UNIFORMLY CONVEX FUNCTIONS ON BANACH SPACES

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Abstract. We study the connection between uniformly convex functions $f : X \to \mathbb{R}$ bounded above by $\|x\|^p$, and the existence of norms on X with moduli of convexity of power type. In particular, we show that there exists a uniformly convex function $f : X \to \mathbb{R}$ bounded above by $\|\cdot\|^2$ if and only if X admits a norm with modulus of convexity of power type 2.

1. Introduction

Uniformly convex functions on Banach spaces were introduced by Levitin and Poljak in [12]. Their properties were studied in depth by Zălinescu [15], and then later Azé and Penot [2] studied their duality with uniformly smooth convex functions; see also [16] for more details. Yet, surprisingly, little precise information seems to be known about when they can exist on Banach spaces. For example, [16, Theorem 3.5.13], shows that a Banach space admitting a uniformly convex function whose domain has nonempty interior is reflexive, and in fact, it can be shown that such a Banach space is superreflexive—see Theorem 2.4 (recall that a Banach space is *superreflexive* if and only if it admits an equivalent uniformly convex norm [9]). On the other hand, [4] shows that if $\|\cdot\|$ is uniformly convex, then the function $f(x) = \|x\|^r$ for $r > 1$ is totally convex which is weaker than f being uniformly convex; see [3, 5, 6] for applications of totally convex and other related convex functions. In this note, we will provide precise information as to when $f(x) = \|x\|^r$ is uniformly convex. We also examine the more general converse problem: if $f : X \to \mathbb{R}$ is uniformly convex and bounded above by $\|\cdot\|^r$, does X admit a norm with a modulus of convexity of power type related to r?

We work with a real Banach space X with dual X^*, and let B_X and S_X denote the closed unit ball and sphere respectively. The *Fenchel conjugate* of f is the function $f^* : X^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{x^*(x) - f(x) : x \in X\}.$$

For a given convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ we define its *modulus of convexity* as the function $\delta_f : \mathbb{R}^+ \to [0, +\infty]$ defined by

$$\delta_f(t) := \inf\left\{\frac{1}{2}f(x) + \frac{1}{2}f(y) - f\left(\frac{x+y}{2}\right) : \|x - y\| = t,\ x, y \in \operatorname{dom} f\right\},$$

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where the infimum over the empty set is $+\infty$. Similarly we consider the *modulus of smoothness* of $f : X \to \mathbb{R}$ as the function $\rho_f : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\rho_f(t) := \sup \left\{ \frac{1}{2} f(x) + \frac{1}{2} f(y) - f\left(\frac{x+y}{2}\right) : \|x-y\| = t \right\}.$$

We say that f is *uniformly convex* when $\delta_f(t) > 0$ for all $t > 0$, and f has a *modulus of convexity of power type* p if there exists $C > 0$ so that $\delta_f(t) \geq Ct^p$ for all $t > 0$. We will say f is *uniformly smooth* if $\lim_{t\to 0^+} \rho_f(t)/t = 0$, and f has a *modulus of smoothness of power type* q if there is a constant $C > 0$, so that $\rho_f(t) \leq Ct^q$ for all $t > 0$. Let us note that these concepts are sometimes defined using the *gage of uniform convexity* and *gage of uniform smoothness* respectively as found in [16]; it is important to note that these alternate definitions using the respective gages are equivalent to those just given; cf. [15, Remark 2.1] and [16, p. 205].

2. Uniform Convexity of Functions and Norms

This section will demonstrate for $1 < p < \infty$ that $f(\cdot) = \|\cdot\|^p$ is uniformly convex if and only if $\|\cdot\|$ has modulus of convexity of power type p.

Lemma 2.1. *Let* $0 < r \leq 1$ *and* $f(t) = t^r$. *Then there exists* $C > 0$ *such that* $|f(t) - f(s)| \leq C\,|t-s|^r$ *for all* $s,\ t \in [0,\infty)$.

Proof. We will show this works with $C = 2^r$. Without loss of generality assume $0 \leq s \leq t$. First, if $s \leq t/2$ we compute

$$t^r - s^r \leq t^r = 2^r \left(\frac{t}{2}\right)^r \leq 2^r (t-s)^r.$$

Otherwise, if $s > t/2$, then by the Mean Value Theorem

$$\begin{aligned} t^r - s^r &= rc^{r-1}(t-s) \text{ for some } s < c < t \\ &= rc^{r-1}(t-s)^r (t-s)^{1-r} \\ &\leq rc^{r-1}(t-s)^r c^{1-r} = r(t-s)^r, \end{aligned}$$

since $c \geq t/2 \geq t-s$ in this case. Hence we can choose $C = \max\{r, 2^r\} = 2^r$. □

