A NON LINEAR MAP FOR MLUR RENORMING

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ABSTRACT. We provide a criterion for MLUR renormability of normed spaces involving the class of σ-slicely continuous maps, recently introduced in [11]. As a consequence of this result, we obtain a theorem of G. Alexandrov concerning the three space problem for MLUR renormings of Banach spaces.

A normed space $(X, \|\cdot\|)$ (or its norm) is said to be midpoint locally uniformly rotund (MLUR for short) if for every $x \in X$ and every sequence $(x_n)_n$ in X such that $\|x_n + x\| \to \|x\|$ and $\|x_n - x\| \to \|x\|$ we have $\|x_n\| \to 0$.

Recall also that X is locally uniformly rotund (LUR for short) if for every $x \in X$ and every sequence $(x_n)_n \subset X$ such that $\lim_n \|x_n\| = \|x\|$ and $\lim_n \|x_n + x\| = 2\|x\|$ we have $\lim_n \|x_n - x\| = 0$, and that X is strictly convex or rotund (R for short) if $x = y$ whenever x and y are elements of X such that $\|x\| = \|y\| = \|(x + y)/2\|$.

It is clear that LUR $\Rightarrow$ MLUR and that MLUR $\Rightarrow$ R. In the paper [5], devoted to the renorming of spaces of continuous functions on trees, R. Haydon provides the first example (the only known to date) of MLUR space with no equivalent LUR renorming. There, he also shows that for every tree Υ, the existence of an equivalent strictly convex norm on $C(\Upsilon)$ implies MLUR renormability on this space. This coincidence is not true in general: an example of strictly convexifiable space without MLUR renorming is ℓ_∞ (see [2, 6]).

Our aim in this paper is to provide a criterion for MLUR renorming of spaces that have images in MLUR spaces through special non linear maps. These are the σ-slicely continuous maps recently introduced in [11], where a non linear transfer technique for LUR renormability has been developed.

Definition 1. *Let X and Y be normed spaces, and let A be a subset of X. A map $\Phi : A \longrightarrow Y$ is said to be σ-slicely continuous if for every $\epsilon > 0$ we may write $A = \bigcup_n A_{n,\epsilon}$ in such a way that for every $x \in A_{n,\epsilon}$ there exists an open half space $H \subset X$ such that $x \in H$ and $\operatorname{diam} \Phi(H \cap A_{n,\epsilon}) < \epsilon$.*

Recall that an open half space of X is a set of the form $\{x : f(x) > q\}$, with $q \in \mathbb{R}$ and $f \in X^* \setminus \{0\}$.

In [9] (see also [12]), there is a characterization of LUR renormable spaces X given in terms of countable decompositions of X by sets which are union of slices with small diameter. From this characterization it follows that, if X or Y

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is LUR renormable, then every linear bounded operator $Q : X \to Y$ is σ-slicely continuous, and the same holds for any composition $\Phi = B\,Q : X \to Z$ with $B : Y \to Z$ continuous (see [11, Proposition 1.14]).

We shall use σ-slicely continuous maps in order to obtain our main result as follows.

Theorem 1. *Let X be a normed space. Suppose that there exist a σ-slicely continuous map $\Phi : X \to X$ and a MLUR renormable subspace Y of X such that $x - \Phi x \in Y$, for all $x \in X$. Then, X admits an equivalent MLUR norm.*

We apply this theorem to the three space problem for MLUR renormability. The three space problem for a property of Banach spaces is the question whether a Banach space X possesses this property provided that for some closed subspace Y of X, both Y and X/Y have it. To admit an equivalent LUR norm is a three space property [4]. In the case of MLUR (and rotund) renormings, the problem was negatively solved in [5]. It was however shown in [1] the following weak version of it, which is a consequence of Theorem 1.

