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SPACES OF FUNCTIONS WITH COUNTABLY MANY DISCONTINUITIES

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ABSTRACT

Let Γ be a Polish space and let K be a separable and pointwise compact set of functions on Γ. Assume further that each function in K has only countably many discontinuities. It is proved that $\mathcal{C}(K)$ admits a $\mathfrak{T}_p$-lower semicontinuous and locally uniformly rotund norm, equivalent to the supremum norm. A slightly more general result is shown and a related conjecture is stated.

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1. Introduction

It is known that, for a wide range of classes of "well-behaved" compact spaces K, the space $\mathcal{C}(K)$ of all continuous real-valued functions on K admits a norm, equivalent to the supremum norm, that is locally uniformly rotund. One approach to this sort of problem, using projectional resolutions of the identity, reached its most general result with the proof in [5] that such a renorming exists when K is a continuous image of a Valdivia compact. One of the present authors [7] recently established LUR renormability for $\mathcal{C}(K)$ when K is a Namioka–Phelps compact. Novel techniques developed in [11] led to a proof of LUR renormability for the space $\mathcal{C}(H)$, where H is the Helly space. The main result of this paper can be seen as a generalization of this last result. We use techniques from both [11] and [7], as well as borrowing a key idea from the recent paper [1] of Argyros et al.

The Helly space is an example of a class of compacta about which there are still a number of open questions. We say that a compact space K is a Rosenthal compact if there exists a Polish space Γ and a homeomorphism from K onto a subset of the space $\mathcal{B}_1(\Gamma)$ of Baire-1 functions on Γ equipped with the pointwise topology. Important (and now classical) results of Rosenthal [13] and Bourgain, Fremlin and Talagrand [2] show that such compact spaces have certain properties which place them close to metrizable compacta and to weakly compact subsets of Banach spaces. It is natural therefore to hope for good results when we look at these compact spaces in the context of renorming theory. In fact, however, as Todorcevic [15] has recently observed, there is a scattered Rosenthal compactification K of a tree space such that C(K) has no LUR renorming, [6]. Now that space K is non-separable and, as other recent work of Todorcevic [16] has shown, it is only from separable Rosenthal compacta that we should expect really good behaviour. We are therefore led to make the conjecture.

CONJECTURE: *If K is a separable Rosenthal compact then $\mathcal{C}(K)$ admits a locally uniformly convex renorming.*

A proof of this conjecture would yield as an immediate corollary that X^* is LUR renormable whenever X is a separable Banach space with no subspace isomorphic to ℓ^1. Indeed, in this case, we may take Γ to be the unit ball of the dual space X^*, which is compact and metrizable (so certainly Polish) under the weak* topology $\sigma(X^*, X)$ and K to be the unit ball of X^{**} under the weak* topology $\sigma(X^{**}, X^*)$. By the results of [12], the elements of K are then of the first Baire class when we regard them as functions on Γ. Moreover, K is separable, since the unit ball of X (which we are assuming to be separable) is dense in K by Goldstine's theorem. Finally, of course, X^* embeds as a closed subspace of $\mathcal{C}(K)$. The main theorem of this paper establishes LUR renormability of $\mathcal{C}(K)$ only for a subclass of separable Rosenthal compacta K, namely those representable as spaces of functions with only countably many discontinuities in the Polish space Γ.

THEOREM 1 (Main Theorem): *Let Γ be a Polish space and let K be a separable and pointwise compact set of functions on Γ. Assume further that each function in K has only countably many discontinuities. Then $\mathcal{C}(K)$ admits a $\mathfrak{T}_p$-lower semicontinuous and locally uniformly rotund norm, equivalent to the supremum norm.*

When K is not assumed to be separable and each element of K has only countably many discontinuities, the fact that $C(K)$ is a σ-fragmentable space for the pointwise convergence topology $\mathfrak{T}_p$ was obtained by I. Kortezov [10] following previous results on the Namioka property by A. Bouziad [3]. In the last section of the paper we present our approach for this result. We believe it could help to deal with the LUR renorming problem on this class of $C(K)$ spaces.

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2. Preliminaries

Our notation and terminology are standard: we write ω for the set $\{0, 1, 2, \ldots\}$ of all natural numbers and $\mathbb{N}$ for the set of all positive integers. When A is a set, we write $\#A$ for the cardinality of A and $[A]^{<\omega}$ for the set of all finite subsets of A. We recall that a topological space is said to be Polish if it separable, metrizable and complete for some metric compatible with the topology. A space is analytic if it is a continuous image of some Polish space.

We shall be considering real-valued functions on a Polish space Γ which have only countably many discontinuities. For such a function s we may introduce the following subsets of Γ, which consist of the "big" discontinuities:

$$J(s,\delta) = \{\gamma \in \Gamma\colon \operatorname{osc}\ s \restriction U > \delta \text{ whenever } U \text{ is open and } \gamma \in U\},$$

the above being defined for all positive real δ. Each of these sets is a countable closed subset of the Polish space Γ and hence a scattered topological space. We recall the Cantor–Bendixon derivation for such spaces: as usual, for a topological space J, we write J' for the derived set, consisting of those points of J which are not isolated; by transfinite recursion, we define successive derived sets $J^{(\xi)}$ for ordinals ξ by

$$J^{(0)} = J, \quad J^{(\eta)} = \bigcap_{\xi<\eta} (J^{(\xi)})' \quad (\eta > 0).$$

The space J is scattered if and only if $J^{(\lambda)} = \emptyset$ for some λ; if this is so, then the smallest such λ is called the derived length, or Cantor–Bendixon index of J. Of course, if J is countable, this index will automatically be a countable ordinal.

The proof of our main theorem divides into two parts. First we establish LUR-renormability of $\mathcal{C}(K)$ without making a separability assumption about K, assuming instead that there is some countable ordinal Ω such that $J(s,\delta)^{(\Omega)} = \emptyset$ for all $\delta > 0$ and all $s \in K$. Then we show that separability of K implies such a uniform bound on derived length. It is in the second of these steps that we use ideas from [1], involving the rank of a well-founded relation, and the so-called Rank Theorem for analytic relations.

In our case, the relation will be the ordering on a certain tree. We now set out some notation and terminology. Recall that a tree is a partially ordered set $(\Upsilon, \prec)$ with the property that for each $v \in \Upsilon$ the set $\{u \in \Upsilon\colon u \prec v\}$ is well-ordered by $\prec$. Any tree Υ may be partitioned into **levels** Υ_ξ, with Υ_0 consisting of the $\prec$-minimal elements. More generally, for $v \in \Upsilon$ we define the **height** $\mathrm{ht}(v)$ to be the order-type of $\{u \in \Upsilon\colon u \prec v\}$, and define the ξ^{th} level Υ_ξ to be $\{v \in \Upsilon\colon \mathrm{ht}(v) = \xi\}$. Following Todorcevic [14], when Υ and Ψ are trees, we write $\Upsilon \otimes \Psi$ for the set $\{(u,v) \in \Upsilon \times \Psi\colon \mathrm{ht}(u) = \mathrm{ht}(v)\}$, which is itself a tree when equipped with the order $(u,v) \preccurlyeq (u',v')$ if and only if $u \preccurlyeq u'$ and $v \preccurlyeq v'$. (The restriction that $\mathrm{ht}(u) = \mathrm{ht}(v)$ is important to ensure that we have a total order on the predecessors of (u',v').)