Theorem 2.2. *For* $1 < q \leq 2$, *the following are equivalent in a Banach space* X.
(a) The norm $\|\cdot\|$ *has modulus of smoothness of power type* q.
(b) The function $f(\cdot) = \|\cdot\|^q$ *has modulus of smoothness of power type* q.
(c) The function $f(\cdot) = \|\cdot\|^q$ *is uniformly smooth.*

Proof. (a) $\Rightarrow$ (b): We assume that $\|\cdot\|$ has modulus of smoothness of power type q. Then it has a (Fréchet) derivative satisfying a $(q-1)$-Hölder-condition on its sphere, see [7, Lemma V.5.1]. Moreover, $f(x) = \|x\|^q$ satisfies $f'(0) = 0$ and $f'(x) = q\,\|x\|^{q-1}\,\phi_x$ where $\phi_x \in J(x)$, the duality map, for $x \neq 0$ (i.e. $\phi_x \in S_X$, and $\phi_x(x) = \|x\|$). Observe that if $x = 0$ or $y = 0$ then $\|f'(x) - f'(y)\| \leq q\,\|x-y\|^{q-1}$. Assuming that $x, y \in X \setminus \{0\}$ we compute,

$$\begin{aligned} f'(x) - f'(y) &= q\,\|x\|^{q-1}\,\phi_x - q\,\|y\|^{q-1}\,\phi_y \\ &= q\,\|x\|^{q-1}\,\phi_x - q\,\|x\|^{q-1}\,\phi_y + q\,\|x\|^{q-1}\,\phi_y - q\,\|y\|^{q-1}\,\phi_y \\ &= q\,\|x\|^{q-1}\,(\phi_x - \phi_y) + \left(q\,\|x\|^{q-1} - q\,\|y\|^{q-1}\right)\phi_y. \end{aligned} \tag{2.1}$$

Using Lemma 2.1 we also compute

$$\left| q\,\|x\|^{q-1} - q\,\|y\|^{q-1} \right| \le 2^{q-1} q \big|\, \|x\| - \|y\| \,\big|^{q-1} \le 2^{q-1} q\,\|x-y\|^{q-1}. \tag{2.2}$$

We now work on an estimate for $q\,\|x\|^{q-1}(\phi_x - \phi_y)$. First, consider the case where $0 < \|y\| \le \|x\| \le 1$. If $\|y\| \le \|x\|/2$, then

$$q\,\|x\|^{q-1}\,\|\phi_x - \phi_y\| \le 2q\,\|x\|^{q-1} \le q2^q\,\|x-y\|^{q-1}.$$

If $\|x\| \le 1$ and $\|y\| \ge \|x\|/2$, consider $x' = \lambda x$ where $\lambda = \|y\|/\|x\|$, so that $\|x'\| = \|y\|$. Then

$$\begin{aligned} \|x'-y\| &\le \|x'-x\| + \|x-y\| \\ &= \|x\| - \|y\| + \|x-y\| \le 2\,\|x-y\|. \end{aligned}$$

Thus, given the Hölder-condition for the derivative on spheres, there is $C > 0$ such that $\|\phi_u - \phi_v\| \le C\,\|u-v\|^{q-1}/\|u\|^{q-1}$ when $\|u\| = \|v\|$, and so we have

$$\|\phi_x - \phi_y\| = \|\phi_{x'} - \phi_y\| \le C\left(\frac{2\,\|x-y\|}{\|y\|}\right)^{q-1}.$$

Consequently,

$$q\,\|x\|^{q-1}\,\|\phi_x - \phi_y\| \le Cq\frac{\|x\|^{q-1}}{\|y\|^{q-1}}2^{q-1}\,\|x-y\|^{q-1} \le C2^{2q-2}q\,\|x-y\|^{q-1}.$$

Hence in either case we have $K > 0$ so that for $x, y \in B_X$ we have

$$q\,\|x\|^{q-1}\,\|\phi_x - \phi_y\| \le K\,\|x-y\|^{q-1}.$$

Now consider the case $\|x\| > 1$, and let $\lambda = \|x\|$. Denote $u = x/\lambda$ and $v = y/\lambda$. Then $u, v \in B_X$ and $\|u-v\| = \|x-y\|/\lambda$. Thus one can write

$$\begin{aligned} q\,\|x\|^{q-1}\,\|\phi_x - \phi_y\| &= q\,\|x\|^{q-1}\,\|\phi_u - \phi_v\| \\ &\le q\,\|x\|^{q-1} K\,\|u-v\|^{q-1} \\ &= q\,\|x\|^{q-1}\frac{1}{\|x\|^{q-1}}K\,\|x-y\|^{q-1} \\ &= Kq\,\|x-y\|^{q-1}. \end{aligned}$$

Consequently, in any case

$$q\,\|x\|^{q-1}\,\|\phi_x - \phi_y\| \le Kq\,\|x-y\|^{q-1}. \tag{2.3}$$

Combining (2.1), (2.2) and (2.3) shows that for $f(x) = \|x\|^q$, $f'(x)$ satisfies a $(q-1)$-Hölder-condition and hence that $\|x\|^q$ has modulus of smoothness of power type q, on appealing to [7, Lemma V.5.1] again.

