



Uniformly smooth renorming of Banach spaces with modulus of convexity of power type 2

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Abstract

An upper bound $q(c)$ for the best, under equivalent renorming, possible power type of the modulus of smoothness of a Banach space with modulus of convexity satisfying $\delta_X(\varepsilon) \geq c\varepsilon^2$, is found. The estimate is asymptotically sharp and is expressed in terms of linear fractional function $q(c)$.

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1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space and $S_X = \{x \in X; \|x\| = 1\}$ be its unit sphere. The *modulus of convexity*, respectively of *smoothness*, of X is defined by

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2}; x, y \in S_X, \|x-y\| = \varepsilon \right\}, \quad \text{for } \varepsilon \in [0, 2];$$

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respectively

$$\rho_X(\tau) = \sup \left\{ \frac{\|x + \tau y\| + \|x - \tau y\| - 2}{2}; x, y \in S_X \right\}, \quad \text{for } \tau \geq 0.$$

We say that the modulus of convexity (respectively of smoothness) has an estimate of power type p if $\delta_X(\varepsilon) \geq c_1 \varepsilon^p$ (respectively $\rho_X(\tau) \leq c_2 \tau^p$). We say that a Banach space X is p -uniformly convex (respectively p -uniformly smooth) renormable if there exists an equivalent norm on X such that the corresponding modulus of convexity (respectively of smoothness) has an estimate of power type p .

From the renorming theorem of Enflo and Pisier (see [11,19]) it follows that any superreflexive Banach space is p -uniformly convex and q -uniformly smooth renormable for some p and q , satisfying $1 < q \leq 2 \leq p < \infty$.

Using Kwapien's characterization [15] (for an elegant proof see also [23]) of Hilbert spaces, Figiel and Pisier [9] prove that each Banach space which is 2-uniformly convex and 2-uniformly smooth renormable is isomorphic to Hilbert space.

Rakov [20] proves that if δ_X is of power type 2 and, more precisely,

$$\delta_X(\varepsilon) \geq c\varepsilon^2,$$

for $c > 0.1076$ and small $\varepsilon > 0$, then X is q -uniformly smooth renormable for each

$$q < \log 2 / \log(\sqrt{2}(c_1 + \sqrt{c_1^2 - 1})), \quad (1)$$

where $c_1 = 1 + (\sqrt{2} - 1)\sqrt{1 - 8c}$. This can be simplified as $2 - q > k\sqrt[4]{1 - 8c}$.

The roots of Rakov result go back to the isometric characterizations of Hilbert spaces in the class of Banach spaces.

It is easy to see that if H is a Hilbert space then

$$\delta_H(\varepsilon) = 1 - \sqrt{1 - \varepsilon^2/4} = \varepsilon^2/8 + o(\varepsilon^2). \quad (2)$$

Nördlander [18] shows that

$$\delta_X(\varepsilon) \leq \delta_H(\varepsilon) \quad (3)$$

for any Banach space X and any $\varepsilon \in [0, 2]$. It is proved in [1] that X is Hilbert space whenever $\delta_X(\varepsilon) = \delta_H(\varepsilon)$ for some $\varepsilon \neq 2 \cos(k\pi/2n)$, $n = 2, 3, \dots$; $k = 1, 2, \dots, n - 1$. An asymptotic version of the latter is the following statement. If

$$\lim_{\varepsilon \rightarrow 0} \frac{\delta_X(\varepsilon)}{\varepsilon^2} = \frac{1}{8}$$

then X is a Hilbert space. This result is obtained independently in [14,20,22] using different approaches.

We improve the estimate (1). This improvement is achieved by combining ideas from [14] (i.e. use of differential inequalities) and [20] (i.e. use of so called John sphere, see, e.g., [4, p. 68], [10], also known as Loewner ellipsoid, [2]). Namely, we prove the following

Theorem 1.1. *There is an absolute constant k_1 such that if the Banach space X satisfies*

$$\liminf_{\varepsilon \rightarrow 0} \frac{\delta_X(\varepsilon)}{\varepsilon^2} \geq \frac{1}{8(1+b)} \tag{4}$$

for some $b \geq 0$, then X is q -uniformly smooth renormable for

$$q = 1 + \frac{1}{1+k_1b}. \tag{5}$$

For the sake of brevity we define for each Banach space X

$$a(X) = 2 \limsup_{\tau \rightarrow 0} \frac{\rho_X(\tau)}{\tau^2} - 1 \quad \text{and} \quad b(X) = \left(8 \liminf_{\varepsilon \rightarrow 0} \frac{\delta_X(\varepsilon)}{\varepsilon^2} \right)^{-1} - 1.$$

From (2) and (3) it follows that $0 \leq b(X) \leq \infty$. The dual relation

$$a(X^*) = b(X) \tag{6}$$

is a direct consequence of Lindenstrauss formula: (see, e.g., [16, p. 61])

$$\rho_{X^*}(\tau) = \sup_{\varepsilon \in [0,2]} \left\{ \frac{\tau\varepsilon}{2} - \delta_X(\varepsilon) \right\}. \tag{7}$$

Indeed, if $b(X) = \infty$ then there are $\varepsilon_n \rightarrow 0$ and $\mu_n \rightarrow 0$ such that $\delta_X(\varepsilon_n) = \mu_n \varepsilon_n^2$. Set $\tau_n = 4\mu_n \varepsilon_n$ and let $\varepsilon = \varepsilon_n$ in the supremum on the right-hand side of (7) to obtain $\rho_{X^*}(\tau_n) \geq \mu_n \varepsilon_n^2$. So, $\rho_{X^*}(\tau_n)/\tau_n^2 \geq 1/(16\mu_n) \rightarrow \infty$ and $a(X^*) = \infty$.