The trees with which we shall be concerned are **well-founded**, that is to say that they contain no strictly $(\prec)$-increasing sequence. Of course this implies that it is only the levels Υ_n $(n \in \omega)$ which are non-empty, but it also enables one to introduce a second ordinal index, which we call the **rank**. An element of a well-founded tree is of rank 0 if it is **$\prec$-maximal**. More generally, we define a derivation on Υ, by setting $A' = A \setminus \max A$, where $\max A$ is the set of all maximal elements of the subset A of Υ; we then define

$$\Upsilon^{[0]} = \Upsilon; \quad \Upsilon^{[\eta]} = \bigcap_{\xi<\eta} \Upsilon^{[\xi]\prime} \quad (\eta \geq 1).$$

An element u of Υ is defined to be of rank ξ if $u \in \Upsilon^{[\xi]} \setminus \Upsilon^{[\xi+1]}$. It is straightforward to check that the rank $r(u)$ satisfies the following identity:

$$r(u) = \sup\{r(v) + 1 \colon v \in u^+\},$$

where u^+ is the set of immediate successors of u in the tree-order $\prec$ (and where we are of course taking the supremum of the empty set of ordinals to be 0). It will be useful to record here an easy result about ranks in the tree $\Upsilon \otimes \Upsilon$.

Lemma 1: *Let Υ and Ψ be a well-founded tree. Then $\Upsilon \otimes \Psi$ is well-founded and the rank of (u, v) in $\Upsilon \otimes \Psi$ is given by*

$$r(u, v) = \min\{r(u), r(v)\}.$$

Proof: The immediate successors of (u, v) in $\Upsilon \otimes \Psi$ are the pairs (u', v') with $u' \in u^+$ and $v' \in v^+$. So the result follows from the identity we noted above. ∎

The important theorem that we shall use is the Rank Theorem. We refer the reader to [9] for a full account and state here the special case which concerns us.

The Rank Theorem: *Let Υ be a well-founded tree and assume that the tree order $\preccurlyeq$ (considered as a subset of $\Upsilon \times \Upsilon$) is an analytic topological space. Then there is a countable ordinal Ω such that $\Upsilon^{[\Omega]} = \emptyset$.*

Finally, we recall the definition of locally uniform convexity and some (probably familiar) convexity arguments. If ν is a non-negative real-valued convex function on a real vector space X and $x, x_n \in X$ $(n \in \mathbb{N})$, we say that the LUR hypothesis holds for ν (and x and the sequence x_n) if $\nu(x_n)$ and $\nu(\frac{1}{2}(x + x_n))$ both tend to the limit $\nu(x)$ as $n \to \infty$. If ν is a norm and the LUR hypothesis implies that $\nu(x - x_n) \to 0$, then we say that ν is locally uniformly rotund.

It is worth noting that the LUR hypothesis holds if and only if

$$\frac{1}{2}\nu(x)^2 + \frac{1}{2}\nu(x_n)^2 - \nu(\frac{1}{2}(x + x_n))^2 \to 0$$

as $n \to \infty$. If $\nu^2 = \mu^2 + \lambda^2$, where μ and λ are themselves non-negative and convex, and if the LUR hypothesis holds for ν, then it holds for λ and μ also.

This is an observation that we shall use a number of times, justifying each such application by the phrase “by convexity”.

3. Construction of a LUR norm

In this section we shall prove the following theorem.

THEOREM 2: *Let Γ be a Polish space and let K be a pointwise compact set of Baire-1 functions on Γ. Assume that there exists a countable ordinal Ω such that for all $s \in K$ and all $\delta > 0$ the Ω^{th} derived set $J(s,\delta)^{(\Omega)}$ is empty. Then the Banach space $\mathcal{C}(K)$ admits an equivalent $\mathfrak{T}_p$-lower semicontinuous and locally uniformly rotund norm.*

As in [6] and [7] we shall employ a method of recursive definitions, combined with the following result, which we refer to as Deville's Lemma. Let us mention that an approach based on countably decomposition in the spirit of [11] is also possible. Notice that, using the language introduced in the previous section, the key assumption in this lemma can be expressed by saying that the LUR hypothesis holds for θ, x and (x_n).

LEMMA 2 ([4], p. 279): *Let $(\phi_i)_{i\in I}$ and $(\psi_i)_{i\in I}$ be two pointwise-bounded families of non-negative, real-valued, convex functions on a real vector space Z. For $i \in I$ and positive integers p define functions $\theta_{i,p}$, θ_p and θ by setting*

$$\begin{aligned} 2\theta_{i,p}(x)^2 &= \phi_i(x)^2 + p^{-1}\psi_i(x)^2, \\ \theta_p(x) &= \sup_{i\in I} \theta_{i,p}(x), \\ \theta(x)^2 &= \sum_{p=1}^{\infty} 2^{-p}\theta_p(x)^2. \end{aligned}$$

Let x and x_r ($r \in \mathbb{N}$) be elements of Z and assume that

$$\frac{1}{2}\theta(x)^2 + \frac{1}{2}\theta(x_r)^2 - \theta(\frac{1}{2}(x + x_r))^2 \to 0$$

as $r \to \infty$. Then there is a sequence (i_r) of elements of I such that

$$\phi_{i_r}(x) \to \sup_{i\in I} \phi_i(x) \quad \textit{and}$$
$$\frac{1}{2}\psi_{i_r}(x)^2 + \frac{1}{2}\psi_{i_r}(x_r)^2 - \psi_{i_r}(\frac{1}{2}(x + x_r))^2 \to 0$$

as $r \to \infty$.

Modifying a little our earlier notation for discontinuity sets, when $s, t \in K$ and $m \in \mathbb{N}$, we shall write $J(s,t,m)$ for the union $J(s, 1/4m) \cup J(t, 1/4m)$. Our hypothesis implies that $J(s,t,m)^{(\Omega)} = \emptyset$ for all $s,t \in K$. Let $\mathcal{B}$ be a countable base for the topology in Γ. Let Q be a dense countable subset of Γ, and write $P = \Gamma \setminus Q$. When $R \subset Q$, $F \subset P$ and $m \in \mathbb{N}$, we define

$$I(R,F,m) = \{(s,t) \in K \times K\colon \|(s-t)\restriction_R\|_\infty \le 1/4m,\ \|(s-t)\restriction_F\|_\infty \le 1/m\}.$$

When R,F,m are as above, $\xi < \Omega$ is an ordinal and $\mathcal{M}$ is a finite subset of $\mathcal{B}$, we define

$$I(R,F,m,\xi,\mathcal{M}) = \{(s,t) \in I(R,F,m)\colon \#J(s,t,m)^{(\xi)} \cap U \le 1 \text{ for all } U \in \mathcal{M}\}.$$

These sets $I(R,F,m,\xi,\mathcal{M})$ will play the role of the index set I in applications of Deville's Lemma. In order to make clear our applications of this lemma we make the (otherwise redundant) definition, which defines the functions $\phi_i(x) = \varphi(x,s,t)$ for $i = (s,t) \in I$,

$$\varphi(x,s,t) = (1/2)|x(s) - x(t)|, \quad s,t \in K,\ x \in C(K),$$

and introduce the suprema

$$\varphi(x,R,F,m) = \sup\{\varphi(x,s,t)\colon (s,t) \in I(R,F,m)\}$$

and

$$\varphi(x,R,F,m,\xi,\mathcal{M}) = \sup\{\varphi(x,s,t)\colon (s,t) \in I(R,F,m,\xi,\mathcal{M})\}.$$

The definition of the convex functions ψ_i is more complicated and some more notation is needed. Without loss of generality we can assume that $K \subset [0,1]^\Gamma$. For finite $R \subset Q$ and $F \subset P$ we choose a finite subset $K(R,F,m)$ of K (of cardinality at most $m^{\#F}(4m)^{\#R}$)such that for all $s \in K$ there exists $t \in K(R,F,m)$ with $(s,t) \in I(R,F,m)$. To construct the functions ψ_i and the promised equivalent norm, we make recursive definitions as set out in the following lemma.