Now (b) $\Rightarrow$ (c) is trivial, so we prove (c) $\Rightarrow$ (a). Suppose $\|\cdot\|$ does not have modulus of smoothness of power type q. Then there are $x_n, y_n \in S_X$ such that $\|x_n - y_n\| \to 0$ while

$$\|\phi_{x_n} - \phi_{y_n}\| \ge n\,\|x_n - y_n\|^{q-1}.$$

Let $\delta_n = \|x_n - y_n\|$ and define $u_n = \frac{1}{\delta_n\sqrt{n}}x_n$ and $v_n = \frac{1}{\delta_n\sqrt{n}}y_n$. Then $\|u_n - v_n\| = \frac{1}{\sqrt{n}} \to 0$. However

$$\begin{aligned}\|f'(u_n) - f'(v_n)\| &= \left\|q\,\|u_n\|^{q-1}\,\phi_{u_n} - q\,\|v_n\|^{q-1}\,\phi_{v_n}\right\| \\ &= \left\|q\,\|u_n\|^{q-1}\,\phi_{x_n} - q\,\|v_n\|^{q-1}\,\phi_{y_n}\right\| \\ &= \frac{q}{\delta_n^{q-1}n^{\frac{q-1}{2}}}\,\|\phi_{x_n} - \phi_{y_n}\| \\ &\geq \frac{q}{\delta_n^{q-1}n^{\frac{q-1}{2}}}\left(n\delta_n^{q-1}\right) = qn^{\frac{3-q}{2}} \to \infty.\end{aligned}$$

It follows that $f(\cdot) = \|\cdot\|^q$ is not a uniformly smooth convex function. □

The results in [2] enable us to derive the dual version of Theorem 2.2 for uniformly convex functions.

Theorem 2.3. *Let X be a Banach space, and let $2 \leq p < \infty$. Then the following are equivalent.*
(a) The norm $\|\cdot\|$ on X has modulus of convexity of power type p.
(b) The function $f(\cdot) = \|\cdot\|^p$ has modulus of convexity of power type p.
(c) The function $f(\cdot) = \|\cdot\|^p$ is uniformly convex.

Proof. (a) ⇒ (b): Let us assume that $\|\cdot\|$ has modulus of convexity of power type p, then the modulus of smoothness of the dual norm on X^*, namely $\|\cdot\|_*$, is of power type q where $\frac{1}{p} + \frac{1}{q} = 1$; see [7]. By Theorem 2.2 the function $g(\cdot) = \frac{1}{q}\|\cdot\|_*^q$ has modulus of smoothness of power type q. The Fenchel conjugate of g is $g^*(\cdot) = \frac{1}{p}\|\cdot\|^p$, see [2, 16]. Now g^*—and hence $\|\cdot\|^p$—has a modulus of convexity of power type p according to [2] (see also [16, Corollary 3.5.11]).

Observe that (b) ⇒ (c) is trivial, so we prove (c) ⇒ (a). Indeed, assuming that $f(\cdot) = \|\cdot\|^p$ is a uniformly convex function, then [2] shows that f^* (and hence $\|\cdot\|_*^q$) is a uniformly smooth function. According to Theorem 2.2, $\|\cdot\|_*$ has modulus of smoothness of power type q; therefore $\|\cdot\|$ has modulus of convexity of power type p, see [7]. □

We close this section by confirming that the spaces with nontrivial uniformly convex functions are the superreflexive spaces.

Theorem 2.4. *Let X be a Banach space. Then the following are equivalent.*
(a) There exists a l.s.c. uniformly convex function $f : X \to (-\infty, +\infty]$ such that the interior of the domain of f is not empty.
(b) X admits an equivalent uniformly convex norm.
(c) There exist $p \geq 2$ and an equivalent norm $\|\cdot\|$ on X so that $f(x) = \|\cdot\|^p$ is uniformly convex.

