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Integral Equations and Operator Theory

On the structure of L^1 of a vector measure via its integration operator

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Abstract. Geometric and summability properties of the integration operator associated to a vector measure m can be translated in terms of structure properties of the space $L^1(m)$. In this paper we study the cases of the integration operator being: (i) p-concave on $L^p(m)$, or (ii) positive p-summing on $L^1(m)$ (where $1 \leq p < \infty$). We prove that (i) is equivalent to saying that $L^1(m)$ contains continuously the L^p space of a (non-negative scalar) control measure for m. On the other hand, we show that (ii) holds if and only if $L^1(m)$ is order isomorphic to the L^1 space of a non-negative scalar measure.

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1. Introduction

Let E be an order continuous Banach function space (over a non-negative scalar measure) having weak order unit. It is known that there is a vector measure m such that E is order isomorphic to $L^1(m)$, see [3, Theorem 8] or [12, Proposition 3.30]. In this case, we say that m *represents* E. This representation is not unique. However, the properties of the integration operator associated to *some/every* vector measure representing E determine some features of E. From this point of view, properties of the integration operator like compactness or weak compactness have already been studied (see [12, Section 3.3] and the references therein).

In this paper we analyze the continuous injection in E of usual Lebesgue spaces $L^r(\lambda)$ (λ being a non-negative scalar measure and $1 \leq r < \infty$). Some

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results in this direction can be found in [12, Chapter 6]. Our arguments are closely related to the ones presented there.

In our main result, Theorem 2.3, we characterize the *vector-valued norm inequalities* that the integration operator associated to certain vector measure m representing E must satisfy in order to have the following property:

There is a (non-negative scalar) control measure λ for m such that

$$L^r(m) \hookrightarrow L^r(\lambda) \hookrightarrow L^1(m) \simeq E.$$

Moreover, in Theorem 2.3 we also prove that the property above is equivalent to saying that the integration operator on $L^r(m)$ is r-concave. Here, the notation $E_1 \hookrightarrow E_2$ (where E_1 and E_2 are Banach function spaces over non-negative scalar measures defined on the same measurable space) means that 'identity' mapping is a well-defined one-to-one operator (*i.e.* linear continuous mapping) from E_1 to E_2.

The last part of the paper is devoted to the 'extreme case': when is E order isomorphic to $L^1(\lambda)$ for some non-negative scalar measure λ? In Theorem 2.7 we show that the *positive p-summability* ($1 \le p < \infty$) of the integration operator associated to *some/every* vector measure representing E provides a complete answer to the previous question.

Terminology and preliminaries. All unexplained terminology can be found in our standard references [6], [9] and [12]. All our vector spaces are real. Given a Banach space Y, the symbol Y' stands for the topological dual of Y and the duality is denoted by $\langle \cdot, \cdot \rangle$. We write B_Y to denote the closed unit ball of Y. If in addition Y is a Banach lattice, we write Y^+ and B_Y^+ for the positive cone of Y and its intersection with B_Y, respectively. A relevant class of Banach lattices is that of Banach function spaces. Given a finite measure space (Ω, Σ, μ), a linear subspace E of $L^0(\mu)$ equipped with a complete norm $\|\cdot\|_E$ is called a *Banach function space over* μ if the following conditions are satisfied: (i) if $f \in L^0(\mu)$ and $g \in E$ are such that $|f| \le |g|$ (for the μ-a.e. order), then $f \in E$ and $\|f\|_E \le \|g\|_E$; (ii) every simple function belongs to E; and (iii) the 'identity' defines a one-to-one operator from E to $L^1(\mu)$.

Throughout this paper X is a Banach space, (Ω, Σ) is a measurable space and $m : \Sigma \to X$ is a (countably additive) vector measure. By a *control measure* for m we mean a non-negative scalar measure λ on (Ω, Σ) such that $\lambda(A) = 0$ if and only if $\|m\|(A) = 0$, where $\|m\|$ denotes the semivariation of m. For each $x' \in X'$ we write $\langle m, x' \rangle$ to denote the scalar measure defined by $\langle m, x' \rangle(A) := \langle m(A), x' \rangle$, for all $A \in \Sigma$. From now on we fix a *Rybakov control measure* for m, that is, a control measure of the form $\mu = |\langle m, x_0' \rangle|$ with $x_0' \in B_{X'}$, *cf.* [6, p. 268]. In this way, a property holds μ-a.e. if and only if it holds $\|m\|$-a.e.

A Σ-measurable function $f : \Omega \to \mathbb{R}$ is m-integrable if f is integrable with respect to $\langle m, x' \rangle$ for every $x' \in X'$ and, for each $A \in \Sigma$, there exists a vector $\int_A f\, dm \in X$ such that $\langle \int_A f\, dm, x' \rangle = \int_A f\, d\langle m, x' \rangle$ for all $x' \in X'$. Given $1 \le p < \infty$, the space $L^p(m)$ is the Banach function space over μ made up of all equivalence classes of functions f such that $|f|^p$ is m-integrable, endowed with the norm

$$\|f\|_{L^p(m)} := \sup_{x' \in B_{X'}} \Big(\int_\Omega |f|^p d|\langle m, x'\rangle| \Big)^{\frac{1}{p}}.$$

The space $L^p(m)$ is p-convex, order continuous and has weak unit. Observe that $\|f\|^p_{L^p(m)} = \||f|^p\|_{L^1(m)}$ for all $f \in L^p(m)$. For the basic properties of this space, we refer the reader to [7] and [12, Chapter 3].