Theorem 2. *Let X be a Banach space, and suppose that there is a closed MLUR renormable subspace Y of X such that the quotient X/Y has an equivalent LUR norm. Then, X admits an equivalent MLUR norm.*

Proof. We consider the quotient map Q, from X onto X/Y. By the Bartle-Graves theorem (see e.g. [3, Chapter VII.3]), there is a continuous selector $B : X/Y \to X$ such that $B\,Qx \in Qx$, for all $x \in X$. If we define $\Phi = B\,Q$, then $x - \Phi x \in Y$, for every $x \in X$. Since X/Y is LUR renormable, Q and Φ are σ-slicely continuous and Theorem 1 gives the MLUR renormability on X. □

Let us mention that a similar result to the previous was obtained in [7] for Kadec renormability (recall that a normed space is Kadec if the norm and the weak topologies coincide on its unit sphere).

An equivalent definition for MLUR norms can be stated in terms of the notion of strongly extreme point of a subset of a normed space, introduced in [8].

Definition 2. *Let A be a subset of a normed space $(X, \|\cdot\|)$, and $\epsilon, \delta > 0$. An element $x \in A$ is said to be an (ϵ, δ)-strongly extreme point of A if*

$$\|u - v\| < \epsilon \text{ whenever } u, v \in A \text{ and } \|x - \tfrac{u+v}{2}\| < \delta$$

The point x is called an ϵ-strongly extreme point of A it there exists $\delta > 0$ such that x is an (ϵ, δ)-strongly extreme point of A.

A normed space is MLUR if, and only if, every element of its unit sphere is an ϵ-strongly extreme point of the unit ball, for each $\epsilon > 0$.

In the proof of Theorem 1 we use the following covering type characterization for the class of MLUR renormable spaces given in [10, Theorem 1]: *A normed space X admits an equivalent MLUR norm if, and only if, for every $\epsilon > 0$ we can write $X = \bigcup_n X_{n,\epsilon}$ in such a way that each $x \in X_{n,\epsilon}$ is an ϵ-strongly extreme point of $co(X_{n,\epsilon})$.*

Proof of Theorem 1. First step. The countable covering. Let us fix $\epsilon > 0$, and take the countable decomposition of the MLUR renormable space Y:

$$Y = \bigcup_{n,m} Y^{n,m}$$

with $n, m \in \mathbb{N}$, $\frac{1}{m} < \frac{\epsilon}{2}$, and such that every point $y \in Y^{n,m}$ is a $(\epsilon, \frac{1}{m})$-strongly extreme point of $co(Y^{n,m})$.

Let denote $\Psi(x) = x - \Phi(x) \in Y$, and lift the decomposition from Y to X,

$$X = \bigcup_{n,m} X^{n,m}$$

where $X^{n,m} = \{x \in X : \Psi(x) \in Y^{n,m}\}$.

Now, we use the fact that Φ is σ-slicely continuous on each $X^{n,m}$ to get countable coverings

$$X^{n,m} = \bigcup_{k,q} C^{n,m}_{k,q}$$

with $k \in \mathbb{N}$ and $q \in \mathbb{Q}$, such that for each $x \in C^{n,m}_{k,q}$ there exist a positive rational number r_x and an half space $H_x = \{y \in X : f_x(y) > q\}$, defined by $f_x \in X^*$ with $\|f_x\| = 1$, that satisfy

$$\operatorname{diam} \Phi(H_x \cap C^{n,m}_{k,q}) < \frac{1}{8\,m} \tag{1}$$

and

$$f_x(x) > q + r_x > q$$

We can assume without loss of generality that Φ is bounded on $C^{n,m}_{k,q}$ having

$$\sup\{\|\Phi(x)\| : x \in C^{n,m}_{k,q}\} < M_{k,q}$$

for some constant $M_{k,q}$.

In order to get more control on the values of functionals f_x we decompose a little more the pieces $C^{n,m}_{k,q}$. At this point we use some technical ideas taken from [10, Proposition 1] and [11, Lemma 4.21].