Proof. (a) ⇒ (b): Shift f so that $0 \in \operatorname{int}(\operatorname{dom} f)$ and let $\phi \in \partial f(0)$, and replace f with $f - (f(0) + \phi)$. Let $g(x) = \frac{f(x)+f(-x)}{2}$ and let $r > 0$ be such that $B_{2r} \subset \operatorname{dom} f$. Now for $\|h\| = r$ we have

$$\frac{1}{2}g(h) + \frac{1}{2}g(0) - g\left(\frac{h}{2}\right) \geq \delta_g(r) > 0.$$

Thus $g(h) \geq 2\delta_g(r)$ for all h such that $\|h\| = r$. Let us consider the norm $|||\cdot|||$ whose unit ball is $B = \{x : g(x) \leq \delta_g(r)\}$. We have shown that $B \subset rB_{(X,\|\cdot\|)}$ and since

0 is a point of continuity of g (see [16, Corollary 2.2.13]) there exists $R > 0$ such that $|||\cdot||| \leq R\,\|\cdot\|$. Therefore $|||\cdot|||$ is an equivalent norm on X.

If $|||x_n||| = |||y_n||| = 1$ and

$$\frac{1}{2}\,|||x_n||| + \frac{1}{2}\,|||y_n||| - \left|\left|\left|\frac{x_n + y_n}{2}\right|\right|\right| \to 0,$$

then, since g is Lipschitz on B by [16, Corollary 2.2.12], $g\left(\frac{x_n+y_n}{2}\right) \to \delta_g(r)$. Consequently $\frac{1}{2}g(x_n) + \frac{1}{2}g(y_n) - g\left(\frac{x_n+y_n}{2}\right) \to 0$, so $\|x_n - y_n\| \to 0$ and hence $|||x_n - y_n||| \to 0$

(b) $\Rightarrow$ (c): According to the well-known Enflo-Pisier theorem ([9, 13]), there exist $p \geq 2$ and an equivalent norm $\|\cdot\|$ whose modulus of convexity is of power type p. Consequently, Theorem 2.3 shows the function $\|\cdot\|^p$ is uniformly convex.

(c) $\Rightarrow$ (a): This is trivial. □

Note that the indicator function of a single point in any Banach space is trivially uniformly convex. Thus, some domain interiority condition is required in Theorem 2.4(a).

3. Growth Rates Uniformly Convex Functions and Renorming

In this section we will construct a uniformly convex norm whose modulus of convexity is related to the growth rate of a given uniformly convex function on the Banach space thus sharpening Theorem 2.4.

Lemma 3.1. *Suppose $f : X \to \mathbb{R}$ is a l.s.c. uniformly convex function. Then there exist a uniformly convex l.s.c. function $h : X \to \mathbb{R}$ that is centrally symmetric with*

$$0 = h(0) = \inf\{h(x) : x \in X\}.$$

Proof. We can assume that f is centrally symmetric and u.c. by replacing f with $\frac{f(x)+f(-x)}{2}$. Observe then that $f(0) = \min\{f(x) : x \in X\}$ and thus, the function $h(x) = g(x) - g(0)$ satisfies our requirements. □

Lemma 3.2. *Let $\|\cdot\|$ be a norm on a Banach space X. Suppose $\|x\| = \|y\| \geq 1$, and $\|x - y\| \geq \delta$ where $0 < \delta \leq 2$. Then $\inf_{t\geq 0} \|x - ty\| \geq \delta/2$.*

Proof. Assume that $\|x - t_0 y\| < \delta/2$ for some $t_0 \geq 0$. Then $|1 - t_0|\,\|y\| < \delta/2$ and so

$$\|x - y\| \leq \|x - t_0 y\| + |1 - t_0|\,\|y\| < \delta.$$

which is a contradiction. □

Lemma 3.3. *Suppose F is convex and l.s.c. with $F(0) = 0$. Suppose for all $n \geq N$, we have that $\{\|\cdot\|_n\}_{n\geq N}$ are norms on $(X, \|\cdot\|)$ with*

$$\frac{K}{\sqrt{F(2^n)}}\,\|\cdot\| \leq \|\cdot\|_n \leq \frac{1}{2^n}\,\|\cdot\|,$$

for some $K > 0$. Assume that $\|x\|_n = \|y\|_n = 1$ with $\|x - y\| \geq 1$ implies that

$$\left\|\frac{x+y}{2}\right\|_n \leq 1 - \frac{C}{F\left(\frac{2}{K}\sqrt{F(2^n)}\right)}$$

for some $C > 0$. Then the modulus of convexity of the equivalent norm $|\cdot| = \sum_{n\geq N} \|\cdot\|_n$ satisfies

$$\delta_{|\cdot|}(t) \geq \frac{R}{\sqrt{F(Mt^{-1})}F\left(\frac{2}{K}\sqrt{F(Mt^{-1})}\right)},$$

for some positive constants R and M.