Lemma 3: *There are functions ν,θ,ψ, defined for $x \in C(K)$, $R \subset Q$, $F \subset P$, $m \in \mathbb{N}$, $\xi < \Omega$, $\mathcal{M} \in [\mathcal{B}]^{<\omega}$, $k \in \mathbb{N}$, s, $t \in I(R,F,m,\mathcal{M})$, and satisfying the* *following:*

$$
\begin{aligned}
3\nu(x,R,F,m)^2 =& \varphi(x,R,F,m)^2 + \frac{1}{\#K(R,F,m)} \sum_{t\in K(R,F,m)} x(t)^2 \\
&+ \sum_{\xi,\mathcal{M}} a(\xi,\mathcal{M})\theta(x,R,F,m,\xi,\mathcal{M})^2;\\
\theta(x,R,F,m,\xi,\mathcal{M})^2 =& \sum_{k=1}^{\infty} 2^{-k}\theta(x,R,F,m,\xi,\mathcal{M},k)^2;\\
2\theta(x,R,F,m,\xi,\mathcal{M},k)^2 =& \sup_{(s,t)\in I(Q,F,m,\xi,\mathcal{M})} [\varphi(x,s,t)^2 \\
&+ k^{-1}\psi(x,s,t,R,F,m,\xi,\mathcal{M})^2];\\
\psi(x,s,t,R,F,m,\xi,\mathcal{M})^2 =& \frac{1}{\#\mathcal{M}} \sum_{U\in\mathcal{M}} \nu(x,R,F\cup(U\cap J(s,t,m)^{(\xi)}),m)^2,
\end{aligned}
$$

where the positive constants $a(\xi,\mathcal{M})$ are chosen so that

$$\sum_{\xi<\Omega,\mathcal{M}\in[\mathcal{B}]^{<\omega}} a(\xi,\mathcal{M}) = 1.$$

Proof: As in [6] and [7], this follows from Banach's Contraction Mapping Theorem applied in a suitable function space. To describe it we take as domain for our functions the set $\mathcal{D} := C(K)\times[Q]^{<\omega}\times[R]^{<\omega}\times\mathbb{N}\times[0,\Omega)\times[\mathcal{B}]^{<\omega}\times\mathbb{N}$ and we consider the set $\mathcal{H}$ of all mappings from $\mathcal{D}$ into $\mathbb{R}^2$, i.e.

$$(\nu,\theta)(x,R,F,m,\xi,\mathcal{M},k) = (\nu(x,R,F,m),\theta(x,R,F,m,\xi,\mathcal{M},k))$$

such that $\nu(\cdot,R,F,m)$ and $\theta(\cdot,R,F,m,\xi,\mathcal{M},k)$ are convex and positively homogeneous in $x\in C(K)$ and they both are bounded above by $\|\cdot\|_\infty$. On $\mathcal{H}$ we consider the complete metric given by $d((\nu,\theta),(\nu',\theta')) = \max\{\sup\{|\nu^2(\delta)-\nu'^2(\delta)|: \delta\in\mathcal{D}\}, \sup\{|\theta^2(\delta)-\theta'^2(\delta)|: \delta\in\mathcal{D}\}\}$ and we define the contractive map $F\colon\mathcal{H}\to\mathcal{H}$, $F(\nu,\theta)=(\nu',\theta')$ where

$$
\begin{aligned}
3\nu'(x,R,F,m)^2 =& \varphi(x,R,F,m)^2 + \frac{1}{\#K(R,F,m)} \sum_{t\in K(R,F,m)} x(t)^2 \\
&+ \sum_{\xi,\mathcal{M}} a(\xi,\mathcal{M}) \sum_{k=1}^{\infty} 2^{-k}\ \theta(x,R,F,m,\xi,\mathcal{M},k)^2;\\
2\theta'(x,R,F,m,\xi,\mathcal{M},k)^2 =& \sup_{(s,t)\in I(Q,F,m,\xi,\mathcal{M})} \Bigg[\varphi(x,s,t)^2 \\
&+ k^{-1}\frac{1}{\#\mathcal{M}} \sum_{U\in\mathcal{M}} \nu(x,R,F\cup(U\cap J(s,t,m)^{(\xi)}),m)^2\Bigg]. \qquad \blacksquare
\end{aligned}
$$

Notice that, because of the definition of $I(Q, F, m, \xi, \mathcal{M})$, each of the sets $F \cup (U \cap J(s,t,m)^{(\xi)})$ which occur in the definition of ψ is either just F, or else F with one extra element appended. Notice also that ν, θ and ψ, when considered as functions of x, are non-negative, positively homogeneous and bounded (by 1) on the unit ball of $\mathcal{C}(K)$.

We define a pointwise lower semicontinuous norm in $\mathcal{C}(K)$ by

$$2\|x\|^2 = \|x\|_\infty^2 + \sum_{m,R} c(R,m)\nu(x,R,\emptyset,m)^2,$$

where $c(R,m)$ are further positive constants such that $\sum_{m,R} c(R,m) = 1$. Our aim is to show that this norm is **LUR**.

So we consider $x \in \mathcal{C}(K)$ and a sequence x_n such that the LUR hypothesis is satisfied for this norm. Notice that, by convexity and the definition of the norm as an ℓ^2 sum, the LUR hypothesis holds for each of the functions $\nu(\cdot, R, \emptyset, m)$.

Now let $\varepsilon > 0$ be given. It will be enough to show that there exists a subsequence (x_{n_k}) such that $\|x - x_{n_k}\|_\infty < 5\varepsilon$ for all k.

Our compact space K is a closed subset of $[0,1]^\Gamma$ equipped with the product topology and our given function x is continuous on K. Hence, there exist $m \in \mathbb{N}$ and a finite subset T of Γ such that $|x(s) - x(t)| \le \varepsilon$ whenever

$$\sup_{\gamma \in T} |s(\gamma) - t(\gamma)| \le 1/m.$$

If we set $S = T \cap P$ we obviously have $|x(s) - x(t)| \le \varepsilon$ whenever $(s,t) \in I(Q, S, m)$. Rather than working with this set S, we choose a finite subset S of P of minimal cardinality subject to the above condition, that is

$$(s,t) \in I(Q,S,m) \Longrightarrow |x(s) - x(t)| \le \varepsilon.$$

Now, by an easy compactness argument, we may also choose a finite subset R of Q such that $|x(s) - x(t)| < 2\varepsilon$ whenever $(s,t) \in I(R,S,m)$. Recalling the definitions given earlier, we see that

$$\varphi(x,R,S,m) < \varepsilon.$$

The proof of Theorem 2 depends on two lemmas. It is perhaps worth emphasizing that from now on x, x_n, ε, m, R and S are all fixed

Lemma 4: *If the LUR hypothesis holds for the function $\nu(\cdot, R, S, m)$ and a subsequence (x_{n_k}) then $\|x - x_{n_k}\|_\infty < 5\varepsilon$ for all large enough k.*