The operator $I^1_m : L^1(m) \to X$ defined by $I^1_m(f) := \int_\Omega f\, dm$ is called the *integration operator* associated to m. Since $L^p(m) \hookrightarrow L^1(m)$ (*cf.* [12, p. 122]), we can also consider the operator on $L^p(m)$ defined by

$$I^p_m : L^p(m) \to X, \quad I^p_m(f) := \int_\Omega f\, dm.$$

The fact that $L^1(m)$ is order continuous ensures that its topological dual $L^1(m)'$ coincides with its Köthe dual $L^1(m)^\times$ (*cf.* [11, Corollary 2.6.5]) and we identify each functional $\varphi \in L^1(m)'$ with the (unique) function $u \in L^1(\mu)$ such that $\langle f, \varphi\rangle = \int_\Omega f u\, d\mu$ for all $f \in L^1(m)$. As usual, we write $u\, d\mu$ to denote the real-valued measure on (Ω, Σ) given by $A \rightsquigarrow \int_A u\, d\mu$.

For simplicity, from now on we just write the symbol $\int$ instead of $\int_\Omega$ to denote any 'integral' over Ω.

Remark 1.1. Let λ be a control measure for m. The following statements are equivalent:

(1) $L^1(m) \hookrightarrow L^1(\lambda)$.
(2) $\lambda = u\, d\mu$ for some $u \in L^1(m)'$, $u \geq 0$.

Proof. (1)⇒(2). We can write $\lambda = u\, d\mu$ for some $u \in L^1(\mu)$, $u \geq 0$. Since the linear functional on $L^1(m)$ given by $f \rightsquigarrow \int f\, d\lambda = \int f u\, d\mu$ is continuous, it follows that u belongs to $L^1(m)'$.

(2)⇒(1). For each $f \in L^1(m)$ we have $f \in L^1(\lambda)$ and

$$\|f\|_{L^1(\lambda)} = \int |f| u\, d\mu \leq \|f\|_{L^1(m)} \|u\|_{L^1(m)'},$$

hence $L^1(m) \hookrightarrow L^1(\lambda)$. □

2. Results

Let $r \geq 1$ and $p, q > 1$ be real numbers such that $1/r = 1/p + 1/q$. Then the product fg belongs to $L^r(m)$ whenever $f \in L^p(m)$ and $g \in L^q(m)$, with $\|fg\|_{L^r(m)} \leq \|f\|_{L^p(m)} \|g\|_{L^q(m)}$, *cf.* [12, (3.88)]. Therefore, we can consider the bilinear continuous mapping $L^p(m) \times L^q(m) \to X$ defined by

$$(f, g) \rightsquigarrow I^r_m(fg) = \int fg\, dm.$$

In [8] this approach has been used to obtain factorization theorems for operators defined between Banach lattices satisfying adequate convexity/concavity properties. In our main result, Theorem 2.3 below, we discuss the r-concavity of the integration operator $I_m^r : L^r(m) \to X$ in terms of the bilinear mapping described above. Moreover, we show that the r-concavity of I_m^r is equivalent to the fact that $L^r(m) \hookrightarrow L^r(\lambda) \hookrightarrow L^1(m)$ for some control measure λ for m. We stress that the 1-concavity of I_m^1 is analyzed in [12] (see Section 3.4 and Chapter 6), where some relevant examples can also be found.

To state our Theorem 2.3 we need the following:

Definition 2.1. *A set $S \subset B^+_{L^1(m)'}$ is positively norming for $L^1(m)$ if*

$$\|f\|_{L^1(m)} = \sup_{\varphi \in S} \langle |f|, \varphi\rangle \quad \text{for all } f \in L^1(m).$$

Remark 2.2. Some examples of positively norming sets:

- $B^+_{L^1(m)'}$ is positively norming for $L^1(m)$.
- If λ is a non-negative scalar measure on (Ω, Σ), then the singleton $\{\chi_\Omega\} \subset B^+_{L^\infty(\lambda)}$ is positively norming for $L^1(\lambda)$.
- Given $x' \in B_{X'}$, define $\varphi_{x'} \in B^+_{L^1(m)'}$ by $\varphi_{x'}(f) := \int f \, d|\langle m, x'\rangle|$. The Radon-Nikodým derivative of $|\langle m, x'\rangle|$ with respect to μ
$$\frac{d|\langle m, x'\rangle|}{d\mu} = \left|\frac{d\langle m, x'\rangle}{d\mu}\right|$$
is the function associated to $\varphi_{x'}$ via the identification $L^1(m)' \simeq L^1(m)^\times$. Clearly, the set $\{\varphi_{x'} : x' \in B_{X'}\}$ is positively norming for $L^1(m)$.