Proof. Since F is convex and $F(0) = 0$ the function $F(r)/r$ is non-decreasing for $r > 0$, [14]. Hence the norm $|\cdot|$ is equivalent to $\|\cdot\|$. Moreover, we can find another scalar $k > 0$ so that the norm $|||\cdot||| = k\,|\cdot|$ satisfies

$$K'\,|||\cdot||| \leq \|\cdot\| \leq |||\cdot||| \text{ and } \frac{K'}{\sqrt{F(2^n)}}\,|||\cdot||| \leq \|\cdot\|_n \leq \frac{1}{2^n}\,\|\cdot\|$$

for some $K' > 0$ and for all $n \geq N$. Now assume that $|||x||| = |||y||| = 1$ and take $n \geq N$ such that

$$\frac{1}{K'2^{n-1}} \leq |||x-y||| < \frac{1}{K'2^{n-2}}.$$

We may without loss of generality assume that $\|x\|_n \leq \|y\|_n$. Now let us denote $a = \|x\|_n^{-1}$ and $b = \|y\|_n^{-1}$. It is clear that $2^n \leq b \leq a \leq \frac{\sqrt{F(2^n)}}{K'}$, therefore $\|ax - ay\| \geq 2$. By Lemma 3.2 $\|ax - by\| \geq 1$, and so we compute

$$\begin{aligned}
\left\|\frac{ax+ay}{2}\right\|_n &\leq \left\|\frac{ax+by}{2}\right\|_n + \frac{1}{2}(a-b)\,\|y\|_n \\
&\leq \frac{1}{2}\|ax\|_n + \frac{1}{2}\|by\|_n + \frac{1}{2}(a-b)\,\|y\|_n - \frac{C}{F\left(\frac{2}{K}\sqrt{F(2^n)}\right)} \\
&= \frac{a}{2}(\|x\|_n + \|y\|_n) - \frac{C}{F\left(\frac{2}{K}\sqrt{F(2^n)}\right)}.
\end{aligned}$$

This inequality implies

$$\left\|\frac{x+y}{2}\right\|_n \leq \frac{1}{2}\|x\|_n + \frac{1}{2}\|y\|_n - \frac{C}{aF\left(\frac{2}{K}\sqrt{F(2^n)}\right)}.$$

Thus, applying the triangle inequality to each norm in the sum for $|||\cdot|||$, we get

$$\left|\left|\left|\frac{x+y}{2}\right|\right|\right| \leq \frac{k}{2}\sum_{n\geq N}\|x\|_n + \frac{k}{2}\sum_{n\geq N}\|y\|_n - \frac{K'kC}{\sqrt{F(2^n)}F\left(\frac{2}{K}\sqrt{F(2^n)}\right)}.$$

Let us denote $M = 4/K'$ and $R = K' \cdot k \cdot C$ and observe that if $|||x-y||| = t \leq \frac{1}{K'2^{n-2}}$ then $F(2^n) \leq F\left(\frac{4}{tK'}\right)$. Therefore

$$\left|\left|\left|\frac{x+y}{2}\right|\right|\right| \leq 1 - \frac{R}{\sqrt{F(Mt^{-1})}F\left(\frac{2}{K}\sqrt{F(Mt^{-1})}\right)},$$

which finishes the proof, since $\delta_{|||\cdot|||} = \delta_{|\cdot|}$. □

In the next result, by the *"maximum slope"*, we mean the supremal one-sided derivative in any direction.

Lemma 3.4. *Suppose $f : X \to (-\infty, +\infty]$ is a l.s.c. uniformly convex function. Then*

(a) $\liminf_{\|x\|\to\infty} \frac{f(x)}{\|x\|^2} > 0$.
(b) *If $f(x) \le F(\|x\|)$ for all x and F is non-decreasing, then the maximum slope of f is bounded above by $F(2\|x\|)/\|x\|$ at any x satisfying $f(x) \ge 0$.*

Proof. Part (a) is shown in [16, Proposition 3.5.8].

(b) Let $x_0 \in X$ and suppose that the directional derivative of f at x_0 in a direction $h \in S_X$ is larger than $F(2\|x_0\|)/\|x_0\|$. Then by the three slope inequality,

$$f(x_0 + \|x_0\| h) > f(x_0) + \frac{F(2\|x_0\|)}{\|x_0\|}\|x_0\| \ge F(2\|x_0\|).$$

Since $\|x_0 + \|x_0\| h\| \le 2\|x_0\|$ and F is non-decreasing,

$$F(2\|x_0\|) \ge F(\|x_0 + \|x_0\| h\|) > F(2\|x_0\|).$$

This is a contradiction. □

Theorem 3.5. *Let X be a Banach space and let $f : X \to \mathbb{R}$ be a non-negative continuous uniformly convex function satisfying $f(x) \le F(\|x\|)$ for all $x \in X$ for some non-negative real function F with $F(0) = 0$. Then X admits an equivalent norm $|\cdot|$ so that*

$$\delta_{|\cdot|}(t) \ge \frac{R}{\sqrt{F(Mt^{-1})}F\left(S\sqrt{F(Mt^{-1})}\right)},$$

for some positive constants R,M and S.

