THE WEAK TOPOLOGY ON L^p OF A VECTOR MEASURE

I. FERRANDO AND J. RODRÍGUEZ

Abstract. Let ν be a countably additive measure defined on a measurable space (Ω, Σ) and taking values in a Banach space X. Let $1 < p < \infty$. In this paper we study some aspects of the weak topology on the Banach lattice $L^p(\nu)$ of all (equivalence classes of) measurable real-valued functions on Ω which are p-th power integrable with respect to ν. We show that every subspace of $L^p(\nu)$ is weakly compactly generated and has weakly compactly generated dual. We prove that a bounded net (f_α) in $L^p(\nu)$ is weakly convergent to $f \in L^p(\nu)$ if and only if $\int_A f_\alpha \, d\nu \to \int_A f \, d\nu$ weakly in X for every $A \in \Sigma$. Finally, we also provide sufficient conditions ensuring that the set of functionals

$$\Big\{ f \mapsto \int_\Omega fg \, d\langle x^*, \nu\rangle : \ g \in B_{L^q(\nu)}, \ x^* \in B_{X^*} \Big\} \subset B_{L^p(\nu)^*}$$

is a James boundary, where $1/p + 1/q = 1$.

1. Introduction

In the classical space of integrable functions $L^p(\mu)$, where μ is a probability measure and $1 \leq p < \infty$, a bounded net (f_α) is weakly convergent to $f \in L^p(\mu)$ if and only if $\int_A f_\alpha \, d\mu \to \int_A f \, d\mu$ for every measurable set A. This is a direct consequence of the duality $L^p(\mu)^* \cong L^q(\mu)$ and the density of simple functions in $L^q(\mu)$ (here $1/p + 1/q = 1$). In general, for the L^p space associated to a vector measure ν taking values in a Banach space X (see Section 2 for the definitions) there is not a 'good' representation of the dual and so the study of the weak topology becomes more difficult. In the case $p = 1$, G. P. Curbera [3] and independently S. Okada [21] showed that, if $L^1(\nu)$ contains no complemented subspace isomorphic to ℓ^1, then the weak convergence of bounded nets in $L^1(\nu)$ is characterized by the weak convergence in X of the integrals over arbitrary measurable sets. For bounded *sequences* in $L^1(\nu)$ such characterization of weak convergence holds whenever the range of ν is norm relatively compact [21], but not in general [4]. Later, G. Manjabacas [19,

2000 *Mathematics Subject Classification.* 28B05, 46E30, 46G10.

Key words and phrases. Vector measure, space of integrable functions, weak convergence, James boundary, weakly compactly generated Banach space.

I. Ferrando was supported by the grants FPI-UPV 2006-07 (Universidad Politécnica de Valencia) and MTM2006-11690-C02-01 (MEC and FEDER). J. Rodríguez was supported by the "Juan de la Cierva" Programme (MEC and FSE) and the grants MTM2005-08379 (MEC and FEDER) and 00690/PI/04 (Fundación Séneca).

Section 4.7] studied weak compactness in $L^1(\nu)$ with the help of the weaker topology $\sigma(L^1(\nu), B)$ of pointwise convergence on the norming set $B \subset B_{L^1(\nu)^*}$ made up of all functionals of the form $f \mapsto \int_\Omega fh \, d\langle x^*, \nu\rangle$, where $h \in B_{L^\infty(\nu)}$ and $x^* \in B_{X^*}$. The key point is that bounded $\sigma(L^1(\nu), B)$-compact sets are weakly compact whenever B is a James boundary for $B_{L^1(\nu)^*}$, and this is the case, for instance, provided that ν has norm relatively compact range.