Proof: By convexity and the expression for ν as an ℓ^2-sum, we get

$$\frac{1}{2}\varphi(x,R,S,m)^2 + \frac{1}{2}\varphi(x_n,R,S,m)^2 - \varphi\Big(\frac{1}{2}(x+x_n),R,S,m\Big)^2 \to 0 \quad \text{as } n \to \infty.$$

Since $\varphi(x, R, S, m) < \varepsilon$, it follows that

$$\varphi(x_n, R, S, m) < \varepsilon \quad \text{for all large enough } n.$$

Looking at the second term in the definition of ν and applying convexity again we see that

$$\frac{1}{2}x(t)^2 + \frac{1}{2}x_n(t)^2 - \Big(\frac{1}{2}(x(t) + x_n(t))\Big)^2 \to 0 \quad \text{as } n \to \infty \text{ for any } t \in K(R, S, m),$$

which implies that

$$\max_{t \in K(R,S,m)} |x(t) - x_n(t)| < \varepsilon \quad \text{for all large enough } n.$$

For any $s \in K$ there exists $t \in K(R, S, m)$ with $(s, t) \in I(R, S, m)$ and so

$$|x(s) - x_n(s)| \le 2\varphi(x, R, S, m) + 2\varphi(x_n, R, S, m) + \max_{t \in K(R,S,m)} |x(t) - x_n(t)| < 5\varepsilon$$

for all large enough n. ∎

LEMMA 5: *Let F be a proper subset of S, let (x_{n_k}) be a subsequence of (x_n) and assume that the LUR hypothesis holds for the subsequence (x_{n_k}) and the function $\nu(\cdot, R, F, m)$. Then there exists $\gamma \in S \setminus F$ and a further subsequence $(x_{n_{k_j}})$ such that the LUR hypothesis holds for $(x_{n_{k_j}})$ and the function $\nu(\cdot, R, F \cup \{\gamma\}, m)$.*

Proof: To simplify notation, avoiding double subscripts, let us assume that the initial subsequence (x_{n_k}) is actually the sequence (x_n) itself. Since $\#F < \#S$ we may use the minimality in the choice of S to see that there is some

$$(s, t) \in I(Q, F, m) \quad \text{with } |x(s) - x(t)| > \varepsilon.$$

By the choice of S, there exists $\gamma \in S$ with $|s(\gamma) - t(\gamma)| > (1/m)$. We claim that any such γ must be in $J(s, t, m)$, so that $S \cap J(s, t, m) \neq \emptyset$. Indeed, otherwise, $\gamma \notin J(s, t, m)$ and there exists an open set $U \ni \gamma$ such that $|s(\delta) - s(\gamma)| \le 1/4m$ and $|t(\delta) - t(\gamma)| \le 1/4m$ for all $\delta \in U$. By density of Q, there is some $\delta \in Q \cap U$ and, since $(s, t) \in I(Q, F, m)$, we have $|s(\delta) - t(\delta)| \le 1/4m$. Combining the above inequalities, we obtain $|s(\gamma) - t(\gamma)| \le 3/4m$, a contradiction.

Since S is finite there is some maximum ordinal $\xi(s, t, m)$ with

$$S \cap J(s, t, m)^{(\xi(s,t,m))} \neq \emptyset.$$

We now assume that we have chosen $(s,t) \in I(R,F,m)$ in such a way that

$$\xi(s,t,m) = \min\{\xi(s',t',m)\colon (s',t') \in I(Q,F,m) \text{ and } |x(s') - x(t')| > \varepsilon\}.$$

Let $\mathcal{M}$ be a finite subset of $\mathcal{B}$, chosen in such a way that $S \subseteq \bigcup \mathcal{M}$ and $\# U \cap J(s,t,m)^{(\xi)} \leq 1$ for all $U \in \mathcal{M}$.

We have $(s,t) \in I(Q,F,m,\xi,\mathcal{M})$ and $\varphi(x,s,t) > \varepsilon/2$, so

$$\varphi(x,Q,F,m,\xi,\mathcal{M}) > \varepsilon/2.$$

If we now look at the third term in the definition of ν and apply the familiar convexity argument we see that the function $\theta(\cdot,R,F,m,\xi,\mathcal{M})$ and the sequence (x_n) satisfy the LUR hypothesis. So by Deville's Lemma there is a sequence $(s_n,t_n) \in I(Q,F,m,\xi,\mathcal{M})$ such that

$$\begin{aligned}\frac{1}{2}\psi(x,s_n,t_n,R,F,m)^2 + \frac{1}{2}\psi(x_n,s_n,t_n,R,F,m)^2& \\ -\psi\Big(\frac{1}{2}(x+x_n),s_n,t_n,R,F,m\Big)^2 &\to 0\end{aligned}$$

Query author: this and the next marked equations were was too long, and we had to split it. Please let me know if you think of a better solution.

and

$$\varphi(x,s_n,t_n) \to \varphi(x,Q,F,m,\xi,\mathcal{M}) > \varepsilon/2.$$

So $\varphi(x,s_n,t_n) > \epsilon/2$, i.e. $|x(s_n) - x(t_n)| > \epsilon$ for all large enough n. Reasoning as before, we get that $S \cap J(s_n,t_n,m) \neq \emptyset$ for such n. From the minimality of ξ it follows that $S \cap J(s_n,t_n,m)^{(\xi)} \neq \emptyset$. Now

$$S \cap J(s_n,t_n,m)^{(\xi)} \subseteq \bigcup_{U \in \mathcal{M}} U \cap J(s_n,t_n,m)^{(\xi)}$$

and each of the sets $U \cap J(s_n,t_n,m)^{(\xi)}$ contains at most one element because $(s_n,t_n) \in I(Q,F,m,\xi,\mathcal{M})$. Thus, for some $U(n) \in \mathcal{M}$ the intersection $U(n) \cap J(s_n,t_n,m)^{(\xi)}$ contains exactly one point γ_n which is in S. If we proceed to a subsequence (n_k), we may assume that U_{n_k} and γ_{n_k} are the same set U in $\mathcal{M}$ and the same element γ of S, for all k. Finally, by looking at the definition of ψ and applying convexity yet again, we see that

$$\begin{aligned}&\frac{1}{2}\nu(x,R,F\cup\{\gamma\},m)^2 + \frac{1}{2}\nu(x_{n_k},R,F\cup\{\gamma\},m)^2 \\ &\quad - \nu(\frac{1}{2}(x+x_{n_k}),R,F\cup\{\gamma\},m)^2 \\ &\qquad \leq \#\mathcal{M}(\frac{1}{2}\psi(x,s_{n_k},t_{n_k},R,F,m)^2 + \frac{1}{2}\psi(x_{n_k},s_{n_k},t_{n_k},R,F,m)^2 \\ &\qquad\quad - \psi(\frac{1}{2}(x+x_{n_k}),s_{n_k},t_{n_k},R,F,m)^2) \to 0 \quad \text{as } k \to \infty.\end{aligned}$$

This is the LUR hypothesis for $\nu(\cdot, R, F \cup \{\gamma\}, m)$ and the subsequence (x_{n_k}). ∎

To complete the proof, we have already noted that the LUR hypothesis holds for (x_n) and the function $\nu(\cdot, R, \emptyset, m)$. A finite number of applications of Lemma 5 yield a subsequence satisfying the LUR hypothesis for the function $\nu(\cdot, R, S, m)$. Lemma 4 now does what we need to complete the proof of Theorem 2.