If $b(X) < \infty$ let $v_1 = \liminf_{\varepsilon \rightarrow 0} \delta_X(\varepsilon)/\varepsilon^2 = [8(1+b(X))]^{-1} > 0$. Pick $\varepsilon_n \rightarrow 0$ and $\mu_n \rightarrow 0$ such that $\delta_X(\varepsilon_n) = v_1(1 + \mu_n)\varepsilon_n^2$. For $\tau_n = 4v_1\varepsilon_n$ we have (setting in (7) $\varepsilon = \varepsilon_n$) $\rho_{X^*}(\tau_n) \geq v_1\varepsilon_n^2(2 - (1 + \mu_n))$, so $\limsup_{\tau \rightarrow 0} \rho_X(\tau)/\tau^2 \geq 1/(16v_1)$. That is, $a(X^*) \geq b(X)$.

On the other hand, for any $v \in (0, v_1)$ there is $\varepsilon_0 > 0$ such that $\delta_X(\varepsilon) > v\varepsilon^2$ for $\varepsilon \in (0, \varepsilon_0)$. Since $\delta_X(\varepsilon)$ is increasing, we have that $\delta_X(\varepsilon_0) > 0$ and

$$\sup_{\varepsilon \in [\varepsilon_0, 2]} \left\{ \frac{\tau\varepsilon}{2} - \delta_X(\varepsilon) \right\} \leq \tau - \delta_X(\varepsilon_0) < 0$$

for small enough $\tau > 0$. Therefore, for τ close to zero

$$\rho_{X^*}(\tau) \leq \sup_{\varepsilon \in [0, \varepsilon_0]} \left\{ \frac{\tau\varepsilon}{2} - v\varepsilon^2 \right\} = \frac{\tau^2}{16v}.$$

So, $a(X^*) \leq 1/(8v_1) - 1 = b(X)$, since $v \in (0, v_1)$ was arbitrary, completing the proof of (6).

In these terms Theorem 1.1 states that X is q -uniformly smooth renormable for $q = 1 + 1/(1+k_1b(X))$.

If we compare this to the known situation of l_p , $1 < p \leq 2$, we find that $b(l_p) = (2 - p)/(p - 1)$, see [13] (for a simple proof see also [17]). So,

$$p = \frac{1}{1 + b(l_p)}.$$

On the other hand l_p is not q -uniformly smooth renormable for $q > p$, see, e.g., [8]. Therefore k_1 from Theorem 1.1 could not possibly be less than one. That is, the estimate (5) is sharp up to the multiplicative constant k_1 , when $b \rightarrow 0$ as well as when $b \rightarrow \infty$. Thus in actual fact Theorem 1.1 reveals that, asymptotically, the behaviour of any space is the same as that of l_p .

In order to establish Theorem 1.1 we use the following statement which is essentially two-dimensional.

Theorem 1.2. *There is an absolute positive constant k_2 such that if X is a Banach space and $x, y \in S_X$ then*

$$\limsup_{\tau \rightarrow 0} \frac{\|x + \tau y\|^2 + \|x - \tau y\|^2 - 2}{2\tau^2} \leq 1 + k_2 a(X). \quad (8)$$

This estimate is not trivial and it depends upon the homogeneity of the norm, even though this fact is somehow implicit in our approach. We also make crucial use of Euclidean geometry on the plane and it seems unclear whether Theorem 1.2 could be established without the aid of the latter.

Remark 1.3. Inequality (8) is much easier to prove for fixed $a_0 > 0$ and $a(X) \in [a_0, \infty)$.

To demonstrate this we would present at the end of the paper a short proof of the following estimate:

$$\limsup_{\tau \rightarrow 0} \frac{\|x + \tau y\|^2 + \|x - \tau y\|^2 - 2}{2\tau^2} \leq 1 + 3 \limsup_{\tau \rightarrow 0} \frac{\rho_X(\tau)}{\tau^2} \quad (9)$$

for all $x, y \in S_X$.

If then $a(X) \geq a_0 > 0$, we can write

$$\limsup_{\tau \rightarrow 0} \frac{\|x + \tau y\|^2 + \|x - \tau y\|^2 - 2}{2\tau^2} \leq 1 + \frac{3}{2}(a(X) + 1) \leq 1 + \frac{3}{2}\left(a(X) + \frac{a(X)}{a_0}\right).$$

So, for $a(X) \geq a_0$

$$\limsup_{\tau \rightarrow 0} \frac{\|x + \tau y\|^2 + \|x - \tau y\|^2 - 2}{2\tau^2} \leq 1 + \frac{3(1 + a_0)}{2a_0} a(X). \quad (10)$$

The advantage of (8) is that it transfers directly to $L_2(X)$: the space of all (equivalence classes of) measurable X -valued functions f on a probabilistic space Ω such that the norm $\|f\|_2 = (\mathbb{E}\|f\|^2)^{1/2}$ is finite. Thus, using some dual arguments, we can demonstrate the following:

Proposition 1.4. *Let X be a Banach space. Then*

$$\delta_{L_2(X)}(\varepsilon) \geq 1 - \sqrt{1 - \frac{\varepsilon^2}{4(1 + k_2 b(X))}}, \quad (11)$$

where k_2 is from Theorem 1.2.

Obviously the above result specifies for the case of power type 2 estimate the classical result of Figiel and Pisier [8,9] (see also [16, p. 68]) which says that δ_X and $\delta_{L_2(X)}$ are equivalent at zero.

The renorming is now a matter of straightforward application of the deep theorems of Gurarij and Gurarij [12] (see also [7, p. 303]) and Pisier [19] (see also [5, p. 149]).

Remark 1.5. It seems interesting whether an analogue of Theorem 1.1 can be stated in terms of the modulus ξ_X introduced in [3].

In the following section we prove Theorem 1.1 and Proposition 1.4, assuming that Theorem 1.2 is known. In the final section we present the proof of Theorem 1.2.

2. Proof of Theorem 1.1

We split the proof of Proposition 1.4 into few lemmas.