The aim of this paper is to discuss some aspects of the weak topology on $L^p(\nu)$ for $1 < p < \infty$. This space plays a relevant role in the theory of Banach lattices and has attracted the attention of many authors in recent years, see [5], [10], [11], [12], [13], [14], [23] and [24]. In contrast with the case of scalar measures, $L^p(\nu)$ is not reflexive in general (see e.g. Example 3.11). However, it turns out that every subspace of $L^p(\nu)$ is weakly compactly generated and has weakly compactly generated dual (Theorem 3.1 and Corollary 3.2). The order continuity of $L^p(\nu)^*$ (Theorem 3.1) paves the way to deal with our main result, Theorem 3.5, stating that *a bounded net* (f_α) *in* $L^p(\nu)$ *is weakly convergent to* $f \in L^p(\nu)$ *if and only if* $\int_A f_\alpha \, d\nu \to \int_A f \, d\nu$ *weakly in* X *for every measurable set* A. Equivalently, the weak topology coincides on any bounded subset of $L^p(\nu)$ with the topology $\sigma(L^p(\nu), \Gamma)$ of pointwise convergence on the norming set

$$\Gamma := \{\gamma_{g,x^*} : \; g \in B_{L^q(\nu)}, \; x^* \in B_{X^*}\} \subset B_{L^p(\nu)^*}, \quad \gamma_{g,x^*}(f) := \int_\Omega fg \; d\langle x^*, \nu\rangle.$$

This answers affirmatively a question implicit in [13, 14, 23] where $\sigma(L^p(\nu), \Gamma)$ had been used, for instance, while trying to finding concrete representations of $L^p(\nu)$ as a dual space. In the last part of the paper we look for conditions ensuring that Γ is a James boundary for $B_{L^p(\nu)^*}$. This happens in each of the following cases:

- ν has norm relatively compact range and $L^p(\nu)$ is reflexive (Theorem 3.9);
- X is a Banach lattice and ν is positive (Theorem 3.12).

2. Preliminaries and notation

All unexplained terminology can be found in our standard references [9] (Banach spaces), [18, 20] (Banach lattices) and [7] (vector measures). All our Banach spaces $(Y, \|\cdot\|)$ are assumed to be real. We denote by B_Y the closed unit ball of Y and Y^* stands for its topological dual. We write w^* to denote the weak* topology on Y^*. The evaluation of $y^* \in Y^*$ at $y \in Y$ is denoted by either $\langle y^*, y\rangle$ or $y^*(y)$. By a 'subspace' of Y we mean a closed linear subspace. A set $C \subset B_{Y^*}$ is *norming* if $\|y\| = \sup\{|y^*(y)| : y^* \in C\}$ for every $y \in Y$; in this case we denote by $\sigma(Y, C)$ the (locally convex Hausdorff) topology on Y of pointwise convergence on C. A set $C \subset B_{Y^*}$ is a *James boundary* for B_{Y^*} if for each $y \in Y$ there is $y^* \in C$ such that $\|y\| = y^*(y)$. The classical example of James boundary is given by the set $\mathrm{Ext}(B_{Y^*})$ of extreme points of B_{Y^*}, cf. [9, Fact 3.45]. Recall that Y is *weakly compactly generated* (*WCG* for short) if there is a weakly compact subset of Y whose linear span is dense in Y. Standard examples of WCG Banach spaces are the separable or reflexive ones.

Throughout the paper (Ω, Σ) is a measurable space, X is a Banach space, $\nu : \Sigma \to X$ is a countably additive vector measure and $1 < p, q < \infty$ satisfy $1/p + 1/q = 1$. The semivariation of ν is the set function defined on Σ by the formula $\|\nu\|(A) = \sup\{|\langle x^*, \nu\rangle|(A) : x^* \in B_{X^*}\}$; as usual, we write $|\langle x^*, \nu\rangle|$ to denote the variation of the scalar measure $\langle x^*, \nu\rangle$ given by $\langle x^*, \nu\rangle(A) = \langle x^*, \nu(A)\rangle$ for every $A \in \Sigma$. Throughout the paper λ stands for a fixed Rybakov control measure of ν, that is, $\lambda = |\langle x^*, \nu\rangle|$ for some $x^* \in B_{X^*}$ and $\lambda(A) = 0$ if and only if $\|\nu\|(A) = 0$, cf. [7, Theorem 2, p. 268].