Condition (3) in Theorem 2.3 involves some spaces of multiplication operators recently studied in [2]. Recall that if E_1 and E_2 are two Banach function spaces over non-negative scalar measures defined on (Ω, Σ), then an operator $T : E_1 \to E_2$ is called a *multiplication operator* if there is (a unique) $h \in E_2$ such that $T(f) = fh$ for all $f \in E_1$; in this case we write $T = M_h$. The space $\mathcal{M}(E_1, E_2)$ of all multiplication operators from E_1 to E_2 becomes a Banach space when endowed with the operator norm, *cf.* [10].

Theorem 2.3. *Let $r \geq 1$ and $p, q > 1$ be such that $1/r = 1/p + 1/q$. Let $S \subset B^+_{L^1(m)'}$ be a weak* compact convex set which is positively norming for $L^1(m)$. The following statements are equivalent:*

(1) *There is a constant $K > 0$ such that the inequality*
$$\Big(\sum_{i=1}^n \Big\| \int f_i g_i \, dm \Big\|^r\Big)^{\frac{1}{r}} \leq K \Big\| \Big(\sum_{i=1}^n |f_i|^p\Big)^{\frac{1}{p}} \Big\|_{L^p(m)} \Big\| \Big(\sum_{i=1}^n |g_i|^q\Big)^{\frac{1}{q}} \Big\|_{L^q(m)}$$
holds for every $f_1, \ldots, f_n \in L^p(m)$ and $g_1, \ldots, g_n \in L^q(m)$, $n \in \mathbb{N}$.

(2) *There exist a constant $K > 0$ and $u_0, v_0 \in S$ such that the inequality*
$$\Big\| \int fg \, dm \Big\| \leq K \Big(\int |f|^p u_0 \, d\mu\Big)^{\frac{1}{p}} \Big(\int |g|^q v_0 \, d\mu\Big)^{\frac{1}{q}}$$

holds for every $f \in L^p(m)$ *and* $g \in L^q(m)$.

(3) *There exist* $u_0, v_0 \in S$ *such that:*
- $u_0\, d\mu$ *and* $v_0\, d\mu$ *are control measures for* m.
- *Each* $f \in L^p(u_0\, d\mu)$ *induces a multiplication operator*
$$M_f \in \mathcal{M}(L^q(v_0\, d\mu), L^1(m)).$$
- *The mapping* $f \rightsquigarrow M_f$ *is a one-to-one operator from* $L^p(u_0\, d\mu)$ *to* $\mathcal{M}(L^q(v_0\, d\mu), L^1(m))$.

(4) *There exist* $u_0, v_0 \in S$ *such that for* $h_0 = u_0^{r/p} v_0^{r/q} \in B^+_{L^1(m)'}$, *we have*
$$L^r(m) \hookrightarrow L^r(h_0\, d\mu) \hookrightarrow L^1(m).$$

(5) *There is a control measure* ν *for* m *such that*
$$L^r(m) \hookrightarrow L^r(\nu) \hookrightarrow L^1(m).$$

(6) *The integration operator* $I_m^r : L^r(m) \to X$ *is* r*-concave, that is, there is a constant* $K > 0$ *such that the inequality*
$$\Big(\sum_{i=1}^n \Big\| \int f_i\, dm \Big\|^r\Big)^{\frac{1}{r}} \le K \,\Big\| \Big(\sum_{i=1}^n |f_i|^r\Big)^{\frac{1}{r}} \Big\|_{L^r(m)}$$
holds for every $f_1, \dots, f_n \in L^r(m)$, $n \in \mathbb{N}$.

Proof. (1)⇒(2). Observe first that $S \times S$ is a convex compact subset of the linear space $L^1(m)' \times L^1(m)'$ endowed with the (locally convex) product topology $\mathfrak{T}$ obtained from $(L^1(m)', \text{weak}^*)$. We now divide the proof of the implication (1)⇒(2) in several steps.

Step 1. Fix $f_1, \dots, f_n \in L^p(m)$ and $g_1, \dots, g_n \in L^q(m)$. Using (1), the fact that S is positively norming and Young's inequality we obtain
$$\begin{aligned}
&\sum_{i=1}^n \Big\| \int f_i g_i\, dm \Big\|^r \le \\
&\qquad \le K^r \sup_{h \in S} \Big(\sum_{i=1}^n \int |f_i|^p h\, d\mu\Big)^{\frac{r}{p}} \sup_{h \in S} \Big(\sum_{i=1}^n \int |g_i|^q h\, d\mu\Big)^{\frac{r}{q}} \le \\
&\qquad \le \frac{K^r\, r}{p} \sup_{h \in S} \Big(\sum_{i=1}^n \int |f_i|^p h\, d\mu\Big) + \frac{K^r\, r}{q} \sup_{h \in S} \Big(\sum_{i=1}^n \int |g_i|^q h\, d\mu\Big). \qquad (2.1)
\end{aligned}$$