Lemma 2.1. *If X is Banach space and $c \geq 1$ is such that for all $x, y \in S_X$*

$$\limsup_{\tau \rightarrow 0} \frac{\|x + \tau y\|^2 + \|x - \tau y\|^2 - 2}{2\tau^2} \leq c \tag{12}$$

then for all $u, v \in X$

$$\|u + v\|^2 + \|u - v\|^2 \leq 2(\|u\|^2 + c\|v\|^2). \tag{13}$$

Proof. Set $\varphi(t) = \|u + tv\|^2 - ct^2\|v\|^2$. Taking into account (12) we get that for all t

$$\limsup_{h \rightarrow 0} \frac{\varphi(t+h) + \varphi(t-h) - 2\varphi(t)}{h^2} \leq 0.$$

Hence φ is concave. Therefore, $\varphi(1) + \varphi(-1) \leq 2\varphi(0)$, which implies (13). \square

Recall (see, e.g., [5, p. 7]), that the duality mapping $J : X \rightarrow 2^{X^*}$ is defined as

$$Jx = \{f \in X^*; f(x) = \|f\|^2 = \|x\|^2\}.$$

Clearly, if the norm is smooth at x then Jx is a single point.

It is easy to check that for each $f \in X^*$

$$\|f\|^2 = \sup_{x \in X} \{2f(x) - \|x\|^2\}. \tag{14}$$

Let us mention that this formula is related to Fenchel transformation (see, e.g., [21, p. 102]). Evidently, for $f \in Jx$ we have

$$\|f\|^2 + \|x\|^2 = 2f(x). \tag{15}$$

Lemma 2.2. *Let for some $c \geq 1$ and all $u, v \in X$ inequality (13) holds. Then for all $f, g \in X^*$ and $z \in Jf$ we have*

$$\|f + g\|^2 \geq \|f\|^2 + 2g(z) + \|g\|^2/c.$$

Proof. We first show that for all $u, v \in X$

$$\|u + v\|^2 \leq \|u\|^2 + 2Ju(v) + c\|u\|^2. \tag{16}$$

Set $\varphi(t) = \|u + tv\|^2 - ct^2\|v\|^2$. From (13) we get that $\|\cdot\|$, and therefore φ , is differentiable. As in the proof of Lemma 2.1 we see that φ is concave. Therefore, $\varphi(1) \leq \varphi(0) + \varphi'(0)$. Taking into account that $\varphi'(0) = 2Ju(v)$, we get (16).

From (15) and (16) we have that

$$\|f\|^2 = 2f(z) - \|z\|^2, \quad \|x\|^2 - \|z\|^2 - 2f(x - z) \leq c\|x - z\|^2.$$

These, (14) and (15) imply

$$\begin{aligned} \|f + g\|^2 &= \sup_{x \in X} \{2f(x) + 2g(x) - \|x\|^2\} \\ &= \sup_{x \in X} \{2f(z) - \|z\|^2 + 2f(x - z) + 2g(x) + \|z\|^2 - \|x\|^2\} \\ &= \|f\|^2 + \sup_{x \in X} \{2g(x) - (\|x\|^2 - \|z\|^2 - 2f(x - z))\} \\ &\geq \|f\|^2 + \sup_{x \in X} \{2g(x) - c\|x - z\|^2\} \\ &= \|f\|^2 + 2g(z) + \sup_{x \in X} \{2g(x - z) - c\|x - z\|^2\} \\ &= \|f\|^2 + 2g(z) + \|g\|^2/c. \quad \square \end{aligned}$$

Lemma 2.3. *Let for some $c \geq 1$ and all $x, y \in X, f \in Jx$*

$$\|x + y\|^2 \geq \|x\|^2 + 2f(y) + \|y\|^2/c. \tag{17}$$

Then

$$\delta_X(\varepsilon) \geq 1 - \sqrt{1 - \varepsilon^2/4c}. \tag{18}$$

Proof. Let $\|x\| = \|y\| = 1$ and $\|x - y\| = \varepsilon$. Pick $f \in J(\frac{x+y}{2})$. We have that $f(x - y) \geq 0$ or $f(y - x) \geq 0$. By swapping if necessary x and y we may assume that $f(y - x) \geq 0$. Using (17) we write

$$\begin{aligned} 1 = \|y\|^2 &= \left\| \frac{x+y}{2} + \frac{y-x}{2} \right\|^2 \geq \frac{\|x+y\|^2}{4} + 2f\left(\frac{y-x}{2}\right) + \frac{\|y-x\|^2}{4c} \\ &\geq \frac{\|x+y\|^2}{4} + \frac{\varepsilon^2}{4c}. \end{aligned}$$

Therefore,

$$1 - \frac{\|x + y\|}{2} \geq 1 - \sqrt{1 - \varepsilon^2/4c}. \quad \square$$

Now, assuming that Theorem 1.2 is true, we can complete the

Proof of Proposition 1.4. Let $b(X) < \infty$ for otherwise the claim is trivial. Then X is reflexive and we have that $L_2(X)$ is reflexive as well and (see, e.g., [6, p. 98])

$$L_2^*(X) = L_2(X^*). \tag{19}$$

From (6), Theorem 1.2 and Lemma 2.1 it follows that for all $f, g \in X^*$

$$\|f + g\|^2 + \|f - g\|^2 \leq 2(\|f\|^2 + c\|g\|^2),$$

where $c = 1 + k_2b(X)$.

Clearly, for all $f, g \in L_2(X^*)$ we have that

$$\|f + g\|_2^2 + \|f - g\|_2^2 \leq 2(\|f\|_2^2 + c\|g\|_2^2). \tag{20}$$

From Lemmas 2.2, 2.3 and Eq. (19) we get

$$\delta_{L_2(X)}(\varepsilon) \geq 1 - \sqrt{1 - \varepsilon^2/4c}. \quad \square$$

The following elementary inequality is used in the proof of Theorem 1.1.

Lemma 2.4. For all $t > 1$

$$g(t) = \frac{2 \log 2}{\log(4 - (t + 1)/t^2)} > 1 + \frac{1}{1 + 6(t - 1)}.$$

Proof. First note that $(t + 1)/t^2 > 2/(2t - 1)$ and therefore

$$g(t) > g_1(t) = \frac{2l}{l + \log(2 - 1/(2t - 1))},$$

where $l = \log 2$. Set $s = 2(t - 1)/(2t - 1)$, so $s \in (0, 1)$, and consider

$$h(s) = (2 + s) \log(1 + s).$$

Since $h''(s) = s/(1 + s)^2$, the function h is strictly convex for $s > 0$. In particular, $h(s) < sh(1) = 3ls$ for $s \in (0, 1)$. So, $\log(1 + s) < 3ls/(2 + s)$ for $s \in (0, 1)$.