4. Boundedness of the Cantor–Bendixon index

We devote this section to a proof of the following result which, together with Theorem 2, clearly yields Theorem 1. Our proof is closely modeled on a recent theorem of Argyros et al. [1].

THEOREM 3: *Let Γ be a Polish space and let K be a pointwise compact set of functions on Γ. Assume that each function $s \in K$ has only countably many discontinuities and that the set K is separable. Then there exists a countable ordinal Ω such that for all $s \in K$ and all $\delta > 0$ the Ω^{th} derived set $J(s, \delta)^{(\Omega)}$ is empty.*

We first establish a little notation: when k is a natural number we write D_k for the set of all finite sequences of 0's and 1's, of length at most k. Thus $D_k = \bigcup_{0 \le j \le k} \{0, 1\}^j$. The set D_0 has just one element, the sequence () of length 0. If $\sigma \in \{0, 1\}^j$ the **length** j of σ will be denoted $|\sigma|$ and we shall write $\sigma.0$ (resp. $\sigma.1$) for the element of $\{0, 1\}^{j+1}$ which extends σ and has 0 (resp. 1) in its last place. We define $0.\sigma$ and $1.\sigma$ analogously.

We fix a metric d on Γ, compatible with the given topology, under which Γ is complete, as well as a countable base $\mathcal{B}$ for the topology of Γ and a sequence (s_m) which is dense in K. We now introduce, for all infinite subsets M of $\mathbb{N}$, all natural numbers p, all nonempty open subsets X of Γ, and all positive real numbers δ, a set $\Upsilon(X, M, p, \delta)$. Elements of this set are tuples

$$(M, k, (U_\sigma)_{\sigma \in D_k}, (\alpha_\sigma)_{\sigma \in D_k}, (\beta_\sigma)_{\sigma \in D_k}),$$

where $k \in \omega$, $U_\sigma \in \mathcal{B}$, $\alpha_\sigma \in \Gamma$ and $\beta_\sigma \in \Gamma$ satisfy the following conditions:

1. $\overline{U_\sigma} \subseteq X$ and $\operatorname{diam} U_\sigma \le 2^{-p-|\sigma|}$ for all $\sigma \in D_k$;
2. $\overline{U_{\sigma.0}} \cap \overline{U_{\sigma.1}} = \emptyset$ and $\overline{U_{\sigma.0}} \cup \overline{U_{\sigma.1}} \subseteq U_\sigma$ whenever $\sigma \in D_{k-1}$;
3. $\alpha_\sigma, \beta_\sigma \in U_\sigma$;
4. $|s_m(\alpha_\sigma) - s_m(\beta_\sigma)| > \delta$ for all large enough $m \in M$.

The union $\bigcup_{M\in[\mathbb{N}]^\omega} \Upsilon(X,M,p,\delta)$ will be denoted $\Upsilon(X,p,\delta)$. This is a tree under the following ordering:

$$(M,k,(U_\sigma)_{\sigma\in D_k},(\alpha_\sigma)_{\sigma\in D_k},(\beta_\sigma)_{\sigma\in D_k}) \\ \prec (M',k',(U'_\sigma)_{\sigma\in D_{k'}},(\alpha'_\sigma)_{\sigma\in D_{k'}},(\beta'_\sigma)_{\sigma\in D_{k'}})$$

if and only if

$$M=M',\quad k<k',\quad U_\sigma=U'_\sigma,\quad \alpha_\sigma=\alpha'_\sigma \quad\text{and}\quad \beta_\sigma=\beta'_\sigma \quad \text{for all } \sigma\in D_k.$$

Notice that the coordinate k equals the height of the element $(M,k,(U_\sigma),(\alpha_\sigma),(\beta_\sigma))$ in the tree.

LEMMA 6: *For each $\delta>0$ and each non-empty open subset X of Γ, the tree $\Upsilon(X,p,\delta)$ is well founded.*

Proof: The reader will probably have realized that elements of the tree $\Upsilon(X,p,\delta)$ can be regarded as finite attempts at constructing a Cantor set of discontinuities for some element of K. The proof of the present lemma makes this idea more explicit.

We have to show that our tree has no infinite branch. So suppose, if possible, such a branch exists. It would consist of a sequence of elements

$$(M,k,(U_\sigma),(\alpha_\sigma),(\beta_\sigma)) \quad (k\geq 1)$$

satisfying (1) to (4).

It follows from completeness of Γ and the conditions (1) and (2) that, for each infinite sequence $z\in\{0,1\}^\omega$, the intersection $\bigcap_{l\in\omega} U_{z\restriction l}$ contains just one point γ_z. If s is any element of K which is a cluster point of $(s_m)_{m\in M}$ then we have

$$|s(\beta_\sigma)-s(\alpha_\sigma)|\geq\delta$$

for all $\sigma\in\{0,1\}^{<\omega}$. Since each of the sequences $(\alpha_{z\restriction l})$ and $(\beta_{z\restriction l})$ converges to γ_z, we see that each γ_z is a discontinuity point of s, contrary to our hypothesis that there are only countably many such discontinuities. ∎

LEMMA 7: *For each non-empty open subset X of Γ, each $p\in\omega$ and each $\delta>0$, the relation $\prec$ on $\Upsilon(X,p,\delta)$ is analytic.*

Proof: Here we follow [1] quite closely. We note that X is open in the Polish space Γ and hence itself Polish, and that $[\omega]^\omega$ is also a Polish space. The countable sets ω and $\mathcal{B}$ will be equipped with the discrete topology. So if we define

$$H = \bigcup_{k\in\omega} ([\omega]^\omega \times \{k\} \times \mathcal{B}^{D_k} \times X^{D_k} \times X^{D_k}),$$

H is a disjoint union of Polish spaces and hence a Polish space. It follows from our description of the elements of $\Upsilon(X,p,\delta)$ that $\Upsilon(X,p,\delta) \subseteq H$. We shall show that the relation $\prec$ is an analytic subset of $H \times H$. Now it is very easy to see that $\prec$ is closed in $\Upsilon(X,p,\delta) \times \Upsilon(X,p,\delta)$, so it will be enough for us to show that $\Upsilon(X,p,\delta)$ is analytic.

Now it is a standard result that we may enhance the topology of the Polish space Γ in such a way that all the sets $U \in \mathcal{B}$ are both open and closed, and all the functions s_m are continuous, while Γ remains a Polish space. Let us write $\tilde{X}$ for X equipped with this enhanced topology, and $\tilde{H}$ for H equipped with a similarly enhanced topology. What we shall show is that $\Upsilon(X,p,\delta)$ is a countable union of closed subsets of $\tilde{H}$.