Let us consider, for each $x \in C^{n,m}_{k,q}$, a number $0 < s_x < \frac{1}{16\,m}\frac{r_x}{M_{k,q}}$, and define for each pair of positive rationals r and s the sets

$$C^{n,m}_{k,q,r,s} = \left\{ x \in C^{n,m}_{k,q} : s < s_x \text{ and } q + r < \sup_{y \in C^{n,m}_{k,q}} \{f_y(x)\} < q + r + s \right\}$$

Observe that for each $x \in C^{n,m}_{k,q,r,s}$ we have $r + s > r_x$.

Second step. A particular slice. Fix $x \in C^{n,m}_{k,q,r,s}$, take $y_x \in C^{n,m}_{k,q}$ such that

$$q + r + s > f_{y_x}(x) > q + r > q \quad (x \in H_{y_x} \cap C^{n,m}_{k,q}), \tag{2}$$

and consider the half space $H'_x = \{z : f_{y_x}(z) > f_{y_x}(x) - r\}$.

For each $z \in C^{n,m}_{k,q,r,s} \cap H'_x$ we have

$$f_{y_x}(z) > f_{y_x}(x) - r > q \quad (z \in H_{y_x} \cap C^{n,m}_{k,q}) \tag{3}$$

$$f_{y_x}(z) < q + r + s < f_{y_x}(x) + s \quad \text{and}$$

$$f_{y_x}(x) - f_{y_x}(z) > -s > -s_x. \tag{4}$$

Inequalities (1), (2) and (3) ensure us that

$$\|\Phi(x) - \Phi(z)\| < \frac{1}{8m}. \tag{5}$$

On the other hand, for $z \in C^{n,m}_{k,q,r,s} \setminus H'_x$ we have

$$f_{y_x}(x) - f_{y_x}(z) \geq r. \tag{6}$$

Third step. A good estimation for convex combinations. Let us consider two convex combinations of vectors $u_i \in C^{n,m}_{k,q,r,s}$ ($i \in I$) and $v_j \in C^{n,m}_{k,q,r,s}$ ($j \in J$): $u = \sum_{i \in I} \lambda_i u_i$ and $v = \sum_{j \in J} \mu_j v_j$, with $\lambda_i, \mu_j > 0$ and $\sum_{i \in I} \lambda_i = \sum_{j \in J} \mu_j = 1$.

Now, for any vector $x \in C^{n,m}_{k,q,r,s}$, we take the half space H'_x given in the above step, and define the subsets of indexes $I_0 = \{i \in I : u_i \notin H'_x\}$ and $J_0 = \{j \in J : v_j \notin H'_x\}$.

As in the proof of [10, Lemma 4], we take advantage of the inequalities (4) and (6) to obtain an upper bound for $\sum_{i \in I_0} \lambda_i + \sum_{j \in J_0} \mu_j$:

$$\begin{aligned}
\left\|x - \tfrac{u+v}{2}\right\| &\geq f_{y_x}\left(x - \tfrac{u+v}{2}\right) \\
&= \tfrac{1}{2}\left(\sum_i \lambda_i (f_{y_x}(x) - f_{y_x}(u_i)) + \sum_j \mu_j (f_{y_x}(x) - f_{y_x}(v_j))\right) \\
&> \tfrac{1}{2}\left(\sum_{i \in I_0} \lambda_i + \sum_{j \in J_0} \mu_j\right) r - \tfrac{1}{2}\left(\sum_{i \notin I_0} \lambda_i + \sum_{j \notin J_0} \mu_j\right) s \\
&= \tfrac{1}{2}\left(\sum_{i \in I_0} \lambda_i + \sum_{j \in J_0} \mu_j\right)(r+s) - s_x \\
&> \tfrac{1}{2}\left(\sum_{i \in I_0} \lambda_i + \sum_{j \in J_0} \mu_j\right) r_x - s_x
\end{aligned}$$

From this inequality we deduce the following upper bound:

$$\tfrac{1}{2}\left(\sum_{i \in I_0} \lambda_i + \sum_{j \in J_0} \mu_j\right) < \left(\left\|x - \tfrac{u+v}{2}\right\| + s_x\right) \tfrac{1}{r_x} \tag{7}$$

Last step. Strongly extreme points. Now, we are going to prove that each point $x \in C^{n,m}_{k,q,r,s}$ is an ϵ-strongly extreme point of $co(C^{n,m}_{k,q,r,s})$. Thus, the cited covering charaterization [10, Theorem 1] will finish the proof.

