if X^* admits an equivalent LUR dual norm. As JT^* is nonseparable, this space does not admit any LUR dual norm.

Another counterexample to the above question may be found in [3], where it was provided an example of Banach space of continuous functions $C(\Upsilon)$, with Υ a scattered compact (more precisely, a tree), such that $C(\Upsilon)^*$ has an equivalent rotund dual norm and $C(\Upsilon)$ does not admit any Fréchet differentiable norm. The duals of $C(K)$ spaces with K scattered are isometric to $\ell_1(K)$, in particular they have the Kadec property. Consequently $C(\Upsilon)^*$ is ALUR dual renormable. Nevertheless, this space does not have any LUR dual norm (in that case, $C(\Upsilon)$ should admit a Fréchet differentiable norm, see e.g. [1, p. 43]).

Therefore, the class of LUR dual renormable Banach spaces is strictly contained in the class of ALUR dual renormable Banach spaces. The aim of this note is to characterize dual Banach spaces with equivalent ALUR dual norms. LUR renormable Banach spaces were characterized in [5] and [7], in terms of countable decompositions of such spaces. In [7] it was also provided a characterization in the dual case, showing in particular that a dual Banach space admits an equivalent LUR dual norm if, and only if, it has an equivalent dual w^*-Kadec norm (weak star and norm topologies coincide on its unit sphere). In this work we give a covering type characterization for the class of ALUR dual renormable Banach spaces.

Theorem 1. *For a dual Banach space $(X, \|\cdot\|)$ the following assertions are equivalent:*

(1) *X admits an equivalent ALUR dual norm.*
(2) *For each $\epsilon > 0$ there is a countable decomposition*

$$X = \bigcup_n X_{n,\epsilon}$$

in such a way that every $x \in X_{n,\epsilon}$ is an ϵ-very strong extreme point of $\overline{\mathrm{conv}(X_{n,\epsilon})}^{w^*}$.
(3) *For each $\epsilon > 0$ there is a countable decomposition*

$$X = \bigcup_n X_{n,\epsilon}$$

in a such way that each $x \in X_{n,\epsilon}$ is an ϵ-denting point of $\overline{\mathrm{conv}(X_{n,\epsilon})}^{w^*}$.

In order to prove the Theorem, we shall need two results. The first shows a connection between the modulus of dentability and very strong extremality defined above.

Lemma 1. *Let K be a closed convex set of a Banach space X. Then every ϵ-very strong extreme point of K is an ϵ-denting point of this set.*

Proof. Let x be an (ϵ, δ)-very strong point of K, for some $\delta > 0$. Suppose that x is not an ϵ-denting point of K. Then, there exist a finite set I and subsets $\{a_i\}_{i\in I} \subset \mathbb{R}^+$ and $\{x_i\}_{i\in I} \subset K$ such that $\sum_i a_i = 1$, $\|x - \sum_i a_i x_i\| < \delta$, and $\|x_i - x\| > \epsilon$ for every $i \in I$. Let us denote by λ de Lebesgue measure on $[0, 1]$, let $\{A_i\}_{i\in I}$ be a partition of $[0, 1]$ with $\lambda(A_i) = a_i$ for every $i \in I$, and set $f = \sum_{i\in I} x_i \chi_{A_i}$. Clearly, f is a Bochner integrable function with values in K such that $\| \int_0^1 f(t)dt - x\| = \| \sum_i a_i x_i - x\| < \delta$. Since x is an (ϵ, δ)-very strong extreme point of K we get that $\int_0^1 \|f(t) - x\|dt < \epsilon$. But

$$\int_0^1 \|f(t) - x\|dt = \sum_i \int_{A_i} \|x_i - x\|dt = \sum_i a_i\|x_i - x\| > \sum_i a_i\epsilon = \epsilon$$

a contradiction. So, x is an ϵ-denting point of K.