Define the function $\phi : S \times S \to \mathbb{R}$ (depending on the f_i's and g_i's) by
$$\phi(u, v) := \sum_{i=1}^n \Big\| \int f_i g_i\, dm \Big\|^r - K^r r \sum_{i=1}^n \Big(\frac{1}{p} \int |f_i|^p u\, d\mu + \frac{1}{q} \int |g_i|^q v\, d\mu \Big) \qquad (2.2)$$
for all $(u, v) \in S \times S$. Clearly, ϕ is affine (hence convex) and $\mathfrak{T}$-continuous. Inequality (2.1) can be read as
$$\inf_{(u,v) \in S \times S} \phi(u, v) \le 0$$

and, since this infimum is attained (bear in mind that $S \times S$ is $\mathfrak{T}$-compact and ϕ is $\mathfrak{T}$-continuous), it follows that $\phi(u_\phi, v_\phi) \leq 0$ for some $(u_\phi, v_\phi) \in S \times S$.

Step 2. Let Φ be the family made up of all ϕ's which can be constructed (as in *Step 1*) from different sets of functions in $L^p(m)$ and $L^q(m)$. We claim that Φ is a convex cone of $\mathbb{R}^{S\times S}$. Indeed, take $\alpha_1, \alpha_2 \geq 0$ and, for each $j \in \{1,2\}$, take $n_j \in \mathbb{N}$ and choose functions $f_{1,j}, \dots, f_{n_j,j} \in L^p(m)$ and $g_{1,j}, \dots, g_{n_j,j} \in L^q(m)$ whose associated function belonging to Φ (via (2.2)) is denoted by ϕ_j. Let $\phi \in \Phi$ be the function associated to the collection:

$$\alpha_j^{1/p} f_{i,j} \in L^p(m), \quad \alpha_j^{1/q} g_{i,j} \in L^q(m), \quad j \in \{1,2\},\ i \in \{1, \dots, n_j\}.$$

A direct computation shows that $\alpha_1\phi_1 + \alpha_2\phi_2 = \phi$. This proves the claim.

Step 3. Ky Fan's lemma (*cf.* [5, Lemma 9.10]) applied to the family Φ ensures the existence of $u_0, v_0 \in S$ such that $\phi(u_0, v_0) \leq 0$ for all $\phi \in \Phi$. In particular, for each $f_1 \in L^p(m)$ and $g_1 \in L^q(m)$ we have

$$\Big\| \int f_1 g_1 \, dm \Big\|^r \leq \frac{K^r r}{p} \Big(\int |f_1|^p u_0 \, d\mu \Big) + \frac{K^r r}{q} \Big(\int |g_1|^q v_0 \, d\mu \Big). \tag{2.3}$$

Take $f \in L^p(m)$ and $g \in L^q(m)$. Suppose without loss of generality that $a := (\int |f|^p u_0 \, d\mu)^{1/p}$ and $b := (\int |g|^q v_0 \, d\mu)^{1/q}$ are non-zero. Inequality (2.3) applied to $f_1 := (1/a) f \in L^p(m)$ and $g_1 := (1/b) g \in L^q(m)$ yields

$$\frac{1}{a^r b^r} \Big\| \int fg \, dm \Big\|^r \leq$$
$$\leq \frac{K^r r}{p\, a^p} \Big(\int |f|^p u_0 \, d\mu \Big) + \frac{K^r r}{q\, b^q} \Big(\int |g|^q v_0 \, d\mu \Big) = \frac{K^r r}{p} + \frac{K^r r}{q} = K^r,$$

hence $\| \int fg \, dm \| \leq K (\int |f|^p u_0 \, d\mu)^{1/p} (\int |g|^q v_0 \, d\mu)^{1/q}$. This completes the proof of the implication (1)$\Rightarrow$(2).

(2)$\Rightarrow$(3). We first show that $u_0 \, d\mu$ is a control measure for m. To this end, take $A \in \Sigma$ with $(u_0 \, d\mu)(A) = 0$, that is, $\int \chi_A u_0 \, d\mu = 0$. Given any $B \subset A$ with $B \in \Sigma$, condition (2) applied to $f = \chi_B$ and $g = 1$ implies that $m(B) = 0$, hence $\|m\|(A) = 0$. Similarly, $v_0 \, d\mu$ is a control measure for m.

Let $Y_1 \subset L^p(u_0 \, d\mu)$ and $Y_2 \subset L^q(v_0 \, d\mu)$ be the linear subspaces made up of all simple functions. Given $f \in Y_1$ and $g \in Y_2$, their product fg is again a simple function, so it belongs to $L^1(m)$. Its norm can be computed as $\|fg\|_{L^1(m)} = \sup_{z \in B_{L^\infty(\mu)}} \| \int fgz \, dm \|$ (*cf.* [12, Lemma 3.11]) and condition (2) yields

$$\|fg\|_{L^1(m)} = \sup_{z \in B_{L^\infty(\mu)}} \Big\| \int fgz \, dm \Big\| \leq K \, \|f\|_{L^p(u_0 \, d\mu)} \, \|g\|_{L^q(v_0 \, d\mu)}.$$

Thus we can define a bilinear continuous mapping $\mathcal{P} : Y_1 \times Y_2 \to L^1(m)$ by $\mathcal{P}(f,g) := fg$. Since Y_1 and Y_2 are dense in $L^p(u_0 \, d\mu)$ and $L^q(v_0 \, d\mu)$, respectively, a standard argument ensures the existence of a bilinear continuous mapping $\overline{\mathcal{P}} : L^p(u_0 \, d\mu) \times L^q(v_0 \, d\mu) \to L^1(m)$ extending $\mathcal{P}$.