Following D. R. Lewis' [17] approach to the Bartle-Dunford-Schwartz integral (cf. [8, IV.10]), we say that a Σ-measurable function $f : \Omega \to \mathbb{R}$ is *ν-integrable* if it is integrable with respect to $\langle x^*, \nu\rangle$ for every $x^* \in X^*$ and for each $A \in \Sigma$ there is $\int_A f\, d\nu \in X$ such that

$$x^*\Big(\int_A f\, d\nu\Big) = \int_A f\, d\langle x^*, \nu\rangle \quad \text{for every } x^* \in X^*.$$

Two Σ-measurable functions $f, g : \Omega \to \mathbb{R}$ are identified if they are equal $\|\nu\|$-a.e. The space $L^1(\nu)$ of all (equivalence classes of) ν-integrable functions becomes a Banach lattice when endowed with the $\|\nu\|$-a.e. order and the norm

$$\|f\|_{L^1(\nu)} = \sup_{x^* \in B_{X^*}} \int_\Omega |f|\, d|\langle x^*, \nu\rangle|.$$

It is known that $L^1(\nu)$ is order continuous and has weak unit, see [2, Theorem 1]. Moreover, G. P. Curbera [2, Theorem 8] showed that any order continuous Banach lattice with weak unit is order isomorphic to the L^1 space of some vector measure. Such a Banach lattice is always WCG, see [2, Theorem 2].

Following E. A. Sánchez-Pérez [23], we now say that a Σ-measurable function $f : \Omega \to \mathbb{R}$ is *p-th power ν-integrable* if $|f|^p$ is ν-integrable. The space $L^p(\nu)$ of all (equivalence classes of) p-th power ν-integrable functions is a p-convex (see [18, Section 1.d] for the definition) order continuous Banach lattice with weak unit when equipped with the $\|\nu\|$-a.e. order and the norm

$$\|f\|_{L^p(\nu)} = \sup_{x^* \in B_{X^*}} \Big(\int_\Omega |f|^p\, d|\langle x^*, \nu\rangle|\Big)^{\frac{1}{p}},$$

see [23, Section 2]. $L^p(\nu)$ is WCG (see the proof of Proposition 3 in [5]) and simple functions are dense in it (see [23, Proposition 4]). Recently, A. Fernández and others [11] proved that any p-convex order continuous Banach lattice having weak unit is order isomorphic to the L^p space of some vector measure.

Recall that the product of a p-th power ν-integrable function and a q-th power ν-integrable one is always ν-integrable, see [23, Section 3]. Given $f \in L^p(\nu)$, we can define an operator (i.e. linear and continuous mapping)

$$M_f : L^q(\nu) \to X, \quad M_f(g) := \int_\Omega fg \, d\nu,$$

with $\|M_f\| = \|f\|_{L^p(\nu)}$, see [13, Proposition 2.1]. As a consequence of the previous equality, the set $\Gamma \subset B_{L^p(\nu)^*}$ defined in the introduction is norming.

Finally, note that $L^p(\nu)$ is a Banach (or Köthe) function space over λ, see [23, Proposition 5]. Since $L^p(\nu)$ is order continuous, $L^p(\nu)^*$ coincides with the Köthe dual of $L^p(\nu)$ (cf. [20, Corollary 2.6.5]), that is, $L^p(\nu)^* = \{\varphi_h : h \in \mathcal{H}\}$ where

$$\mathcal{H} := \{h : \Omega \to \mathbb{R} \ \Sigma\text{-measurable} : \ fh \in L^1(\lambda) \text{ for all } f \in L^p(\nu)\}$$

and the duality is given by $\langle \varphi_h, f\rangle := \int_\Omega fh \, d\lambda$.