We set

$$\begin{aligned}\Upsilon_n = \{(M, k, (U_\sigma), (\alpha_\sigma)_{\sigma\in D_k}, (\beta_\sigma))_{\sigma\in D_k}) \in \Upsilon(X,p,\delta)&:\\ |s_m(\alpha_\sigma) - s_m(\beta_\sigma)| \geq \delta \text{ whenever } \sigma \in D_k \text{ and } n \leq m \in M\}&\end{aligned}$$

and shall show that each Υ_n is closed in $\tilde{H}$. Suppose then that

$$(M^l, k^l, (U^l_\sigma)_{\sigma\in D_{k^l}}, (\alpha^l_\sigma)_{\sigma\in D_{k^l}}, (\beta^l_\sigma)_{\sigma\in D_{k^l}}) \quad (l \in \omega)$$

is a sequence of elements of Υ_n which converges in $\tilde{H}$ to the limit

$$(M, k, (U_\sigma)_{\sigma\in D_k}, (\alpha_\sigma)_{\sigma\in D_k}, (\beta_\sigma)_{\sigma\in D_k}).$$

Since we have equipped ω and $\mathcal{B}$ with the discrete topology, $k^l = k$ and $U^l_\sigma = U_\sigma$ for all large enough l. Since α^l_σ and β^l_σ converge respectively to α_σ and β_σ in the topology of $\tilde{X}$, and since U_σ is closed in that topology, we have $\alpha_\sigma, \beta_\sigma \in U_\sigma$. If $m \in M$ and $m \geq n$ then $m \in M^l$ for all l large enough, so that $|s_m(\alpha^l_\sigma) - s_m(\beta^l_\sigma)| \geq \delta$ for all l large enough. Since we have arranged for s_m to be continuous in the topology of $\tilde{X}$, we see that $|s_m(\alpha_\sigma) - s_m(\beta_\sigma)| \geq \delta$. We have finished showing that Υ_n is closed in $\tilde{H}$. ∎

To finish the proof of our theorem, we need to show that the rank of the tree $\Upsilon(X,p,\delta)$ dominates the derived length of the set $J(s,\delta)$ when $s \in K$. We do this using two final lemmata, the first of which expresses an obvious idea in what is perhaps over-pedantic notation.

LEMMA 8: *Let X be a non-empty open subset of Γ, let M be an infinite subset of ω, let p be a natural number and let δ be a positive real number. Let Y and Z be disjoint non-empty open subsets of X and assume that there exists $U \in \mathcal{B}$ with $Y \cup Z \subseteq U \subseteq X$ and $\operatorname{diam} U \leq 2^{-p}$. If $\Upsilon(Y, M, p+1, \delta)^{[\xi]}$ and $\Upsilon(Z, M, p+1, \delta)^{[\xi]}$ are both non-empty, then $\Upsilon(X, M, p, \delta)^{[\xi+1]}$ is also non-empty.*

Proof: We shall show how to embed the tree $\Upsilon(Y, M, p+1, \delta) \otimes \Upsilon(Z, M, p+1, \delta)$ into the set of non-minimal elements of $\Upsilon(X, M, p, \delta)$, that is to say, the elements of height at least 1. Our hypothesis, together with Lemma 1, will then tell us that $\Upsilon(X, M, p, \delta)^{[\xi]}$ contains a non-minimal element, which in turn implies that $\Upsilon(X, M, p, \delta)^{[\xi+1]} \neq \emptyset$.

Since $Y \subseteq U$ and $\Upsilon(Y, M, p+1, \delta) \neq \emptyset$ there certainly exist $\alpha, \beta \in U$ with $|s_m(\alpha) - s_m(\beta)| \geq \delta$ for all large enough $m \in M$. We define $U_{()} = U$, $\alpha_{()} = \alpha$ and $\beta_{()} = \beta$. Now let $(M, k, (V_\sigma)_{\sigma \in D_k}, (\kappa_\sigma)_{\sigma \in D_k}, (\lambda_\sigma)_{\sigma \in D_k})$ and $(M, k, (W_\sigma)_{\sigma \in D_k}, (\mu_\sigma)_{\sigma \in D_k}, (\nu_\sigma)_{\sigma \in D_k})$ be height k elements of $\Upsilon(Y, M, p+1, \delta)$ and $\Upsilon(Z, M, p+1, \delta)$ respectively. If we define

$$\begin{aligned} U_{0.\sigma} &= V_\sigma, & U_{1.\sigma} &= W_\sigma, \\ \alpha_{0.\sigma} &= \kappa_\sigma, & \alpha_{1.\sigma} &= \mu_\sigma, \\ \beta_{0.\sigma} &= \lambda_\sigma, & \beta_{1.\sigma} &= \nu_\sigma, \end{aligned}$$

then it is easy to check that $(M, k+1, (U_\sigma)_{\sigma \in D_{k+1}}, (\alpha_\sigma)_{\sigma \in D_{k+1}}, (\beta_\sigma)_{\sigma \in D_{k+1}})$ is a height $k+1$ element of $\Upsilon(X, M, p, \delta)$. This defines the promised embedding. ∎

LEMMA 9: *Let X be a non-empty open subset of Γ and let δ be a positive real number. Let $s \in K$ be the pointwise limit $s = \lim_{M \ni m \to \infty} s_m$ along the subsequence M, and assume that $X \cap J(s, \delta)^{(\xi)} \neq \emptyset$. Then, for every $p \in \omega$, $\Upsilon(X, M, p, \delta)^{[\xi]} \neq \emptyset$.*

Proof: We proceed by induction on the ordinal ξ, starting with $\xi = 0$: if $J(s, \delta) \cap X \neq \emptyset$ we choose any γ in this set and then select $U \in \mathcal{B}$ with $\gamma \in U$, $\overline{U} \subseteq X$, $\operatorname{diam} U \leq 2^{-p}$. Since $\gamma \in J(s, \delta)$ we have $\operatorname{osc}(s \restriction U) > \delta$ so we can choose $\alpha, \beta \in U$ with $|s(\alpha) - s(\beta)| > \delta$. If we set $U_{()} = U$, $\alpha_{()} = \alpha$, $\beta_{()} = \beta$, then $(M, 0, (U_\sigma)_{\sigma \in D_0}, (\alpha_\sigma)_{\sigma \in D_0}, (\beta_\sigma)_{\sigma \in D_0}) \in \Upsilon(X, M, p, \delta)$.

Now suppose that $X \cap J(s, \delta)^{(\xi+1)} \neq \emptyset$ and that the result is true for ξ. Choose $\gamma \in X \cap J(s, \delta)^{(\xi+1)}$ and an element U of $\mathcal{B}$ with $\gamma \in U$, $\operatorname{diam} U \leq 2^{-p}$. Since γ is a limit point of $J(s, \delta)^{(\xi)}$, we may find $\zeta \in U \cap J(s, \delta)^{(\xi)}$ with $\zeta \neq \gamma$, and then choose disjoint open Y and Z containing γ and ζ respectively. By our inductive hypothesis (which of course applies to *all* p), $\Upsilon(Y, M, p+1, \delta)^{[\xi]}$ and $\Upsilon(Z, M, p+1, \delta)^{[\xi]}$ are both non-empty. By Lemma 8 we now have $\Upsilon(X, M, p, \delta)^{[\xi+1]} \neq \emptyset$.

Finally, let η be a limit ordinal, with $X \cap J(s,\delta)^{(\eta)} \neq \emptyset$, and assume that the result is true for all $\xi < \eta$. As before, we choose $\gamma \in X \cap J(s,\delta)^{(\eta)}$, $U \in \mathcal{B}$ with $\gamma \in U$, $\operatorname{diam} U \leq 2^{-p}$ and $\alpha, \beta \in U$ with $|s_m(\alpha) - s_m(\beta)| \geq \delta$ for all large enough $m \in M$. We use these to define a height 0 element $(M, 0, U, \alpha, \beta)$ of $\Upsilon(X, M, p, \delta)$. For any $\xi < \eta$, γ is a limit point of $X \cap J(s,\delta)^{(\xi)}$, so we can find disjoint open subsets Y, Z of U such that $Y \cap J(s,\delta)^{(\xi)}$ and $Z \cap J(s,\delta)^{(\xi)}$ are both nonempty. By inductive hypothesis, both $\Upsilon(Y, M, p+1, \delta)^{[\xi]}$ and $\Upsilon(Z, M, p+1, \delta)^{[\xi]}$ are both non-empty. The proof of Lemma 8 shows that our already constructed height 0 element $(M, 0, U, \alpha, \beta)$ is in $\Upsilon(X, M, p, \delta)^{[\xi+1]}$. Since this is true for all $\xi < \eta$ we have $\Upsilon(X, M, p, \delta)^{[\eta]} \neq \emptyset$ as claimed. ∎

The proof of Theorem 3 is now complete.