We claim that $fg \in L^1(m)$ and $fg = \overline{\mathcal{P}}(f,g)$ whenever $f \in L^p(u_0 \, d\mu)$ and $g \in L^q(v_0 \, d\mu)$. Indeed, choose sequences (f_n) in Y_1 and (g_n) in Y_2 such that

$\|f_n - f\|_{L^p(u_0\,d\mu)} \to 0$ and $\|g_n - g\|_{L^q(v_0\,d\mu)} \to 0$. We can assume without loss of generality that $f_n \to f$ $u_0\,d\mu$-a.e. and $g_n \to g$ $v_0\,d\mu$-a.e. Then $f_n g_n \to fg$ μ-a.e. On the other hand, the continuity of $\overline{\mathcal{P}}$ ensures that $\overline{\mathcal{P}}(f_n, g_n) = f_n g_n \to \overline{\mathcal{P}}(f,g)$ in $L^1(m)$, and so we have $f_n g_n \to \overline{\mathcal{P}}(f,g)$ in $L^1(\mu)$ as well (because $\mu = |\langle m, x_0'\rangle|$ for some $x_0' \in B_{X'}$). It follows that $fg \in L^1(m)$ and $fg = \overline{\mathcal{P}}(f,g)$.

Therefore, for each $f \in L^p(u_0\,d\mu)$ we can define a multiplication operator

$$M_f : L^q(v_0\,d\mu) \to L^1(m), \quad M_f(g) := fg,$$

with norm $\|M_f\| \le \|\overline{\mathcal{P}}\|\,\|f\|_{L^p(u_0\,d\mu)}$. The natural mapping $f \rightsquigarrow M_f$ is a one-to-one operator from $L^p(u_0\,d\mu)$ to $\mathcal{M}(L^q(v_0\,d\mu), L^1(m))$, as required.

(3)⇒(4). Set $h_0 := u_0^{r/p} v_0^{r/q}$. Since $0 \le h_0 \le \frac{r}{p}u_0 + \frac{r}{q}v_0$ (by Young's inequality) and $\frac{r}{p}u_0 + \frac{r}{q}v_0 \in B^+_{L^1(m)'}$, we also have $h_0 \in B^+_{L^1(m)'}$. Moreover, since $u_0\,d\mu$ and $v_0\,d\mu$ are control measures for m, we can assume without loss of generality that $u_0 > 0$ and $v_0 > 0$ pointwise.

Fix $h \in L^r(h_0\,d\mu)$. Set

$$f := \operatorname{sign}(h)|h|^{\frac{r}{p}}\Big(\frac{v_0}{u_0}\Big)^{\frac{r}{pq}} \in L^p(u_0\,d\mu) \quad \text{and} \quad g := |h|^{\frac{r}{q}}\Big(\frac{u_0}{v_0}\Big)^{\frac{r}{pq}} \in L^q(v_0\,d\mu).$$

According to (3), $h = fg \in L^1(m)$ and

$$\|h\|_{L^1(m)} \le \|M_f\|\,\|g\|_{L^q(v_0\,d\mu)} \le K\,\|f\|_{L^p(u_0\,d\mu)}\,\|g\|_{L^q(v_0\,d\mu)}$$

for some constant $K > 0$ independent of h. But

$$\|f\|_{L^p(u_0\,d\mu)}\,\|g\|_{L^q(v_0\,d\mu)} = \Big(\int |h|^r h_0\,d\mu\Big)^{\frac{1}{p}}\Big(\int |h|^r h_0\,d\mu\Big)^{\frac{1}{q}} = \|h\|_{L^r(h_0\,d\mu)},$$

hence $\|h\|_{L^1(m)} \le K\|h\|_{L^r(h_0\,d\mu)}$.

This shows that the 'identity' mapping from $L^r(h_0\,d\mu)$ to $L^1(m)$ is a well-defined one-to-one operator. In particular, $h_0\,d\mu$ is a control measure for m and so $L^1(m) \hookrightarrow L^1(h_0\,d\mu)$, hence $L^r(m) \hookrightarrow L^r(h_0\,d\mu)$.

(4)⇒(5). Just bear in mind that the condition $L^r(m) \hookrightarrow L^r(h_0\,d\mu)$ implies that $h_0\,d\mu$ is a control measure for m.