3. The results

Our starting point is the following theorem.

Theorem 3.1. *$L^p(\nu)^*$ is order continuous and has weak unit. In particular, $L^p(\nu)^*$ is WCG.*

Proof. Since $L^p(\nu)$ is p-convex and ℓ^1 is not p-convex, we can apply [18, Proposition 1.d.9] to conclude that no sublattice of $L^p(\nu)$ is order isomorphic to ℓ^1. Equivalently, $L^p(\nu)^*$ is order continuous, cf. [20, Theorem 2.4.14]. On the other hand, since $L^p(\nu)$ is an order continuous Banach function space over λ, the space $L^p(\nu)^*$ has weak unit (namely, the functional φ_{χ_Ω}). Therefore, according to the comments in Section 2, $L^p(\nu)^*$ is order isomorphic to the L^1 space of some vector measure and so it is WCG. □

Subspaces of WCG Banach spaces are not WCG in general. The first example showing this phenomenon was built by H.P. Rosenthal [22] (cf. [6, Chapter 5, §10]) over the L^1 space of certain probability measure. However, the property of being WCG is always inherited by subspaces having WCG dual, according to a result of W.B. Johnson and J. Lindenstrauss [16] (cf. [6, Chapter 5, §8]). Since $L^p(\nu)$ is WCG (see Section 2) and the dual of any subspace of $L^p(\nu)$ is WCG (because it is a quotient of the WCG space $L^p(\nu)^*$), we have the following corollary.

Corollary 3.2. *Every subspace of $L^p(\nu)$ is WCG.*

A result of T. Kuo (cf. [7, Corollary 7, p. 83]) states that every dual WCG Banach space has the Radon-Nikodým property. On the other hand, it is well known that a dual Banach space Y^* has the Radon-Nikodým property if and only if every separable subspace of Y has separable dual, cf. [7, Corollary 8, p. 198]. Bearing in mind these facts and Theorem 3.1, we get the following corollary. For further characterizations of the separability of $L^p(\nu)$, see [11].

Corollary 3.3. *Every separable subspace of $L^p(\nu)$ has separable dual. In particular, $L^p(\nu)$ is separable if and only if $L^p(\nu)^*$ is separable.*

In order to prove Theorem 3.5 below we need the following lemma which might be folklore. We include a proof here for the sake of completeness.

Lemma 3.4. *Let Y be a Banach lattice such that both Y and Y^* are order continuous. Let $C \subset Y^*$ be a set which separates the points of Y. Then the ideal $\mathcal{I} \subset Y^*$ generated by C is norm dense in Y^*.*

Proof. The norm closure $\mathcal{I}'$ of $\mathcal{I}$ in Y^* is an ideal, cf. [20, Proposition 1.2.3]. Since Y^* is order continuous, every closed ideal of Y^* is a band, [20, Corollary 2.4.4]. On the other hand, the order continuity of Y ensures that any band of Y^* is w^*-closed, cf. [20, Corollary 2.4.7]. It follows that $\mathcal{I}'$ is w^*-closed. Finally, since $\mathcal{I}'$ is a linear subspace of Y^* which separates the points of Y, an appeal to the Hahn-Banach theorem allows us to conclude that $\mathcal{I}' = Y^*$. □

The proof of the next result is inspired by some of the ideas in [3, Theorem 4].