Since $g_1(t) = 2l/(l + \log(1 + s))$, we have that

$$g_1(t) > \frac{2}{1 + 3(t - 1)/(3t - 2)} = \frac{2(3t - 2)}{6t - 5} = 1 + \frac{1}{1 + 6(t - 1)}. \quad \square$$

With each basic sequence $\{u_i\}_{i=1}^\infty$ the following quantity is associated:

$$\Delta(\{u_i\}_{i=1}^\infty) = \inf\{\|x - y\|; x \in E_{1,k}, y \in E_{k+1,l}, \|x\| = \|y\| = 1, k < l\},$$

where $E_{i,j} = \text{span}\{u_i, u_{i+1}, \dots, u_j\}$.

Lemma 2.5. *If for some $c \geq 1$ and all $x, y \in X, f \in Jx$ (17) holds and $\{u_i\}_{i=1}^\infty$ is a monotone basic sequence in X then*

$$\Delta(\{u_i\}_{i=1}^\infty) \geq \sqrt{1 + c^{-1}}.$$

Proof. Pick $k < l$ and let x, y be such that $\|x\| = \|y\| = 1$ and $x \in E_{1,k}, y \in E_{k+1,l}$. Since the basis is monotone, we have that $\|x + ty\| \geq 1$ for all $t \in \mathbb{R}$. Therefore, there is $f \in Jx$ such that $f(y) = 0$. So, (17) reads

$$\|x - y\|^2 \geq 1 + c^{-1}. \quad \square$$

Recall the following result from [12] (see also [7, p. 303]):

Proposition 2.6. *Let $\{u_i\}_{i=1}^\infty$ be a basic sequence in the Banach space X , such that $\Delta(\{u_i\}_{i=1}^\infty) \geq d, \delta_X(d) > 0$, and let $\lambda = 2(1 - \delta_X(d))$. Then for each $q < \log 2 / \log \lambda$ there exists $A = A(q) > 0$ such that*

$$\left\| \sum_{i=1}^n u_i \right\| \leq A \left(\sum_{i=1}^n \|u_i\|^q \right)^{1/q}, \quad n = 1, 2, \dots$$

We use also the following result of Pisier [19].

Proposition 2.7. *Assume that for some constants $C > 0$ and $q \geq 1$ all X -valued Walsh–Paley martingales $\{M_i\}_{i \geq 0}$ satisfy*

$$\sup_n \|M_n\|_2 \leq C \left(\sum_{i \geq 0} \|dM_i\|_2^q \right)^{1/q},$$

where $dM_0 = M_0, dM_i = M_i - M_{i-1}, i \geq 1$, are the increments of the martingale $\{M_i\}_{i \geq 0}$ and $\|\cdot\|_2$ is the norm in $L_2(X)$.

Then X is q -uniformly smooth renormable.

Proof of Theorem 1.1. If $b(X) = 0$ then X is Hilbert space, see [14,20,22], so we assume that $b(X) > 0$.

Set $Y = L_2(X), c = 1 + k_2 b(X)$, where k_2 is from Theorem 1.2, and

$$\lambda = 2(1 - \delta_Y(\sqrt{(c+1)/c})).$$

From (11) we get

$$\lambda \leq \sqrt{4 - (c+1)/c^2}. \tag{21}$$

Let $\{M_i\}_{i \geq 0}$ be arbitrary X -valued Walsh–Paley martingale. Since $\{dM_i\}_{i \geq 0}$ is monotone basic sequence in Y and (20) is fulfilled for the same reasons as in the proof of Proposition 1.4, we get from Lemma 2.5 that

$$\Delta(\{dM_i\}_{i \geq 0}) \geq \sqrt{(c + 1)/c}.$$

From Propositions 2.6 and 2.7 it follows that X is q -uniformly smooth renormable for each

$$q < \log 2 / \log \lambda.$$

Set $d = 2 \log 2 / \log(4 - (c + 1)c^{-2})$. From (21) it follows that $d \leq \log 2 / \log \lambda$ and hence X is q -uniformly smooth renormable for each $q < d$ and in particular for

$$q = 1 + \frac{1}{1 + 6(c - 1)} = \frac{1}{1 + 6k_2b(X)} \tag{22}$$

due to Lemma 2.4. \square

3. The smoothness of the square of the norm

In order to demonstrate Theorem 1.2 we use the nice differentiability properties of the norm when the modulus of smoothness has an estimate of power type 2.

Lemma 3.1. *Let X be such that $a(X) < \infty$. Then for each two linearly independent $u, v \in X$ the function*

$$r(\sigma) = \|\cos \sigma u + \sin \sigma v\|$$

has first derivative, which is Lipschitz continuous.

Proof. Set $f(t) = \|u + tv\|$. From the proof of [5, Lemma IV.5.1, p. 158], we get that f' is a Lipschitz function on \mathbb{R} . Since $r(\sigma) = |\cos \sigma| f(\tan \sigma)$, we obtain that r' is Lipschitz on any closed interval $I \subset (-\pi/2, 3\pi/2)$, such that $\pi/2 \notin I$. In the same manner, considering $\|v + tu\|$, we get that r' is Lipschitz on any closed interval $I_1 \subset (0, 2\pi)$ such that $\pi \notin I_1$. \square

Lemma 3.2. *Let e_1, e_2 be an orthonormal basis in \mathbb{R}^2 with respect to the standard inner product and let $\|\cdot\|$ be some norm in \mathbb{R}^2 . Let*

$$r(\sigma) = \|\cos \sigma e_1 + \sin \sigma e_2\|$$

and let $x = r^{-1}(\theta)(\cos \theta e_1 + \sin \theta e_2)$, $y = r^{-1}(\phi)(\cos \phi e_1 + \sin \phi e_2)$, that is, $x, y \in S_X$.