(5)⇒(6). By Remark 1.1 we can write $\lambda = h\,d\mu$ for some $h \in L^1(m)'$ with $h \ge 0$. We can assume further that $h \in B^+_{L^1(m)'}$. Let $K > 0$ be a constant such that $\|f\|_{L^1(m)} \le K\|f\|_{L^r(\lambda)}$ for all $f \in L^r(\lambda)$. Given simple functions $f_1, \ldots, f_n \in L^r(m)$, we have

$$\begin{aligned}\Big(\sum_{i=1}^n \Big\|\int f_i\,dm\Big\|^r\Big)^{\frac{1}{r}} &\le \Big(\sum_{i=1}^n \|f_i\|^r_{L^1(m)}\Big)^{\frac{1}{r}} \le \\ &\le K\Big(\sum_{i=1}^n \|f_i\|^r_{L^r(\lambda)}\Big)^{\frac{1}{r}} = K\Big(\int\Big(\sum_{i=1}^n |f_i|^r\Big)\,d\lambda\Big)^{\frac{1}{r}} = \\ &= K\Big(\int\Big(\sum_{i=1}^n |f_i|^r\Big)h\,d\mu\Big)^{\frac{1}{r}} \le K\Big\|\Big(\sum_{i=1}^n |f_i|^r\Big)^{\frac{1}{r}}\Big\|_{L^r(m)}.\end{aligned}$$

Since simple functions are dense in $L^r(m)$ (because this space is order continuous, *cf.* [12, Proposition 3.28]), the r-concavity of I_m^r can be deduced easily from the previous chain of inequalities. To this end, it suffices to bear in mind that the mapping from $L^r(m)$ to $L^1(m)$ given by $f \rightsquigarrow |f|^r$ is continuous because, as in the case of scalar measures (*cf.* [13, Chapter 3, Exercise 24]), the inequality

$$\big\| |f|^r - |g|^r \big\|_{L^1(m)} \leq r \left(\|f\|_{L^r(m)}^{r-1} + \|g\|_{L^r(m)}^{r-1} \right) \|f-g\|_{L^r(m)}$$

holds for all $f, g \in L^r(m)$.

(6)⇒(1). Given $f_1, \ldots, f_n \in L^p(m)$ and $g_1, \ldots, g_n \in L^q(m)$, each product $f_i g_i$ belongs to $L^r(m)$ and the r-concavity of I_m^r yields

$$\Big(\sum_{i=1}^n \Big\| \int f_i g_i \, dm \Big\|^r \Big)^{\frac{1}{r}} \leq K \Big\| \Big(\sum_{i=1}^n |f_i g_i|^r \Big)^{\frac{1}{r}} \Big\|_{L^r(m)}. \tag{2.4}$$

By Hölder's inequality (for real numbers!) we have

$$\sum_{i=1}^n |f_i g_i|^r \leq \Big(\sum_{i=1}^n |f_i|^p \Big)^{\frac{r}{p}} \Big(\sum_{i=1}^n |g_i|^q \Big)^{\frac{r}{q}},$$

hence for each $x' \in B_{X'}$ the inequality

$$\int \Big(\sum_{i=1}^n |f_i g_i|^r \Big) \, d|\langle m, x' \rangle| \leq \int \Big(\sum_{i=1}^n |f_i|^p \Big)^{\frac{r}{p}} \Big(\sum_{i=1}^n |g_i|^q \Big)^{\frac{r}{q}} \, d|\langle m, x' \rangle|$$

holds and again Hölder's inequality (now for integrals!) applied to the right hand side of the previous inequality allows us to conclude that

$$\begin{aligned} \int \Big(\sum_{i=1}^n |f_i g_i|^r \Big) \, d|\langle m, x' \rangle| &\leq \\ &\leq \Big(\int \Big(\sum_{i=1}^n |f_i|^p \Big) \, d|\langle m, x' \rangle| \Big)^{\frac{r}{p}} \Big(\int \Big(\sum_{i=1}^n |g_i|^q \Big) \, d|\langle m, x' \rangle| \Big)^{\frac{r}{q}} \leq \\ &\leq \Big\| \Big(\sum_{i=1}^n |f_i|^p \Big)^{\frac{1}{p}} \Big\|_{L^p(m)}^r \Big\| \Big(\sum_{i=1}^n |g_i|^q \Big)^{\frac{1}{q}} \Big\|_{L^q(m)}^r. \end{aligned}$$

As $x' \in B_{X'}$ is arbitrary, it follows that

$$\Big\| \Big(\sum_{i=1}^n |f_i g_i|^r \Big)^{\frac{1}{r}} \Big\|_{L^r(m)} \leq \Big\| \Big(\sum_{i=1}^n |f_i|^p \Big)^{\frac{1}{p}} \Big\|_{L^p(m)} \Big\| \Big(\sum_{i=1}^n |g_i|^q \Big)^{\frac{1}{q}} \Big\|_{L^q(m)},$$

which combined with (2.4) yields the inequality in (1). The proof of the theorem is over. □

Remark 2.4. In the previous theorem, the equivalence (1)⇔(2) can also be obtained as a particular case of a result of Defant [4, Theorem 1]. The equivalence (4)⇔(6) can be found essentially in [12, Section 6.4], see in particular Lemma 6.39, Proposition 6.40 and Theorem 6.41.