Theorem 3.5. *The weak topology and $\sigma(L^p(\nu), \Gamma)$ coincide on any bounded subset of $L^p(\nu)$. Consequently, a bounded net (f_α) in $L^p(\nu)$ converges weakly to $f \in L^p(\nu)$ if and only if $\int_A f_\alpha \, d\nu \to \int_A f \, d\nu$ weakly in X for every $A \in \Sigma$.*

Proof. Fix a bounded net (f_α) in $L^p(\nu)$ converging to $f \in L^p(\nu)$ in the topology $\sigma(L^p(\nu), \Gamma)$. We will show that $f_\alpha \to f$ weakly. Let $\mathcal{I} \subset L^p(\nu)^*$ be the ideal generated by Γ. Since $L^p(\nu)$ and $L^p(\nu)^*$ are order continuous (the latter by Theorem 3.1), we can apply Lemma 3.4 to conclude that $\mathcal{I}$ is norm dense in $L^p(\nu)^*$. Bearing in mind that (f_α) is bounded, it is clear that in order to prove that $f_\alpha \to f$ weakly it suffices to check that $\langle \varphi, f_\alpha \rangle \to \langle \varphi, f \rangle$ for every $\varphi \in \mathcal{I}$.

To this end, fix $\varphi \in \mathcal{I}$. There exist $g_1, \dots, g_n \in L^q(\nu)$ and $x_1^*, \dots, x_n^* \in X^*$ such that $|\varphi| \le \sum_{i=1}^n |\gamma_{g_i, x_i^*}|$. An easy computation shows that $\gamma_{g_i, x_i^*} = \varphi_{h_i}$, where

$$h_i := g_i \frac{d\langle x_i^*, \nu \rangle}{d\lambda} \in \mathcal{H} \quad \text{for all } 1 \le i \le n.$$

As usual, $\frac{d\langle x_i^*, \nu \rangle}{d\lambda}$ denotes the Radon-Nikodým derivative of $\langle x_i^*, \nu \rangle$ with respect to λ. Take $g \in \mathcal{H}$ satisfying $\varphi = \varphi_g$. Then $\varphi_{|g|} = |\varphi| \le \sum_{i=1}^n \varphi_{|h_i|} = \varphi_{\sum_{i=1}^n |h_i|}$ and therefore

$$(1) \qquad |g| \le \sum_{i=1}^n |h_i| \quad \lambda\text{-a.e.}$$

Let us consider the non-negative finite measures defined on Σ by $\mu(A) := \int_A |g|\ d\lambda$ and $\mu_i(A) := \int_A |h_i|\ d\lambda$ for all $1 \leq i \leq n$. Taking $\tilde{\mu} := \sum_{i=1}^n \mu_i$, inequality (1) ensures that $\mu \leq \tilde{\mu}$ and so we can define an operator $T : L^1(\tilde{\mu}) \to L^1(\mu)$ by $T(h) = h$. Notice that $f_\alpha, f \in L^1(\tilde{\mu})$ because $f_\alpha, f \in L^1(\mu_i)$ for all $1 \leq i \leq n$.

Claim.- $f_\alpha \to f$ weakly in $L^1(\mu_i)$ for every $1 \leq i \leq n$. Indeed, since (f_α) is bounded in $L^1(\mu_i)$ (because it is bounded in $L^p(\nu)$), we only have to check that $\int_A f_\alpha\ d\mu_i \to \int_A f\ d\mu_i$ for every $A \in \Sigma$. To this end, let us consider a Hahn decomposition $\{G, \Omega \setminus G\}$ of $\langle x_i^*, \nu\rangle$, that is, $G \in \Sigma$ and

$$|\langle x_i^*, \nu\rangle|(E) = \langle x_i^*, \nu\rangle(E \cap G) - \langle x_i^*, \nu\rangle(E \setminus G) \quad \text{for all } E \in \Sigma.$$

We have

$$\begin{aligned}
&\int_A f_\alpha d\mu_i = \int_A f_\alpha |g_i|\ d|\langle x_i^*, \nu\rangle| \\
&= \int_\Omega f_\alpha(|g_i|\chi_{A\cap G} - |g_i|\chi_{A\setminus G})\ d\langle x_i^*, \nu\rangle \to \int_\Omega f(|g_i|\chi_{A\cap G} - |g_i|\chi_{A\setminus G})\ d\langle x_i^*, \nu\rangle \\
&\qquad = \int_A f|g_i|\ d|\langle x_i^*, \nu\rangle| = \int_A f d\mu_i,
\end{aligned}$$

because $|g_i|\chi_{A\cap G} - |g_i|\chi_{A\setminus G} \in L^q(\nu)$ and $f_\alpha \to f$ in the topology $\sigma(L^p(\nu), \Gamma)$. This proves the *Claim*.