(i) *If r is twice differentiable at θ , then*

$$\lim_{\tau \rightarrow 0} \frac{\|x + \tau y\| + \|x - \tau y\| - 2}{\tau^2} = \frac{\sin^2(\phi - \theta)}{r^2(\phi)} r(\theta)(r(\theta) + r''(\theta)). \tag{23}$$

(ii) If r is differentiable at θ and for some κ and small enough $|\eta|$

$$r(\theta + \eta) \leq r(\theta) + r'(\theta)\eta + \kappa\eta^2, \tag{24}$$

then

$$\limsup_{\tau \rightarrow 0} \frac{\|x + \tau y\|^2 + \|x - \tau y\|^2 - 2}{2\tau^2} \leq \frac{r^2(\theta) + r'^2(\theta) + r(\theta)(|r'(\theta)| + 2\kappa)}{r^2(\phi)}. \tag{25}$$

Proof. Each vector $z = z_1 e_1 + z_2 e_2 \in \mathbb{R}^2$ can be represented as $z = |z|(\cos \sigma e_1 + \sin \sigma e_2)$, where $|\cdot|$ is the Euclidean norm in \mathbb{R}^2 , i.e. $|z| = \sqrt{z_1^2 + z_2^2}$. So, we have

$$\|z\| = |z|r(\sigma), \quad \tan \sigma = z_2/z_1. \tag{26}$$

Case 1. $\theta = 0$. Then $x = r^{-1}(0)e_1$, so

$$\begin{aligned} x + \tau y &= (r^{-1}(0) + r^{-1}(\phi)\tau \cos \phi)e_1 + (r^{-1}(\phi)\tau \sin \phi)e_2 \\ &= r^{-1}(0)[(1 + (r(0)r^{-1}(\phi)\tau) \cos \phi)e_1 + ((r(0)r^{-1}(\phi)\tau) \sin \phi)e_2]. \end{aligned}$$

Let

$$t = r(0)r^{-1}(\phi)\tau, \quad l = r^{-1}(0). \tag{27}$$

Then

$$x + \tau y = l((1 + t \cos \phi)e_1 + (t \sin \phi)e_2) \tag{28}$$

and

$$|x + \tau y|^2 = l^2(1 + 2t \cos \phi + t^2). \tag{29}$$

As $\sqrt{1+w} = 1 + w/2 - w^2/8 + o(w^2)$, we have that

$$\begin{aligned} |x + \tau y| &= l\left(1 + t \cos \phi + \frac{t^2}{2} - \frac{1}{8}(2t \cos \phi + t^2)^2\right) + o(t^2) \\ &= l\left(1 + t \cos \phi + \frac{t^2}{2}(1 - \cos^2 \phi)\right) + o(t^2). \end{aligned}$$

That is,

$$|x + \tau y| = l\left(1 + t \cos \phi + \frac{t^2}{2} \sin^2 \phi\right) + o(t^2). \tag{30}$$

For small enough τ there is $\theta_\tau \in (-\pi/2, \pi/2)$ such that

$$x + \tau y = |x + \tau y|(\cos \theta_\tau e_1 + \sin \theta_\tau e_2). \tag{31}$$

Keeping in mind (26) and (28) we get

$$\tan \theta_\tau = \frac{t \sin \phi}{1 + t \cos \phi}.$$

Thus

$$\theta_\tau = \arctan \frac{t \sin \phi}{1 + t \cos \phi} = \frac{t \sin \phi}{1 + t \cos \phi} + o(t^2),$$

since $\arctan w = w + o(w^2)$.

So, $\theta_\tau = (t \sin \phi)(1 - t \cos \phi + o(t)) + o(t^2)$, or

$$\theta_\tau = t \sin \phi - \frac{t^2}{2} \sin 2\phi + o(t^2). \tag{32}$$

Since by assumption $r''(0)$ exists, the Taylor formula gives

$$r(\theta_\tau) = r(0) + r'(0)\theta_\tau + r''(0)\frac{\theta_\tau^2}{2} + o(\theta_\tau^2).$$

From (32) it follows that $|\theta_\tau| \leq 2|t|$ when $|t|$ is small enough. Therefore,

$$r(\theta_\tau) = r(0) + r'(0)\left(t \sin \phi - \frac{t^2}{2} \sin 2\phi\right) + \frac{r''(0)}{2}\left(t \sin \phi - \frac{t^2}{2} \sin 2\phi\right)^2 + o(t^2)$$

which yields

$$r(\theta_\tau) = r(0) + r'(0)(\sin \phi)t + (r''(0) \sin^2 \phi - r'(0) \sin 2\phi)\frac{t^2}{2} + o(t^2). \tag{33}$$

From (31) and the definition of r it follows that

$$\|x + \tau y\| = r(\theta_\tau)|x + \tau y|.$$

This, (27), (30) and (33) imply

$$\|x + \tau y\| = 1 + \xi t + \zeta t^2 + o(t^2),$$

where

$$\begin{aligned} \zeta &= \frac{\sin^2 \phi}{2} + lr'(0)\frac{\sin 2\phi}{2} + \frac{l}{2}(r''(0) \sin^2 \phi - r'(0) \sin 2\phi) \\ &= \frac{\sin^2 \phi}{2} + \frac{l}{2}r''(0) \sin^2 \phi = \frac{\sin^2 \phi}{2}(1 + r^{-1}(0)r''(0)). \end{aligned}$$

From this and (27), which rewrites $\tau^{-2} = r^2(0)r^{-2}(\phi)t^{-2}$, it follows that

$$\begin{aligned} \lim_{\tau \rightarrow 0} \frac{\|x + \tau y\| + \|x - \tau y\| - 2}{\tau^2} &= r^2(0)r^{-2}(\phi) \lim_{\tau \rightarrow 0} \frac{\|x + \tau y\| + \|x - \tau y\| - 2}{t^2} \\ &= 2r^2(0)r^{-2}(\phi)\zeta = \frac{\sin^2 \phi}{r^2(\phi)}r(0)(r(0) + r''(0)). \end{aligned}$$