Given $1 \leq p < \infty$, a Banach function space E is called p-concave (resp. p-convex) if the identity operator on E is p-concave (resp. p-convex), that is, there is a constant $K > 0$ such that the inequality

$$\Big(\sum_{i=1}^{n} \|z_i\|_E^p\Big)^{\frac{1}{p}} \leq K \Big\| \Big(\sum_{i=1}^{n} |z_i|^p\Big)^{\frac{1}{p}} \Big\|_E \quad \text{(resp. the reverse one)}$$

holds for every $z_1, \dots, z_n \in E$, $n \in \mathbb{N}$. It is known that E *is order isomorphic to the* L^p *space of a non-negative scalar measure whenever it is simultaneously* p*-concave and* p*-convex*, *cf.* [9, p. 59]. As $L^p(m)$ is always p-convex, the following result (*cf.* [12, Proposition 3.74]) can be seen as a specialized version of the previous statement.

Corollary 2.5. *The following statements are equivalent:*
(1) $L^p(m)$ *is* p*-concave for some* $1 \leq p < \infty$.
(2) $L^1(m)$ *is* 1*-concave.*
(3) $L^p(m)$ *is* p*-concave for every* $1 \leq p < \infty$.
(4) *The integration operator* $I_m^1 : L^1(m) \to X$ *is* 1*-concave.*
(5) *There is* $h_0 \in B^+_{L^1(m)'}$ *such that the 'identity' map from* $L^1(m)$ *to* $L^1(h_0\, d\mu)$ *is an isomorphism.*
(6) *There is a control measure* λ *for* m *such that* $L^1(m)$ *is order isomorphic to* $L^1(\lambda)$.

Proof. The equivalence (1)⇔(2)⇔(3) follows from a simple computation. (2)⇒(4) is straightforward. (4)⇒(5) follows from the implication (6)⇒(4) in Theorem 2.3 (taking there $r = 1$). For (5)⇒(6) just observe that $h_0\, d\mu$ is a control measure for m. Finally, the implication (6)⇒(2) is a consequence of the 1-concavity of $L^1(\lambda)$ and the general fact that p-concavity is preserved by order isomorphisms (*cf.* [9, Proposition 1.d.9]). □

Following [1], we say that an operator T from a Banach function space E to X is *positive* p*-summing* (where $1 \leq p < \infty$) if there is a constant $K > 0$ such that the inequality

$$\Big(\sum_{i=1}^{n} \|Tz_i\|^p\Big)^{\frac{1}{p}} \leq K \sup_{z' \in B_{E'}} \Big(\sum_{i=1}^{n} |\langle z_i, z'\rangle|^p\Big)^{\frac{1}{p}}$$

holds for every $z_1, \dots, z_n \in E^+$. This property lies strictly between being absolutely p-summing and being p-concave, see [1]. For more information about this subject, we refer the reader to [5]. We will need the following folk characterization of positive p-summing operators.

Remark 2.6. *Let* T *be an operator from a Banach function space* E *to* X *and let* $1 \leq p < \infty$. *Then* T *is positive* p*-summing if and only if there is a constant* $K > 0$ *such that the inequality*

$$\Big(\sum_{i=1}^{n} \|Tz_i\|^p\Big)^{\frac{1}{p}} \leq K \sup_{z' \in B_{E'}} \Big(\sum_{i=1}^{n} |\langle |z_i|, z'\rangle|^p\Big)^{\frac{1}{p}}$$

holds for every $z_1, \dots, z_n \in E$.

We arrive at the last result of the paper.

Theorem 2.7. *Let E be an order continuous Banach function space having weak order unit. The following statements are equivalent:*

(1) *For every vector measure ν representing E and every $1 \le p < \infty$, the integration operator I^1_ν is positive p-summing.*
(2) *There exist a vector measure ν representing E and $1 \le p < \infty$ such that I^1_ν is positive p-summing.*
(3) *There exist a vector measure ν representing E and $1 \le p < \infty$ such that I^1_ν is absolutely p-summing.*
(4) *E is order isomorphic to the L^1 space of a non-negative scalar measure.*

Proof. (1)⇒(2) and (3)⇒(2) are obvious. For the implication (4)⇒(3), just bear in mind that the integration operator of the L^1 space of a non-negative scalar measure has rank 1 and, therefore, it is absolutely p-summing for any $1 \le p < \infty$.