From the previous *Claim* it follows that $f_\alpha \to f$ weakly in $L^1(\tilde{\mu})$. Since T is weak-weak continuous, we infer that $f_\alpha \to f$ weakly in $L^1(\mu)$.

Set $A := \{\omega \in \Omega : g(\omega) \geq 0\} \in \Sigma$. Then

$$\begin{aligned}
\langle \varphi, f_\alpha\rangle &= \int_\Omega f_\alpha g\ d\lambda = \int_A f_\alpha |g|\ d\lambda - \int_{\Omega\setminus A} f_\alpha |g|\ d\lambda \\
&= \int_A f_\alpha\ d\mu - \int_{\Omega\setminus A} f_\alpha\ d\mu \to \int_A f\ d\mu - \int_{\Omega\setminus A} f\ d\mu = \int_\Omega fg\ d\lambda = \langle \varphi, f\rangle.
\end{aligned}$$

This finishes the proof of the first assertion of the theorem. The last part follows immediately bearing in mind that simple functions are dense in $L^q(\nu)$. □

We stress that *a set $\mathcal{F} \subset L^p(\nu)$ is bounded if and only if the set of integrals $\{\int_\Omega fg\ d\nu : f \in \mathcal{F}\} \subset X$ is bounded for every $g \in L^q(\nu)$*. This is a direct consequence of the Uniform Boundedness Principle applied to the family $\{M_f : f \in \mathcal{F}\}$ of operators from $L^q(\nu)$ to X.

Corollary 3.6. *A sequence (f_n) in $L^p(\nu)$ converges weakly to $f \in L^p(\nu)$ if and only if $f_n \to f$ in the topology $\sigma(L^p(\nu), \Gamma)$.*

The rest of the paper is essentially devoted to presenting a couple of sufficient conditions (Theorems 3.9 and 3.12 below) ensuring that Γ is a James boundary for $B_{L^p(\nu)^*}$. We do not know whether this is always the case. Our interest is somehow motivated by the following comment.

A result of G. Godefroy (see [15, Theorem III.3]) ensures that if a dual Banach space Y^* is WCG, then

$$B_{Y^*} = \overline{\mathrm{co}(C)}^{\mathrm{norm}} \tag{2}$$

for every James boundary $C \subset B_{Y^*}$. Other cases where the previous equality holds can be found in [1]. Note that (2) implies that $\sigma(Y, C)$ coincides with the weak topology on any bounded subset of Y. Bearing in mind that $L^p(\nu)^*$ is WCG (Theorem 3.1), we get the following corollary which, in particular, provides a different proof of Theorem 3.5 when Γ is a James boundary for $B_{L^p(\nu)^*}$.

Corollary 3.7. *Let C be a James boundary for $B_{L^p(\nu)^*}$. Then $\sigma(L^p(\nu), C)$ and the weak topology coincide on any bounded subset of $L^p(\nu)$.*

Lemma 3.8. *Suppose ν has norm relatively compact range. Let $f \in L^p(\nu)$. Then the operator M_f is compact.*

Proof. The norm relative compactness of $\nu(\Sigma)$ ensures that M_{χ_Ω} is compact, see [11, Theorem 3.6]. Clearly, this implies that M_{χ_A} is compact for every $A \in \Sigma$ and, consequently, M_f is compact whenever f is a simple function. For the general case, let (f_n) be a sequence of simple functions converging to f in the norm topology of $L^p(\nu)$. Then (M_{f_n}) is a sequence of compact operators converging to M_f in the operator norm and, therefore, M_f is compact too. □