Case 2. $\theta \neq 0$. Consider the rotated basis (\hat{e}_1, \hat{e}_2) :

$$\hat{e}_1 = \cos \theta e_1 + \sin \theta e_2, \quad \hat{e}_2 = -\sin \theta e_1 + \cos \theta e_2.$$

If $\hat{r}(\sigma) = \|\cos \sigma \hat{e}_1 + \sin \sigma \hat{e}_2\|$ then

$$\hat{r}(\sigma) = r(\sigma + \theta),$$

because

$$\begin{aligned} \cos \sigma \hat{e}_1 + \sin \sigma \hat{e}_2 &= (\cos \sigma \cos \theta - \sin \sigma \sin \theta)e_1 + (\cos \sigma \sin \theta + \sin \sigma \cos \theta)e_2 \\ &= \cos(\sigma + \theta)e_1 + \sin(\sigma + \theta)e_2. \end{aligned}$$

Set $\gamma = \phi - \theta$. Since

$$x = \hat{r}^{-1}(0)\hat{e}_1 \quad \text{and} \quad y = \hat{r}^{-1}(\gamma)((\cos \gamma)\hat{e}_1 + (\sin \gamma)\hat{e}_2),$$

from Case 1 it follows that

$$\begin{aligned} \lim_{\tau \rightarrow 0} \frac{\|x + \tau y\| + \|x - \tau y\| - 2}{\tau^2} &= \frac{\sin^2 \gamma}{\hat{r}^2(\gamma)}\hat{r}(0)(\hat{r}(0) + \hat{r}''(0)) \\ &= \frac{\sin^2(\phi - \theta)}{r^2(\phi)}r(\theta)(r(\theta) + r''(\theta)), \end{aligned}$$

because $\hat{r}(\gamma) = r(\theta + \gamma) = r(\phi)$.

(ii) Let r be differentiable and (24) hold. Denote $s = r^2$.

Assume that $\theta = 0$.

As the left-hand side of (24) is positive, taking squares gives

$$s(\eta) \leq s(0) + s'(0)\eta + (r'^2(0) + 2r(0)\kappa)\eta^2 + o(\eta^2).$$

This and (32) imply

$$s(\theta_\tau) \leq s(0) + s'(0)t \sin \phi + \left((r'^2(0) + 2r(0)\kappa) \sin^2 \phi - s'(0) \frac{\sin 2\phi}{2} \right) t^2 + o(t^2).$$

From the latter, (26) and (29), we get

$$\|x + \tau y\|^2 = s(\theta_\tau)|x + \tau y|^2 \leq 1 + \mu t + \nu t^2 + o(t^2),$$

where

$$v = 1 + s^{-1}(0)s'(0) \sin 2\phi + s^{-1}(0) \left((r'^2(0) + 2r(0)\kappa) \sin^2 \phi - s'(0) \frac{\sin 2\phi}{2} \right),$$

i.e.

$$\begin{aligned} v &= 1 + s^{-1}(0) \left((r'^2(0) + 2r(0)\kappa) \sin^2 \phi + s'(0) \frac{\sin 2\phi}{2} \right) \\ &\leq 1 + s^{-1}(0) \left(r'^2(0) + 2r(0)\kappa + \frac{|s'(0)|}{2} \right). \end{aligned}$$

Now, recalling (27) we write:

$$\limsup_{\tau \rightarrow 0} \frac{\|x + \tau y\|^2 + \|x - \tau y\|^2 - 2}{2\tau^2} \leq \limsup_{t \rightarrow 0} \frac{vt^2 + o(t^2) \frac{s(0)}{s(\phi)}}{t^2} = v \frac{s(0)}{s(\phi)}.$$

Since $s'(0) = 2r(0)r'(0)$, we have that

$$vs(0) \leq s(0) + r'^2(0) + 2r(0)\kappa + r(0)|r'(0)| = r^2(0) + r'^2(0) + r(0)(|r'(0)| + 2\kappa).$$

The case $\theta \neq 0$ is derived in the same way as in the proof of (i). \square

Proof of Theorem 1.2. We can assume that x and y are linearly independent. Let Y be the two-dimensional subspace of X spanned over x and y . Let Y be realized on the plane \mathbb{R}^2 in such a way that the Euclidean sphere $S = \{(z_1, z_2) \in \mathbb{R}^2; z_1^2 + z_2^2 = 1\}$ is the John sphere for B_Y . That is, the Euclidean norm $|\cdot| \geq \|\cdot\|$ and there is no ellipse of area greater than π contained in B_Y . It is well known (see, e.g., [4, p. 68], or [10]) that $|\cdot| \leq \sqrt{2}\|\cdot\|$. Let e_1, e_2 be the unit vector basis in \mathbb{R}^2 and $r(\sigma) = \|\cos \sigma e_1 + \sin \sigma e_2\|$. Then $S_Y = \{r^{-1}(\sigma)(\cos \sigma, \sin \sigma); \sigma \in [-\pi, \pi]\}$. Since $\|\cdot\| \leq |\cdot| \leq \sqrt{2}\|\cdot\|$ we get

$$1/\sqrt{2} \leq r(\sigma) \leq 1 \tag{34}$$

for all σ .

Lemma 1 [20] shows that at each arc of $S = \{z \in \mathbb{R}^2; |z| = 1\}$ of Euclidean length $\pi/2$ there is a point of contact $w \in S \cap S_Y$. So, for any σ there exists σ_1 such that

$$r(\sigma_1) = 1 \quad \text{and} \quad |\sigma - \sigma_1| \leq \pi/4. \tag{35}$$

Let us mention that (35) implies $r(\sigma) \geq 1/\sqrt{2}$.

From Lemma 3.1 we know that r' is absolutely continuous and hence r'' exists almost everywhere. From the definition of $a(X)$ and (23) it follows that for almost all θ and all ϕ

$$\frac{\sin^2(\phi - \theta)}{r^2(\phi)} r(\theta)(r(\theta) + r''(\theta)) \leq 1 + a(Y).$$

Setting in the above (for each fixed θ) $\phi = \theta + \pi/2$, we derive

$$r(\theta)(r(\theta) + r''(\theta)) \leq r^2(\phi)(1 + a(Y)) \leq 1 + a(Y),$$

since $r \leq 1$. So,

$$r(r + r'') \leq 1 + a(Y) \tag{36}$$

almost everywhere. Since Y is a subspace of X we have that $a(Y) \leq a(X)$.