Lemma 2. *Let $(X, \|\cdot\|)$ be a Banach space and denote by B_X its unit ball. Let $\epsilon, \delta, \eta > 0$ such that $\eta < \min\{\epsilon, \frac{\delta}{3}\}$. If x is an (ϵ, δ)-very strong extreme point of $\|x\|B_X$, then it is a 2ϵ-very strong extreme point of $(\|x\| + \eta)B_X$.*

Proof. Let $f : [0, 1] \longrightarrow (\|x\| + \eta)B_X$ be a Bochner integrable function such that

$$\|x - \int_0^1 f(t)dt\| < \eta$$

and let $g = \frac{\|x\|}{\|x\|+\eta} f$. Then g is a Bochner integrable function with values in $\|x\|B_X$, and from the last inequality it follows that

$$\|x - \int_0^1 g(t)dt\| = \frac{1}{\|x\|+\eta}\|\eta x + \|x\|(x - \int_0^1 f(t)dt)\| < 2\eta < \delta$$

Since x is an (ϵ, δ)-very strong extreme point of the set $\|x\|B_X$ it follows that $\int_0^1 \|g(t) - x\|dt < \epsilon$, and consequently,

$$\int_0^1 \|f(t) - x\|dt \leq \int_0^1 \|g(t) - x\|dt + \frac{\eta}{\|x\|}\int_0^1 \|g(t)\|dt < \epsilon + \eta < 2\epsilon$$

as we wanted.

Proof of Theorem 1. $(1) \Longrightarrow (2)$ We may assume that the norm of X is ALUR, and let us fix $\epsilon > 0$. For every positive rational number r define

$$X_{r,\epsilon} = \{x \in X : x \text{ is an } \epsilon\text{-very strong extreme point of } rB_X\}$$

Since rB_X is w^*-compact and $X_{r,\epsilon} \subseteq \overline{\text{conv}(X_{r,\epsilon})}^{w^*} \subseteq rB_X$ we get that every $x \in X_{r,\epsilon}$ is an ϵ-very strong extreme point of the set $\overline{\text{conv}(X_{r,\epsilon})}^{w^*}$. It remains to prove that

$$X = \{0\} \cup \bigcup_{r \in \mathbb{Q}^+} X_{r,\epsilon}$$

Let $x \in X \setminus \{0\}$. As the norm of X is ALUR there is $\delta > 0$ such that x is an $(\frac{\epsilon}{2}, \delta)$-very strong extreme point of the set $\|x\|B_X$. Let η be a positive number such that $\eta < \min\{\frac{\epsilon}{2}, \frac{\delta}{3}\}$ and $\|x\| + \eta$ is a rational, and set $r = \|x\| + \eta$. Then, from Lemma 2 it follows that x is an (ϵ, η)-very strong extreme point of the set $\|x\|B_X + \eta B_X = rB_X$. So $x \in X_{r,\epsilon}$.

$(2) \Longrightarrow (3)$ This implication follows inmediately from Lemma 1.

$(3) \Longrightarrow (1)$ According to Proposition 1 it is enough to construct on X an equivalent dual rotund norm with the Kadec property. We follow some arguments of [6] and [7]. For each $m \in \mathbb{N}$, let $\epsilon_m = \frac{1}{m}$. There is a decomposition

$$X = \bigcup_n X_{n,\epsilon_m}$$

in such a way that each $x \in X_{n,\epsilon_m}$ is an ϵ_m-denting point of $\overline{\text{conv}(X_{n,\epsilon_m})}^{w^*}$. Let $\{A_k\}_k$ be an enumeration of the set $\{\overline{\text{conv}(X_{n,\epsilon_m})}^{w^*} + \frac{1}{p}B_X : m, n, p \in \mathbb{N}\}$. Note that each A_k is convex, weak star closed and has non empty interior in the norm topology. Observe also that if x is an ϵ-denting point of a set A, then x is a 2ϵ-denting point of $A + \theta B_X$ for enough small $\theta > 0$. Therefore, for every $x \in X$ and every $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that x is an interior ϵ-denting point of A_k. For every k, take $a_k \in int(A_k)$ and let F_k be the Minkowski functional of A_k respect to a_k. Let $(\lambda_k)_k$ be a sequence of positive real numbers such that the series