Proof. First of all, replacing f with $\frac{f(x)+f(-x)}{2}$ we can assume that f is centrally symmetric. Using Fenchel conjugation we obtain $f(x) \le F^{**}(\|x\|)$ for all $x \in X$. According to Lemma 3.4 we choose $N \in \mathbb{N}$ and $K > 0$ so that $f(x) \ge K^2\|x\|^2$ whenever $\|x\| \ge N$. Thus we have

$$K^2\|x\|^2 \le f(x) \le F^{**}(\|x\|) \text{ whenever } \|x\| \ge N.$$

For $n \ge N$, let $|\cdot|_n$ have unit ball $B_n = \{x : f(x) \le F^{**}(2^n)\}$. For any $x \in X \setminus \{0\}$, $f(x/|x|_n) = F^{**}(2^n)$. Hence $F^{**}(\|x\|/|x|_n) \ge F^{**}(2^n)$. Since $F(0) = 0$, F^{**} and $F^{**}(s)/s$ are non-decreasing. This implies that $\|x\| \ge 2^n|x|_n$. Analogously, using that $K^2\|x/|x|_n\|^2 \le f(x/|x|_n)$ one obtains $\sqrt{F^{**}(2^n)}|x|_n \ge K\|x\|$. Consequently,

$$\frac{K}{\sqrt{F^{**}(2^n)}}\|x\| \le |x|_n \le \frac{1}{2^n}\|x\|.$$

Now suppose $|x|_n = |y|_n = 1$, and $\|x-y\| \ge 1$. Then we have

$$f\left(\frac{x+y}{2}\right) \le F^{**}(2^n) - \delta_f(1).$$

The uniform convexity of f ensures that $\delta_f(1) > 0$. Because

$$\left\|\frac{x+y}{2}\right\| \le \frac{\sqrt{F^{**}(2^n)}}{K},$$

then the maximum slope of f in any direction at $\frac{x+y}{2}$ is bounded above by

$$F^{**}\left(2\frac{\sqrt{F^{**}(2^n)}}{K}\right) / \frac{\sqrt{F^{**}(2^n)}}{K},$$

since $F^{**}(r)/r$ is non-decreasing for $r > 0$, and on using Lemma 3.4. Consequently, the distance from $\frac{x+y}{2}$ to $S_{|\cdot|_n}$ measured with respect to $\|\cdot\|$ is at least

$$\frac{\delta_f(1)\frac{\sqrt{F^{**}(2^n)}}{K}}{F^{**}\left(2\frac{\sqrt{F^{**}(2^n)}}{K}\right)}.$$

Since the "radius" of B_n measured with $\|\cdot\|$ is at most $\frac{\sqrt{F^{**}(2^n)}}{K}$, we have

$$\left|\frac{x+y}{2}\right|_n \leq 1 - \frac{\delta_f(1)\frac{\sqrt{F^{**}(2^n)}}{K}}{F^{**}\left(2\frac{\sqrt{F^{**}(2^n)}}{K}\right)} \cdot \frac{K}{\sqrt{F^{**}(2^n)}} = 1 - \frac{\delta_f(1)}{F^{**}\left(\frac{2}{K}\sqrt{F^{**}(2^n)}\right)}.$$

Applying Lemma 3.3, and noting $F^{**} \leq F$, we obtain that

$$\delta_{|\cdot|}(t) \geq \frac{R}{\sqrt{F^{**}(Mt^{-1})}F^{**}\left(\frac{2}{K}\sqrt{F^{**}(Mt^{-1})}\right)} \geq \frac{R}{\sqrt{F(Mt^{-1})}F\left(\frac{2}{K}\sqrt{F(Mt^{-1})}\right)}.$$

□

Corollary 3.6. *Let X be a Banach space and $f : X \to \mathbb{R}$ a non-negative continuous uniformly convex function satisfying that $f(x) \leq \|x\|^p$ for some $p \geq 2$ and for all $x \in X$. Then X admits a norm with modulus of convexity of power type $\frac{p}{2}(p+1)$.*

Proof. Applying Theorem 3.5 for $F(t) = t^p$ we obtain an equivalent norm $|\cdot|$ and positive constants R, M and S such that

$$\begin{aligned}\delta_{|\cdot|}(t) &\geq \frac{R}{\sqrt{(Mt^{-1})^p}\left(S\sqrt{(Mt^{-1})^p}\right)^p} = \frac{R}{\left(\frac{M}{t}\right)^{\frac{p}{2}} S^p \left(\frac{M}{t}\right)^{\frac{p^2}{2}}} \\ &= \frac{R}{S^p M^{\frac{p}{2}(p+1)}} t^{\frac{p}{2}(p+1)},\end{aligned}$$

i.e., there exists a positive constant K such that $\delta_{|\cdot|}(t) \geq K t^{\frac{p}{2}(p+1)}$. □

4. A Sharp Result for $p = 2$

In this section, we will sharpen the result from Corollary 3.6 in the case $p = 2$ to obtain the optimal result that if X has a uniformly convex function bounded above by $\|\cdot\|^2$ then there is an equivalent norm on X with a modulus of convexity of power type 2. We refer to [8] for some related information on this case. We begin with some preliminary results.