Let us fix $x \in C^{n,m}_{k,q,r,s}$ and take $0 < \delta_x < \frac{1}{16\,m} \min\{1, \frac{r_x}{M_{k,q}}\}$. We will prove that if u and v are in $co(C^{n,m}_{k,q,r,s})$ and $\|x - \frac{u+v}{2}\| < \delta_x$, then $\|u - v\| < \epsilon$.

We take $u = \sum_{i \in I} \lambda_i u_i$ and $v = \sum_{j \in J} \mu_j v_j$ with the same notation as in the previous step. Using that the identity map is just the sum $\Phi + \Psi$, we get

$$\begin{aligned}
\|u - v\| &= \|\textstyle\sum_i \lambda_i (\Phi(u_i) + \Psi(u_i)) - \sum_j \mu_j (\Phi(v_j) + \Psi(v_j))\| \\
&\leq \|\textstyle\sum_i \lambda_i \Phi(u_i) - \sum_j \mu_j \Phi(v_j)\| + \|\sum_i \lambda_i \Psi(u_i) - \sum_j \mu_j \Psi(v_j)\|
\end{aligned} \tag{8}$$

The first member of this last sum is less than $\frac{\epsilon}{2}$ because of the σ-slice continuity. In fact, using the inequalities (5) and (7) we have

$$\begin{aligned}
&\| \sum_i \lambda_i \Phi(u_i) - \sum_j \mu_j \Phi(v_j)\| \\
&\quad \leq \| \sum_i \lambda_i(\Phi(u_i) - \Phi(x))\| + \| \sum_j \mu_j(\Phi(v_j) - \Phi(x))\| \\
&\quad \leq \left(\sum_{i\notin I_0} \lambda_i + \sum_{j\notin J_0} \mu_j\right) \frac{1}{8\,m} + \left(\sum_{i\in I_0} \lambda_i + \sum_{j\in J_0} \mu_j\right) 2M_{k,q} \\
&\quad < \frac{1}{4\,m} + 2(\delta_x + s_x)\frac{1}{r_x} 2M_{k,q} \\
&\quad < \frac{1}{4\,m} + \frac{1}{2m} = \frac{3}{4\,m} < \frac{\epsilon}{2}.
\end{aligned}$$

To obtain the same bound for the second summand in (8) we use the MLUR decomposition in the beginning of the proof. Let us consider the vectors in $co(Y^{n,m})$: $u' = \sum_i \lambda_i \Psi(u_i)$ and $v' = \sum_j \mu_j \Psi(v_j)$. Now, bearing in mind that $\Psi(x)$ is an $(\frac{\epsilon}{2}, \frac{1}{m})$-strongly extreme point of $co(Y^{n,m})$, if we are able to prove that $\|\Psi(x) - \frac{u'+v'}{2}\| < \frac{1}{m}$, we can conclude that $\|u' - v'\| < \frac{\epsilon}{2}$ and finish the proof.

The next inequalities give the estimation wanted:

$$\begin{aligned}
&\|\Psi(x) - \frac{u'+v'}{2}\| = \\
&\quad = \|x - \frac{u+v}{2} - \frac{1}{2}\sum_i \lambda_i(\Phi(x) - \Phi(u_i)) - \sum_j \mu_j(\Phi(x) - \Phi(v_j))\| \\
&\quad \leq \|x - \frac{u+v}{2}\| + \frac{1}{2}\| \sum_i \lambda_i(\Phi(u_i) - \Phi(x)) + \sum_j \mu_j(\Phi(v_j) - \Phi(x))\| \\
&\quad < \delta_x + \frac{3}{4\,m} < \frac{1}{m}.
\end{aligned}$$

□

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