Theorem 3.9. *Suppose ν has norm relatively compact range and $L^p(\nu)$ is reflexive. Then:*

(i) Γ *is w^*-closed in $L^p(\nu)^*$.*

(ii) $\mathrm{Ext}(B_{L^p(\nu)^*}) \subset \Gamma$. *In particular, Γ is a James boundary for $B_{L^p(\nu)^*}$.*

Proof. Since Γ is norming and symmetric, the Hahn-Banach theorem ensures that $B_{L^p(\nu)^*} = \overline{\mathrm{co}(\Gamma)}^{w^*}$. This equality and the so-called "converse" of the Krein-Milman theorem (cf. [8, Lemma 5, p. 440]) yield $\mathrm{Ext}(B_{L^p(\nu)^*}) \subset \overline{\Gamma}^{w^*}$.

Since $\mathrm{Ext}(B_{L^p(\nu)^*})$ is a James boundary for $B_{L^p(\nu)^*}$, it only remains to prove that Γ is w^*-closed. To this end, let $(\gamma_{g_\alpha, x^*_\alpha})$ be a net in Γ which converges to some $\varphi \in B_{L^p(\nu)^*}$ in the w^*-topology. We will check that $\varphi \in \Gamma$. By the reflexivity of $L^p(\nu)$, the space $L^q(\nu)$ is reflexive as well, see [11, Corollary 3.10]. Since $B_{L^q(\nu)}$ is weakly compact and B_{X^*} is w^*-compact, we can assume without loss of generality that $g_\alpha \to g \in B_{L^q(\nu)}$ weakly and $x^*_\alpha \to x^* \in B_{X^*}$ in the w^*-topology. We claim that $\varphi = \gamma_{g,x^*}$.

To this end, fix $f \in L^p(\nu)$ and set $x_\alpha := \int_\Omega g_\alpha f \, d\nu \in X$ for every α. Since $g_\alpha \to g$ weakly in $L^q(\nu)$, we have

$$x^*(x_\alpha) = \int_\Omega g_\alpha f \, d\langle x^*, \nu\rangle \to \int_\Omega g f \, d\langle x^*, \nu\rangle = \gamma_{g,x^*}(f).$$

On the other hand, the set $\{x_\alpha\}$ is norm relatively compact (by Lemma 3.8), $x_\alpha^* \to x^*$ in the w^*-topology and (x_α^*) is bounded, so we have

$$|x_\alpha^*(x_\alpha) - x^*(x_\alpha)| \to 0.$$

Since $|x_\alpha^*(x_\alpha) - \gamma_{g,x^*}(f)| \leq |x_\alpha^*(x_\alpha) - x^*(x_\alpha)| + |x^*(x_\alpha) - \gamma_{g,x^*}(f)|$ for every α, we conclude that

$$\varphi(f) = \lim_\alpha \gamma_{g_\alpha, x_\alpha^*}(f) = \lim_\alpha x_\alpha^*(x_\alpha) = \gamma_{g,x^*}(f).$$

As $f \in L^p(\nu)$ is arbitrary, $\varphi = \gamma_{g,x^*}$ and the proof is over. □

Remark 3.10. Under the assumptions of the previous theorem, the fact that Γ is a James boundary for $B_{L^p(\nu)^*}$ can be deduced in a more direct way. Namely, given $f \in L^p(\nu)$, the operator $M_f : L^q(\nu) \to X$ is weak-weak continuous, hence the convex set $M_f(B_{L^q(\nu)})$ is weakly compact and, in particular, norm closed. The compactness of M_f now ensures that $M_f(B_{L^q(\nu)})$ is norm compact, thus there is $g \in B_{L^q(\nu)}$ such that $\|M_f(g)\| = \|M_f\| = \|f\|_{L^p(\nu)}$. Clearly, we have $\|M_f(g)\| = \gamma_{g,x^*}(f)$ for some $x^* \in B_{X^*}$, and the conclusion follows.