Let X be a Banach space. We can associate to X the following modulus

$$\widetilde{\delta}_X(\varepsilon) = \sup_{\tau \geq 0}\left\{\frac{1}{2}\tau\varepsilon - \rho_{X^*}(\tau)\right\},$$

where $\varepsilon \in [0, 2]$. By Lindenstrauss' formula $\widetilde{\delta}_X \leq \delta_X$ while $\widetilde{\delta}_X(\varepsilon) \geq \delta_X(\varepsilon/2)$, see [10].

The modulus of smoothness associated with X satisfies the following property which characterizes those functions being a modulus of smoothness of a Banach space (see [11] for a dual result).

Proposition 4.1. [10, Proposition 10] *Let $(X, \|\cdot\|)$ be a Banach space. If $0 < \tau \leq \sigma$, then*

$$\rho_X(\sigma)/\sigma^2 \leq L\rho_X(\tau)/\tau^2,$$

where L is a constant smaller than $2\prod_{n=0}^{\infty}(1 + 2^{-n}/3) \approx 3.6591297\ldots$.

Lemma 4.2. *Let X be a Banach space. Suppose $\{\|\cdot\|_n\}_{n\geq N}$ are norms on $(X, \|\cdot\|)$ so that for some $K > 0$ and all $n \geq N$, one has*

$$K\|\cdot\| \leq \|\cdot\|_n \leq \|\cdot\|.$$

Then there exists an equivalent norm $|\cdot|$ such that for all $n \geq N$

$$\delta_{|\cdot|}(t) \geq R_0 \delta_{\|\cdot\|_n}(t),$$

where $R_0 > 0$ is a universal constant.

Proof. A norm with the required property can be defined by the formula

$$|x|^2 = \sum_{n\geq N} a_n \|x\|_n^2,$$

where a_n satisfies $\sum_{n\geq N} a_n = \left(\frac{K}{2K_0L}\right)^2$ and where L is as in Proposition 4.1.

Let us fix $n \geq N$ and denote $Y = \ell_2(X, \|\cdot\|_n)$. Applying [10, Prop. 19] with $M(t) = t^2$ and $X_i = (X, \|\cdot\|_n)^*$ one has that

$$\rho_{Y^*}(\tau) \leq K_0 \sup_{\tau\leq u\leq 1} \rho_{(X,\|\cdot\|_n)^*}(\tau/u)u^2,$$

where K_0 depends neither on X nor on $\|\cdot\|_n$. Now, applying Proposition 4.1 we obtain

$$\rho_{Y^*}(\tau) \leq K_0 L \rho_{(X,\|\cdot\|_n)^*}(\tau),$$

and by duality

$$\delta_Y(\varepsilon) \geq \widetilde{\delta}_Y(\varepsilon) \geq K_0 L \widetilde{\delta}_{(X,\|\cdot\|_n)}(\varepsilon/K_0L) \geq K_0 L \delta_{(X,\|\cdot\|_n)}(\varepsilon/2K_0L),$$

for all $0 \leq \varepsilon < 2$.

From the proof of [10, Prop. 18] one has that $\delta_{(X,|\cdot|)}(\varepsilon) \geq \frac{1}{2}\delta_Y(c\varepsilon)$, where $c = 2K_0L$. Therefore

$$\delta_{(X,|\cdot|)}(\varepsilon) \geq \frac{K_0L}{2}\delta_{(X,\|\cdot\|_n)}(\varepsilon),$$

which finishes the proof. □

We can now complete our final result.

Theorem 4.3. *Let X be a Banach space and $f : X \to \mathbb{R}$ a continuous uniformly convex function satisfying $f(x) \leq \|x\|^2$ for all $x \in X$. Then X admits a norm with modulus of convexity of power type 2.*

Proof. Again replacing f with $\frac{f(x)+f(-x)}{2}$ clearly preserves the uniform convexity of f and allows us to assume $f(-x) = f(x)$ for all $x \in X$. According to Lemma 3.4 we may choose $N \in \mathbb{N}$ and $K > 0$ so that $f(x) \geq K^2\|x\|^2$ whenever $\|x\| \geq N$. Thus we have

$$K^2\|x\|^2 \leq f(x) \leq \|x\|^2 \text{ whenever } \|x\| \geq N.$$

For $n \geq N$, let $|\cdot|_n$ have unit ball $B_n = \{x : f(x) \leq 2^{2n}\}$. Without lost of generality we can assume that $2^N - N \geq K^{-1}$.