Set

$$a = a(X), \quad c = 1 + a, \quad d = \min r(\theta) \quad \text{and} \quad \kappa = (c - d^2)/(2d)$$

(since r is continuous and π -periodic it attains its minimum).

As a first step we will show that for all θ

$$r'^2(\theta) \leq 4\kappa(r(\theta) - d). \tag{37}$$

Fix arbitrary σ . If $r'(\sigma) = 0$ then (37) holds since $\kappa > 0$ (as $c \geq 1 \geq d \geq d^2$) and $r(\sigma) \geq d$. Let $r'(\sigma) \neq 0$. Since (36) is not affected by the change of variables $\theta \leftrightarrow -\theta$, we can assume that $r'(\sigma) > 0$.

As r is periodic and r' is continuous there is $\sigma_0 < \sigma$, such that $r'(\sigma_0) = 0$ and $r'(\theta) > 0$ for all $\theta \in (\sigma_0, \sigma)$. Multiplying the inequality $r'' + r \leq c/r$, derived from (36), by $r'(\theta) > 0$, we see that for almost all $\theta \in (\sigma_0, \sigma)$

$$(r'^2(\theta) + r^2(\theta))' / 2 \leq c(\log r(\theta))'$$

Integrating the above from σ_0 to σ we see that

$$r'^2(\sigma) \leq 2c \log \frac{r(\sigma)}{r(\sigma_0)} - (r^2(\sigma) - r^2(\sigma_0)).$$

Let $r_0 = r(\sigma_0)$ and $r_1 = r(\sigma)$. Note that $r_1 > r_0$, since $r_1 - r_0 = \int_{\sigma_0}^{\sigma} r'(\theta) d\theta > 0$. Clearly,

$$\log \frac{r_1}{r_0} = \log \left(1 + \frac{r_1 - r_0}{r_0} \right) \leq \frac{r_1 - r_0}{r_0}$$

and thus

$$r'^2(\sigma) \leq g(r_0, r_1),$$

where $g(u, v) = 2c(v - u)u^{-1} - (v^2 - u^2)$. Consider g in the triangle $T: \{(u, v); d \leq u \leq v \leq r_1 \leq 1\}$ (r_1 is no greater than 1 because of (34)). We have that

$$g'_u = -2cvu^{-2} + 2u = 2u^{-2}(u^3 - cv) \leq 0,$$

because $cv \geq v \geq u \geq u^3$, since $u \leq 1 \leq c$; and

$$g'_v = 2cu^{-1} - 2v \geq 0.$$

That is, in the triangle T the function g is increasing on v and decreasing on u , and so attains its maximum in T at $(u, v) = (d, r_1)$. In particular, as $(r_0, r_1) \in T$,

$$r'^2(\sigma) \leq g(d, r_1) = 2c(r_1 - d)d^{-1} - (r_1^2 - d^2).$$

Note that $r_1^2 - d^2 = (r_1 - d)(r_1 + d) \geq 2d(r_1 - d)$, since $r_1 \geq d$. Therefore,

$$r'^2(\sigma) \leq (r_1 - d)(2cd^{-1} - 2d) = 4((c - d^2)/(2d))(r_1 - d) = 4\kappa(r_1 - d).$$

Next, we show that

$$d \geq 1 - (\sqrt{2} + 1)a. \tag{38}$$

If $d = 1$ then (38) is trivial. Let $d < 1$. From (35) it follows that there exist θ_d, θ_1 such that $r(\theta_d) = d, r(\theta_1) = 1$ and $|\theta_1 - \theta_d| \leq \pi/4$. Changing if necessary the variable θ with $-\theta$ we may assume that $\theta_d < \theta_1$. We may also assume that $r(\theta) > d$ for all $\theta \in (\theta_d, \theta_1)$. From (37) it follows that

$$\frac{r'(\theta)}{2\sqrt{r(\theta) - d}} \leq \frac{|r'(\theta)|}{2\sqrt{r(\theta) - d}} \leq \sqrt{\kappa}$$

for all $\theta \in (\theta_d, \theta_1)$. By integrating from θ_d to θ_1 we obtain

$$\frac{\pi}{4} \sqrt{\kappa} \geq (\theta_1 - \theta_d)\sqrt{\kappa} = \int_{\theta_d}^{\theta_1} \sqrt{\kappa} d\theta \geq \frac{1}{2} \int_{\theta_d}^{\theta_1} \frac{r'(\theta)}{\sqrt{r(\theta) - d}} d\theta = \frac{1}{2} \int_d^1 \frac{dr}{\sqrt{r - d}} = \sqrt{1 - d}.$$

Since $\pi^2/16 < 2/3$, we derive

$$1 - d \leq 2\kappa/3 = (c - d^2)/(3d).$$

That is, $0 \leq 2d^2 - 3d + c = (2d - 1)(d - 1) + c - 1$. Recalling that $2d - 1 \geq \sqrt{2} - 1$, see (34), and $c - 1 = a$, we derive $a \geq (\sqrt{2} - 1)(1 - d)$, or $d \geq 1 - a/(\sqrt{2} - 1)$, which implies (38).

Finally, we apply part (ii) of Lemma 3.2 in order to complete the proof.

From (36) it follows that $r'' \leq 2\kappa$ almost everywhere (for, (36) $\Rightarrow r + r'' \leq cd^{-1}$, so $r'' \leq cd^{-1} - d = (c - d^2)d^{-1} = 2\kappa$). From Taylor formula it is clear that (24) is satisfied with this κ . Therefore, inequality (25) holds.