As we mentioned in the introduction, $L^p(\nu)$ is not reflexive in general. We next present a simple example. Recall first that $L^p(\nu)$ is reflexive if (and only if) it does not contain subspaces isomorphic to c_0 (combine [11, Corollary 3.10] and [20, Theorem 2.4.12]). For further characterizations of the reflexivity of $L^p(\nu)$, see [5] and [11].

Example 3.11. *A non-reflexive $L^p(\nu)$.*

Proof. Take $\Omega := \mathbb{N}$, let Σ be the set of all subsets of $\mathbb{N}$ and consider the countably additive vector measure $\nu : \Sigma \to c_0$ given by $\nu(A) = \sum_{n \in A} (1/n) e_n$, where (e_n) is the canonical basis of c_0. It is not difficult to check that

$$L^p(\nu) = \left\{ f \in \mathbb{R}^{\mathbb{N}} : \ (n^{-1/p} f(n))_{n \in \mathbb{N}} \in c_0 \right\}$$

with $\|f\|_{L^p(\nu)} = \sup\{n^{-1/p} |f(n)| : n \in \mathbb{N}\}$ for all $f \in L^p(\nu)$. Clearly, c_0 is isomorphic to $L^p(\nu)$ and this space is not reflexive. □

Recall that a vector measure ϑ taking values in a Banach lattice Y is said to be *positive* if $\vartheta(\cdot) \geq 0$. In this case, we have $|\langle y^*, \vartheta \rangle| \leq \langle |y^*|, \vartheta \rangle$ for every $y^* \in Y^*$ and the semivariation of ϑ can be computed in a simple way, namely, $\|\vartheta\|(\cdot) = \|\vartheta(\cdot)\|$. This observation will be needed in the proof of the next result.

Theorem 3.12. *Suppose X is a Banach lattice and ν is positive. Then Γ is a James boundary for $B_{L^p(\nu)^*}$.*

Proof. Fix $f \in L^p(\nu) \setminus \{0\}$. Since ν is positive, the vector measure $\vartheta : \Sigma \to X$ given by $\vartheta(A) := \int_A |f|^p \, d\nu$ is positive as well. The comments preceding the theorem can be applied to ϑ ensuring that

$$\|f\|_{L^p(\nu)}^p = \|\vartheta\|(\Omega) = \|\vartheta(\Omega)\| = \Big\| \int_\Omega |f|^p \, d\nu \Big\|.$$

Take $x^* \in B_{X^*}$ such that $\|f\|_{L^p(\nu)}^p = x^*(\int_\Omega |f|^p \, d\nu) = \int_\Omega |f|^p \, d\langle x^*, \nu\rangle$. Set $h := \mathrm{sign}(f)|f|^{p-1}$ and note that $h \in L^q(\nu)$ and $\|h\|_{L^q(\nu)}^q = \|f\|_{L^p(\nu)}^p$. Define $g := (1/\|h\|_{L^q(\nu)})h \in B_{L^q(\nu)}$. We claim that $\gamma_{g,x^*}(f) = \|f\|_{L^p(\nu)}$. Indeed:

$$\begin{aligned}\int_\Omega fg \, d\langle x^*, \nu\rangle &= \Big(\int_\Omega fh \, d\langle x^*, \nu\rangle\Big) \cdot \|h\|_{L^q(\nu)}^{-1} \\ &= \Big(\int_\Omega |f|^p \, d\langle x^*, \nu\rangle\Big) \cdot \|f\|_{L^p(\nu)}^{-(p/q)} = \|f\|_{L^p(\nu)}^p \cdot \|f\|_{L^p(\nu)}^{-(p/q)} = \|f\|_{L^p(\nu)}.\end{aligned}$$

This finishes the proof. □

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