5. σ-Fragmentability of $C(K)$ for the non-separable case

Let us recall that space $C(K)$ with the pointwise topology $\mathfrak{T}_p$ is said to be *σ-fragmentable* by its norm if for every $\varepsilon > 0$ we can decompose $C(K) = \bigcup_{n=1}^{\infty} C_{n,\varepsilon}$ in such a way that for every $n \in \mathbb{N}$ and every non-empty subset $T \subset C_{n,\varepsilon}$ there exists a $\mathfrak{T}_p$-open set V such that $V \cap T$ is non-empty and has norm diameter less than ε. This notion was introduced and studied in [8] where among other things it is proved that $C(K)$ is σ-fragmentable if for every $\varepsilon > 0$ we can decompose $C(K) = \bigcup_{n=1}^{\infty} C_{n,\varepsilon}$ in such a way that for every $n \in \mathbb{N}$ and every non-empty subset $T \subset C_{n,\varepsilon}$ there exists a $\mathfrak{T}_p$-open V such that $V \cap T$ is non-empty and covered by countably many sets of diameter less than ε; see [8, Theorem 4.1].

In our last section we shall prove the following

THEOREM 4: *Let Γ be a Polish space and let K be a pointwise compact set of functions on Γ such that each function $s \in K$ has only countably many discontinuities. Then $(C(K), \mathfrak{T}_p)$ is σ-fragmentable by its norm.*

We first establish a little notation: For $s \in K$ let $J(s) = \bigcup_{\delta > 0} J(s,\delta)$, i.e. the set of all the discontinuity points of s. A finite sequence $\{(s_i, t_i)\}_{i=1}^{n}$, $(s_i, t_i) \in K \times K$, $1 \leq i \leq n$, is said to be *fitted* whenever for every i, $1 \leq i \leq n$, we have

$$(1) \qquad s_i(\gamma) = t_i(\gamma) \text{ for all } \gamma \in Q \cup \left(\bigcup_{j<i} (J(s_j) \cup J(t_j)) \right),$$

and the fitted sequence $\{(s_i, t_i)\}_{i=1}^{n}$ is said to have *length* n.

Given $\varepsilon > 0$ we say that $x \in C(K)$ *ε-jumps* a fitted sequence $\{(s_i, t_i)\}_{i=1}^n$ whenever

$$|x(s_i) - x(t_i)| > \varepsilon \quad \text{for every } 1 \leq i \leq n. \tag{2}$$

Theorem 4 will follow from some lemmata.

LEMMA 10: *Given $x \in C(K)$ and $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that x does not ε-jump any fitted sequence in $K \times K$ which has length strictly bigger than n.*

Proof: Since K is compact x is uniformly continuous and there exists $\delta > 0$ and a finite subset F of Γ such that

$$(\sup_{\gamma \in F} |s(\gamma) - t(\gamma)| < \delta \Longrightarrow |x(s) - x(t)| < \varepsilon) \quad \text{for all } s, t \in K. \tag{3}$$

Let $n := \#F$; we claim that if $\{(s_i, t_i)\}_{i=1}^m$ is a fitted sequence which is ε-jumped by x, then we have $m \leq n$. Indeed, since Q is dense in Γ from (1) it follows that $s_1(\gamma) = t_1(\gamma)$ for every point of continuity γ of s_1 and t_1. Then from (1), (2) and (3) it follows that

$$F \cap (J(s_1) \cup J(t_1)) \setminus Q \neq \emptyset.$$

An obvious induction argument gives that

$$F \cap (J(s_i) \cup J(t_i)) \setminus \left(Q \cup \left(\bigcup_{j<i} (J(s_j) \cup J(t_j)) \right) \right) \neq \emptyset, \quad 1 \leq i \leq m. \tag{4}$$

Now the statement follows from (4). ∎

It might be worth remarking that (4) shows that if a fitted sequence $\{(s_i, t_i)\}_{i=1}^n$ is ε-jumped by some $x \in C(K)$, then $(s_i, t_i) \neq (s_j, t_j)$ for $1 \leq i \neq j \leq n$.

Given $x \in C(K)$ and $\varepsilon > 0$, let $j(x, \varepsilon)$ be the minimum of the natural numbers for which the thesis of Lemma 10 holds for ε and x. Thus for any $j > j(x, \varepsilon)$ the function x cannot ε-jump any fitted sequence of length j, and there exists a fitted sequence ε-jumped by x of length equal to $j(x, \varepsilon)$ whenever $j(x, \varepsilon) > 0$.

Given $x \in C(K)$ and $\varepsilon > 0$, a subset $S \subset \Gamma$ is said to *ε-control* x whenever there exist a finite subset $F \subset S$ and $\delta > 0$ for which (3) holds.

LEMMA 11: *Given $x \in C(K)$ and $\varepsilon > 0$, if $\{(s_i, t_i)\}_{i=1}^{j(x,\varepsilon/2)}$ is a fitted sequence which is $(\varepsilon/2)$-jumped by x then $Q \cup (\bigcup_{i \leq j(x,\varepsilon/2)} (J(s_i) \cup J(t_i)))$ ε-controls x.*

Proof: Otherwise for any finite set F, $F \subset Q \cup (\bigcup_{i \le j(x,\varepsilon/2)} (J(s_i) \cup J(t_i)))$, and any $n \in \mathbb{N}$ we can choose $s(F,n)$, $t(F,n) \in K$ such that

$$|s(F,n)(\gamma) - t(F,n)(\gamma)| < \frac{1}{n} \text{ for all } \gamma \in F \text{ whereas } |x(s(F,n)) - x(t(F,n))| \ge \varepsilon.$$

Since K is compact there must exist an adherent point $(\tilde{s}, \tilde{t})$ to the net $(s(F,n), t(F,n))$. Now we have

$$|x(\tilde{s}) - x(\tilde{t})| \ge \varepsilon > \frac{\varepsilon}{2} \text{ and } \tilde{s}(\gamma) = \tilde{t}(\gamma) \text{ for all } \gamma \in Q \cup \Bigg(\bigcup_{i \le j(x,\frac{\varepsilon}{2})} (J(s_i) \cup J(t_i)) \Bigg).$$

Then $(s_1, t_1), \ldots, (s_{j(x,\frac{\varepsilon}{2})}, t_{j(x,\frac{\varepsilon}{2})})$, $(\tilde{s}, \tilde{t})$ is a fitted sequence of length $1 + j(x, \frac{\varepsilon}{2})$ which x $(\varepsilon/2)$-jumps, a contradiction. ∎