$$F^2(x) = \|x\|^2 + \sum_k \lambda_k(F_k^2(x) + F_k^2(-x))$$

converges uniformly on bounded sets and $B = \{x \in X : F(x) \leq 1\}$ contains 0 as an interior point. Clearly F is a convex w^*-lower semicontinuous function, so that B is a convex and w^*-closed set, and consequently, its Minkowski functional $|||\cdot|||$ defines an equivalent dual norm on X.

Now, we prove that the norm $|||\cdot|||$ is rotund and has the Kadec property.

Suppose that $|||\cdot|||$ is not rotund. Then there are $x_1, x_2 \in X$ with

$$|||x_1||| = |||x_2||| = |||\frac{x_1+x_2}{2}||| \tag{1}$$

and

$$\|x_1 - x_2\| > 2\epsilon, \text{ for some } \epsilon > 0 \tag{2}$$

We can find $k \in \mathbb{N}$ such that

$$\frac{x_1+x_2}{2} \in int(A_k) \tag{3}$$

and $\frac{x_1+x_2}{2}$ is an ϵ-denting point of A_k. Since F is uniformly continuous on bounded sets, from (1) we have

$$F(x_1) = F(x_2) = F(\frac{x_1+x_2}{2})$$

and using a convex argument (see [1, Fact II. 2.3]), it follows that

$$F_k(x_1) = F_k(x_2) = F_k(\frac{x_1+x_2}{2})$$

From this and (3) we have $F_k(x_i) < 1$, that is, $x_i \in A_k$, for $i = 1, 2$. Let $H \subset X$ be a weak open half space such that $\frac{x_1+x_2}{2} \in H$ and $\|\cdot\| - diam(A_k \bigcap H) < \epsilon$. Because of (2) we have $x_i \in X \setminus H, i = 1, 2$, and by the convexity of $X \setminus H$ it follows that $\frac{x_1+x_2}{2} \in X \setminus H$, a contradiction. So $|||\cdot|||$ is a rotund norm.

It remains to prove that $|||\cdot|||$ has the Kadec property. Note first that since F is uniformly continuous on bounded sets we have

$$\lim_\alpha F(x_\alpha) = 1 \text{ whenever } (x_\alpha)_{\alpha\in A} \subset X \text{ is a net such that } \lim_\alpha |||x_\alpha||| = 1 \tag{4}$$

Let $(x_\alpha)_{\alpha\in A} \subset X$ be a net and $x \in X$ such that $|||x_\alpha||| = |||x||| = 1$ for every α, and $x_\alpha \longrightarrow x$ in the weak topology of X. Then, by the weak lower semicontinuity of $|||\cdot|||$ we deduce that $|||\frac{x_\alpha+x}{2}||| \longrightarrow 1$, and from (4) we get

$$F(x_\alpha) \longrightarrow 1 \text{ and } F(\frac{x_\alpha+x}{2}) \longrightarrow 1$$

Using again the convex arguments it follows that

$$F_k(x_\alpha) \longrightarrow F_k(x), \text{ for every } k \in \mathbb{N} \tag{5}$$

Let $\epsilon > 0$. As before we take $k_0 \in \mathbb{N}$ and a weak open half space $H \subset X$ such that $x \in int(A_{k_0}) \cap H$ and $\|\cdot\| - diam(A_{k_0} \cap H) < \epsilon$. Because of (5) there is $\alpha_0 \in A$ such that $F_{k_0}(x_\alpha) < 1$, and therefore $x_\alpha \in A_{k_0}$, for every $\alpha \geq \alpha_0$. Since H is a weak open neighbourhood of x we may also assume that $x_\alpha \in H$ for every $\alpha \geq \alpha_0$, and consequently $\|x_\alpha - x\| < \epsilon$, as we wanted.

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