Since $r(\theta) - d \leq 1 - d \leq c - d^2 \leq (c - d^2)/d = 2\kappa$, we deduce from (37) that $|r'(\theta)| \leq 2\sqrt{2}\kappa$ (of course, this is weaker than (37), but we will use it in the following estimate for simplicity, while for $r'^2(\theta)$ we use (37) as it is). So,

$$\begin{aligned} r'^2(\theta) + r(\theta)(|r'(\theta)| + 2\kappa) &\leq 4\kappa(r(\theta) - d) + 1 \cdot (2\sqrt{2}\kappa + 2\kappa) \\ &\leq 2\kappa(2(1 - d) + \sqrt{2} + 1) \leq 6\kappa, \end{aligned}$$

since $r \leq 1$ and $-2d \leq -\sqrt{2}$.

Since $r^2(\theta) \leq 1$ and $r^{-2}(\phi) \leq 1/d^2$, the above and (25) imply:

$$\limsup_{\tau \rightarrow 0} \frac{\|x + \tau y\|^2 + \|x - \tau y\|^2 - 2}{2\tau^2} \leq \frac{1 + 6\kappa}{d^2} = \frac{1}{d^2} + 3\frac{c - d^2}{d^3}. \tag{39}$$

We finish the proof by showing that the right-hand side is less than $1 + k_2a$ for some $k_2 > 0$.

From (34) we have that $1/\sqrt{2} \leq d \leq 1$. Using this and (38), which rewrites $1 - d \leq (\sqrt{2} + 1)a$, we get

$$\begin{aligned} 1 - d^2 &= (1 + d)(1 - d) \leq 2(\sqrt{2} + 1)a \leq 5a, \\ d^{-2} &= 1 + (1 - d^2)d^{-2} \leq 1 + 5ad^{-2} \leq 1 + 10a. \end{aligned}$$

Similarly, recalling that $c = 1 + a$:

$$\frac{c - d^2}{d^3} = \frac{1}{d^3}(a + (1 - d^2)) \leq 2\sqrt{2}(a + 5a). \quad \square$$

Proof of (9). Fix $x, y \in S_X$ and pick $f \in Jx$. From the definition of ρ_X and $f(x) = \|f\| = 1$ it follows that

$$\begin{aligned} 2\rho_X(\tau) &\geq \|x + \tau y\| + \|x - \tau y\| - 2 \geq \|x + \tau y\| + f(x - \tau y) - 2 \\ &= \|x + \tau y\| - 1 - \tau f(y). \end{aligned}$$

Similarly, $2\rho_X(\tau) \geq \|x - \tau y\| - 1 + \tau f(y)$. Thus

$$\|x \pm \tau y\| \leq 1 \pm \tau f(y) + 2\rho_X(\tau),$$

or

$$\|x \pm \tau y\|^2 \leq \|x \pm \tau y\|(1 \pm \tau f(y) + 2\rho_X(\tau)).$$

Therefore,

$$\begin{aligned} \|x + \tau y\|^2 + \|x - \tau y\|^2 - 2 &\leq \|x + \tau y\| + \|x - \tau y\| - 2 + \tau f(y)(\|x + \tau y\| - \|x - \tau y\|) \\ &\quad + 2\rho_X(\tau)(\|x + \tau y\| + \|x - \tau y\|) \\ &\leq 2\rho_X(\tau) + 2\tau^2 + 4(1 + \tau)\rho_X(\tau) \\ &\leq 2\tau^2 + 6(1 + \tau)\rho_X(\tau), \end{aligned}$$

which implies (9). \square

Remark 3.3. Finally, we check the least possible constant k_1 , which could be obtained by the method presented in the paper.

From (22) we get that $k_1 \leq 6k_2$. So, we should find an upper estimate of k_2 . Set $\lambda = 1 + \sqrt{2}$. Fix $a_0 \in (0, \lambda^{-1})$. From (10), (39) and (38) we get

$$\limsup_{\tau \rightarrow 0} \frac{\|x + \tau y\|^2 + \|x - \tau y\|^2 - 2}{2\tau^2} \leq \begin{cases} 1 + \frac{3(1+a_0)}{2a_0}a, & a \geq a_0, \\ f(c, d), \end{cases}$$

where $f(c, d) = d^{-2} + 3(c - d^2)d^{-3}$, $c = 1 + a$, $a = a(X)$, and d is a real number satisfying $1 \geq d \geq 1 - \lambda a > 0$ for $a < a_0$.

Let $g(a) = (1 - \lambda a)^{-2} + 3(1 + a - (1 - \lambda a)^2)(1 - \lambda a)^{-3}$. Since $f'_d(c, d) \leq 0$ for $c \geq 1 \geq d$, we obtain

$$f(c, d) \leq f(1 + a, 1 - \lambda a) = g(a).$$

Since $(1 - \lambda a)^{-2}$ is convex as a function of a , we have that

$$(1 - \lambda a)^{-2} \leq 1 + \frac{(1 - \lambda a_0)^{-2} - 1}{a_0}a = 1 + \frac{2\lambda - \lambda^2 a_0}{(1 - \lambda a_0)^2}a, \quad a \in [0, a_0].$$

Also,

$$1 + a - (1 - \lambda a)^2 = (1 + 2\lambda - \lambda^2 a)a \leq (1 + 2\lambda)a,$$

and $(1 - \lambda a)^{-3} \leq (1 - \lambda a_0)^{-3}$. Therefore, for $a \in [0, a_0]$

$$g(a) \leq 1 + \left(\frac{\lambda(2 - \lambda a_0)}{(1 - \lambda a_0)^2} + 3 \frac{1 + 2\lambda}{(1 - \lambda a_0)^3} \right) a.$$

So, we may choose

$$k_2 = \inf_{0 < a_0 < \lambda^{-1}} \left(\max \left\{ \frac{3(1 + a_0)}{2a_0}, \frac{\lambda(2 - \lambda a_0)}{(1 - \lambda a_0)^2} + 3 \frac{1 + 2\lambda}{(1 - \lambda a_0)^3} \right\} \right).$$

For example, for $a_0 = (8\lambda)^{-1} \approx 0.05$ we have that

$$\frac{3(1 + a_0)}{2a_0} \approx 30.5,$$

while the second term in the above right-hand side is approximately 32.01, so $k_2 < 33$ and $k_1 < 200$.

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