(2)⇒(4). The case $p = 1$ follows from Corollary 2.5 since I^1_ν is p-concave. Assume now that $p > 1$ and let $q > 1$ such that $1/p + 1/q = 1$. By Corollary 2.5, we only have to check that $L^p(\nu)$ is p-concave. To this end, fix $f_1, \dots, f_n \in L^p(\nu)$. Take arbitrary $g_1, \dots, g_n \in B_{L^q(\nu)}$ and denote by μ_0 a fixed Rybakov control measure for ν. Since I^1_ν is positive p-summing, Remark 2.6 ensures that

$$\sum_{i=1}^n \Big\| \int f_i g_i \, d\nu \Big\|^p \le K^p \sup_{h \in B_{L^1(\nu)'}} \sum_{i=1}^n |\langle |f_i g_i|, h\rangle|^p =$$
$$= K^p \sup_{h \in B_{L^1(\nu)'}} \sum_{i=1}^n \langle |f_i g_i|, |h|\rangle^p = K^p \sup_{h \in B_{L^1(\nu)'}} \sum_{i=1}^n \Big(\int |f_i g_i||h| \, d\mu_0 \Big)^p \quad (2.5)$$

for some constant $K > 0$ which depends only on I^1_ν. For each $1 \le i \le n$ and each $h \in B_{L^1(\nu)'}$, Hölder's inequality implies

$$\int |f_i g_i||h| \, d\mu_0 \le \Big(\int |f_i|^p |h| \, d\mu_0 \Big)^{\frac{1}{p}} \Big(\int |g_i|^q |h| \, d\mu_0 \Big)^{\frac{1}{q}} \le$$
$$\le \Big(\int |f_i|^p |h| \, d\mu_0 \Big)^{\frac{1}{p}} \|g_i\|_{L^q(\nu)} \le \Big(\int |f_i|^p |h| \, d\mu_0 \Big)^{\frac{1}{p}},$$

which combined with (2.5) yields

$$\sum_{i=1}^n \Big\| \int f_i g_i \, d\nu \Big\|^p \le K^p \sup_{h \in B_{L^1(\nu)'}} \sum_{i=1}^n \int |f_i|^p |h| \, d\mu_0 =$$
$$= K^p \sup_{h \in B_{L^1(\nu)'}} \int \Big(\sum_{i=1}^n |f_i|^p \Big) |h| \, d\mu_0 \le K^p \Big\| \Big(\sum_{i=1}^n |f_i|^p \Big)^{\frac{1}{p}} \Big\|^p_{L^p(\nu)}.$$

Since each $\|f_i\|_{L^p(\nu)}$ can be computed as the supremum of $\|\int f_i g\,d\nu\|$ where g runs over $B_{L^q(\nu)}$ (*cf.* [12, (3.64)]), we conclude that

$$\Big(\sum_{i=1}^n \|f_i\|_{L^p(\nu)}^p\Big)^{\frac{1}{p}} \le K\Big\|\Big(\sum_{i=1}^n |f_i|^p\Big)^{\frac{1}{p}}\Big\|_{L^p(\nu)}.$$

It follows that $L^p(m)$ is p-concave, as required.

(4)⇒(1). Let $T : L^1(\lambda) \to L^1(\nu)$ be an order isomorphism, where λ is a non-negative scalar measure. We can assume without loss of generality that $\|T^{-1}\| = 1$. Then the functional $h \in B^+_{L^1(\nu)'}$ defined by the formula $\langle f, h\rangle := \int T^{-1}(f)\,d\lambda$ satisfies

$$\langle |f|, h\rangle = \int T^{-1}(|f|)\,d\lambda = \int |T^{-1}(f)|\,d\lambda = \|T^{-1}(f)\|_{L^1(\lambda)} \ge \frac{\|f\|_{L^1(\nu)}}{\|T\|}$$

for all $f \in L^1(\nu)$. Given $f_1, \ldots, f_n \in L^1(\nu)^+$, we have

$$\sum_{i=1}^n \Big\|\int f_i\,d\nu\Big\| \le \sum_{i=1}^n \|f_i\|_{L^1(\nu)} \le \|T\| \sum_{i=1}^n \langle f_i, h\rangle \le \|T\| \sup_{z' \in B_{L^1(\nu)'}} \sum_{i=1}^n |\langle f_i, z'\rangle|.$$

Therefore, the operator I^1_ν is positive 1-summing. By [1, Proposition 2], I^1_ν is also positive p-summing for all $1 \le p < \infty$. The proof is over. □

Remark 2.8. For an order continuous Banach function space E having weak order unit, in general the statements of Theorem 2.7 are not equivalent to the following one:

For every vector measure ν representing E and every $1 \le p < \infty$, the integration operator I^1_ν is absolutely p-summing.

Indeed, observe that the E-valued measure $A \rightsquigarrow \chi_A$ (the characteristic function of A) represents E and its corresponding integration operator is just the identity mapping on E, which is not absolutely p-summing (for any $1 \le p < \infty$) whenever E is infinite-dimensional.

We finish the paper with two questions:

1. We have shown that concavity type properties for the integration operator characterize the continuous injection of Lebesgue spaces in $L^1(m)$. Is it possible to generalize these ideas to characterize the continuous injection of other classical spaces (Lorentz, Orlicz, etc.) in $L^1(m)$?
2. Lozanovskii lattice interpolation spaces obtained from Lebesgue spaces and $L^1(m)$ are a well described class of Banach function spaces. Which are the vector-valued norm inequalities for the integration operator that characterize the continuous injection of such spaces in $L^1(m)$?

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