LEMMA 12: *For every $\varepsilon > 0$ there exists a decomposition $C(K) = \bigcup_{n \in \mathbb{N}} C_{n,\varepsilon}$ such that for every $n \in \mathbb{N}$ and any $x \in C_{n,\varepsilon}$ there exist a weak open set W and a countable set $N \subset \Gamma$ such that $x \in W \cap C_{n,\varepsilon}$ and N ε-controls every $y \in W \cap C_{n,\varepsilon}$.*

Proof: Let $\varepsilon > 0$. Let $C_{n,\varepsilon} := \{x \in C(K) \colon j(x, \varepsilon/2) = n\}$. According to the proof of Lemma 11, if $n = 0$ then the set Q ε-controls every function $y \in C_{0,\varepsilon}$. Suppose $n > 0$, and let us fix a fitted sequence $\{(s_i, t_i)\}_{i=1}^{n}$ which is $\varepsilon/2$-jumped by x. Let $\delta > 0$ such that

$$|x(s_i) - x(t_i)| > \frac{\varepsilon}{2} + \delta, \quad 1 \le i \le n.$$

Then every element y in the weak open set

$$W := \{y \in C(K) \colon |x(s_i) - y(s_i)| < \delta/2 \;\&\; |x(t_i) - y(t_i)| < \delta/2,\; 1 \le i \le n\}$$

$\varepsilon/2$-jumps the above fitted sequence. Then from Lemma 11 we conclude that the set

$$Q \cup \Bigg(\bigcup_{i \le j(x,\varepsilon/2)} (J(s_i) \cup J(t_i)) \Bigg)$$

ε-controls every $y \in C_{n,\varepsilon} \cap W$. ∎

The proof of Theorem 4 now follows from the remarks in the beginning of this section together with the above lemma and the next proposition, which is nothing else than a quantitative version of Ascoli's theorem.

Proposition 5: *Let K be a compact space embedded in a cube $[0,1]^\Gamma$ and let $\mathcal{F}$ be a bounded subset of $C(K)$. If $\mathcal{F}$ is ε-equicontinuous, i.e. there is a finite set $F \subset \Gamma$ and $\delta > 0$ such that $|x(s) - x(t)| < \varepsilon$ whenever $|s(\gamma) - t(\gamma)| < \delta$ for all $\gamma \in F$ and x in $\mathcal{F}$, then $\mathcal{F}$ can be covered by finitely many sets of norm diameter less than 3ε.*

Proof: Let us assume that $\mathcal{F}$ is bounded by m and let us split up the interval $[-m, m]$ into a finite number of sets of diameter less than ε,

$$[-m,m] = \bigcup_{i=1}^{\ell} I_i; \quad \operatorname{diam}(I_i) < \varepsilon,\ 1 \le i \le \ell. \tag{5}$$

Given a subset $S \subset \Gamma$ the symbol $\pi_S\colon [0,1]^\Gamma \to [0,1]^S$ will stand for the canonical projection. For the finite set F we can split $\pi_F(K)$ up into finitely many sets which are included into a cartesian product of intervals of length strictly less than δ, i.e.

$$\pi_F(K) = \bigcup_{j=1}^{p} (L_j \cap \pi_F(K))$$

where $L_j \cap \pi_F(K) \neq \emptyset$ and each L_j is a cube whose factor intervals have a length strictly less than δ. Choose $s_1, \ldots, s_p \in K$ such that $\pi_F(s_j) \in L_j$, $1 \le j \le p$. Then we can cover

$$\mathcal{F} = \bigcup_{h=1}^{N} T_h$$

in such a way that each T_h has the property that $y(s_j)$ belongs to the same interval I_i from (5) for all $y \in T_h$ and all j, $1 \le j \le p$.

We claim that the diameter of every T_h is less than or equal to 3ε. Indeed, given y_1, $y_2 \in T_h$ and $s \in K$ there exists j, $1 \le j \le p$, such that $\pi_F(s) \in L_j$. From the choice of L_j we get

$$|s(\gamma) - s_j(\gamma)| < \delta \quad \text{for all } \gamma \in F.$$

Now from our condition of ε-equicontinuity it follows that

$$|y_i(s) - y_i(s_j)| < \varepsilon, \quad i = 1, 2. \tag{6}$$

From the choice of T_h we have that $y_1(s_j)$ and $y_2(s_j)$ belong to the same interval I_i from (5), so

$$|y_1(s_j) - y_2(s_j)| < \varepsilon. \tag{7}$$

From (6) and (7) we conclude that

$$|y_1(s) - y_2(s)| < 3\varepsilon,$$

being the reasoning valid for every $s \in K$ the norm-diameter of T_h is not bigger than 3ε. ∎

To complete the proof of Theorem 4 let us observe that the decomposition from Lemma 12 gives us sets $C_{n,\varepsilon}$ such that for every $x \in C_{n,\varepsilon}$ the weak open set W containing x gives us the set $W \cap C_{n,\varepsilon}$ which is a countable union of ε-equicontinuous sets, so it can be covered by countably many sets of norm-diameter less than 3ε by the former proposition. The conclusion now follows from [8, Theorem 4.1].

author please update [1], [7], [11], [15] if possible

References

[1] S. Arygros, P. Dodos and V. Kanellopoulos, *Tree structures associated to a family of functions,* Journal of Symbolic Logic, to appear.

[2] J. Bourgain, D. H. Fremlin and M. Talagrand, *Pointwise compact sets of Baire-measurable functions,* American Journal of Mathematics **100** (1978), 845–886.

[3] A. Bouziad, *L'espace de Helly à la propiété de Namioka,* Comptes Rendus de l'Académie des Sciences, Paris, Série I **317** (1993), 841–843.

[4] R. Deville, G. Godefroy and V. Zizler, *Smoothness and Renorming in Banach Spaces,* Pitman Monographs and Surveys in Pure and Applied Mathematics 64, Longman Scientific & Technical, Harlow, 1993.

[5] R. Deville and G. Godefroy, *Some applications of projective resolutions of identity,* Proceedings of the London Mathematical Society **67** (1993), 183–199.

[6] R. G. Haydon, *Trees in renorming theory,* Proceedings of the London Mathematical Society **78** (1999), 549–584.

[7] R. G. Haydon, *Locally uniformly rotund norms in Banach spaces and their duals,* submitted.

[8] J. E. Jayne, I. Namioka and C. A. Rogers, *Topological properties of Banach spaces,* Proceedings of the London Mathematical Society **66** (1993), 651–672.

[9] A. S. Kechris, *Classical Descriptive Set Theory,* Springer-Verlag, New York, 1995.

[10] I. Kortezov, *The function space over the Helly compact is sigma-fragmentable,* Topology and its Applications **106** (2000), 69–75.

[11] A. Moltó, J. Orihuela, S. Troyanski and M. Valdivia, *A non-linear transfer technique for renorming,* submitted.

[12] E. Odell and H. P. Rosenthal, *A double dual characterization of separable Banach spaces containing ℓ^1,* Israel Journal of Mathematics **20** (1975), 375–384.

[13] H. P. Rosenthal, *Point-wise compact subsets of the first Baire class,* American Journal of Mathematics **99** (1977), 362–378.

[14] S. Todorcevic, *Trees and linearly ordered sets,* in *Handbook of Set-Theoretic Topology*, North-Holland, Amsterdam, 1984, pp. 235–293.

[15] S. Todorcevic, *Representing trees as relatively compact subsets of the first Baire class,* Preprint.

[16] S. Todorcevic, *Compact subsets of the first Baire class,* Journal of the American Mathematical Society **12** (1999), 1179–1212.