For any $x \in X \setminus \{0\}$, $f(x/|x|_n) = 2^{2n}$. Hence, using $f(x) \le \|x\|^2$, we obtain $\|x\| \ge 2^n |x|_n$. Analogously, using that $K^2 \|x/|x|_n\|^2 \le f(x/|x|_n)$ one obtains $2^n |x|_n \ge K \|x\|$. Consequently,

$$\frac{K}{2^n} \|x\| \le |x|_n \le \frac{1}{2^n} \|x\| .$$

Let us consider $|x|_n = |y|_n = 1$, this is $f(x) = f(y) = 2^{2n}$, with $|x-y|_n \ge 1/2^n$. Then $\|x-y\| \ge 1$, so

$$f\left(\frac{x+y}{2}\right) \le 2^{2n} - \delta_f(1).$$

The uniform convexity of f ensures that $\delta_f(1) > 0$. Because

$$N \le 2^N - \frac{1}{K} \le \left\|\frac{x+y}{2}\right\| \le \frac{2^n}{K},$$

and applying Lemma 3.4, the maximum slope of f in any direction at $\frac{x+y}{2}$ is bounded above by $4\,2^n/K$. Consequently, the distance from $\frac{x+y}{2}$ to $S_{|\cdot|_n}$ measured with respect to $\|\cdot\|$ is at least $\delta_f(1)\,K/2^{n+2}$. Since the "radius" of B_n when measured with $\|\cdot\|$ is at most $2^n/K$, we thus have

$$\left|\frac{x+y}{2}\right|_n \le 1 - \delta_f(1)\frac{K}{2^{n+2}} \cdot \frac{K}{2^n} = 1 - \frac{C}{2^{2n}}.$$

where $C = \delta_f(1)K^2/4$. This implies that

$$\delta_{|\cdot|_n}\left(\frac{1}{2^n}\right) \ge C\left(\frac{1}{2^n}\right)^2.$$

For a fixed $n \ge N$ let us consider $k = 1, 2, \ldots, 2^n$ and the constant $R = \frac{CR_0}{4L}$ where L is the Figiel's constant of Proposition 4.1. Then

$$C = \frac{C}{2^{2n}} \cdot \frac{1}{2^{-2n}} \le \frac{\delta_{|\cdot|_n}(2^{-n})}{2^{-2n}} \le 4L\frac{\delta_{|\cdot|_n}(k2^{-n})}{k^2 2^{-2n}}.$$

This implies

$$\delta_{|\cdot|_n}\left(\frac{k}{2^n}\right) \ge \frac{R}{R_0}\left(\frac{k}{2^n}\right)^2.$$

For each $n \ge N$, let us consider the new norm $\|\cdot\|_n = 2^n |\cdot|_n$. These new norms satisfy

$$K\|\cdot\| \le \|\cdot\|_n \le \|\cdot\| \text{ and } \delta_{|\cdot|_n}(\cdot) = \delta_{\|\cdot\|_n}(\cdot).$$

Applying Lemma 4.2 we obtain an equivalent norm $|\cdot|$ on X such that $\delta_{|\cdot|}(t) \ge R_0 \delta_{\|\cdot\|_n}(t)$ for $n \ge N$.

Finally, let us fix n_0 and $k \le 2^{n_0}$. For any $n \ge n_0$ we have that $\frac{k}{2^{n_0}} = \frac{k2^{n-n_0}}{2^n}$. Therefore $\delta_{\|\cdot\|_n}\left(\frac{k}{2^{n_0}}\right) = \delta_{\|\cdot\|_n}\left(\frac{k2^{n-n_0}}{2^n}\right) \ge \frac{R}{R_0}\left(\frac{k}{2^{n_0}}\right)^2$, which implies that

$$\delta_{|\cdot|}\left(\frac{k}{2^{n_0}}\right) \ge R\left(\frac{k}{2^{n_0}}\right)^2.$$

In the previous paragraph we have shown that $\delta_{|\cdot|}(t) \ge Rt^2$ for all t lying in $\mathcal{D} = \left\{\frac{k}{2^n} : n \in \mathbb{N}, 1 \le k \le 2^n\right\}$. Since $\mathcal{D}$ is dense in $[0,1]$ and as $\delta_{|\cdot|}(\cdot)$ is continuous, we have $\delta_{|\cdot|}(t) \ge Rt^2$, which finishes the proof. □

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