ON THE LOCAL MODULI OF SQUARENESS

ANTONIO J. GUIRAO

ABSTRACT. We introduce the notions of pointwise modulus of squareness and local modulus of squareness of a normed space X. This answers a question of C. Benitez, K. Przesławski and D. Yost about the definition of a sensible localization of the modulus of squareness. Geometrical properties of the norm of X (Fréchet smoothness, Gâteaux smoothness, local uniform convexity or strict convexity) are characterized in terms of the behaviour of these moduli.

1. INTRODUCTION

Let us recall the modulus of squareness, which was originally defined in [7], where it arose naturally from studying Lipschitz continuous set-valued functions. Given a normed space X, one observes that for any x, $y \in X$ with $\|y\| < 1 < \|x\|$, there is a unique $z = z(x, y)$ in the line segment $[x, y]$ with $\|z\| = 1$. We put

$$\omega(x,y) = \frac{\|x - z(x,y)\|}{\|x\| - 1}$$

and define $\xi = \xi_X : [0,1) \to [1,\infty]$ by

$$\xi(\beta) = \sup\{\omega(x,y) : \|y\| \leq \beta < 1 < \|x\|\}.$$

It is shown in [7] that for an inner product space, $\xi(\beta) = \xi_2(\beta) = 1/\sqrt{1-\beta^2}$, and that for any normed space containing $l_1(2)$, $\xi(\beta) = \xi_1(\beta) = (1+\beta)/(1-\beta)$. The following theorem [1, Theorem O] puts together all the known properties of this modulus.

Theorem 1.1. *Let X be any normed space, ξ its modulus of squareness. Then*

(a) $\xi(\beta) = \sup\{\xi_M(\beta) : M \subset X, \dim M = 2\}$,
(b) *ξ is strictly increasing and convex,*
(c) *$\xi < \xi_1$ everywhere on $(0,1)$, unless X contains arbitrarily close copies of $l_1(2)$,*
(d) *$\xi' \leq \xi_1'$ almost everywhere on $(0,1)$,*
(e) *$\xi > \xi_2$ everywhere on $(0,1)$, unless X is an inner product space,*
(f) *X is uniformly convex if and only if $\lim_{\beta\to1}(1-\beta)\xi(\beta) = 0$,*
(g) *X is uniformly smooth if and only if $\xi'(0) = 0$,*
(h) *$\xi_{X^*}(\beta) = 1/\xi^{-1}(1/\beta)$, for $\beta \in [0,1)$,*
(i) *if $\xi(\beta) < 1/(1-\beta)$ for some β, then X has uniformly normal structure.*

2000 *Mathematics Subject Classification.* 46B20, 46B04.

Key words and phrases. Banach Spaces; Locally Uniformly Rotund Norms; Fréchet Smooth Norms; Strict Convex Norms; Gâteaux Smooth Norms; Modulus of Squareness.

Research supported by the grants MTM2005-08379 of MECD (Spain), 00690/PI/04 of Fundación Séneca (CARM, Spain) and AP2003-4453 of MECD (Spain).

The proof of these properties can be found in [1, 7] and also some of them as well as a more geometrical characterization of ξ in [9, 10, 11].

Let us observe in particular that the behaviour of ξ near one is related to convexity, and that the behaviour of ξ near zero is related to smoothness.

The question of the existence of a sensible *localization* of the modulus of squareness was posed in [1]. In order to answer this question we define here two new moduli.

From now on and for the sake of clearness, for any norm one vector x, $\lambda > 0$ and y with $\|y\| < 1$, we will put $\omega_x(\lambda, y) = \omega((1+\lambda)x, y)$ and $z_x(\lambda, y) = z((1+\lambda)x, y)$. Therefore $\omega_x(\lambda, y) = \|(1+\lambda)x - z_x(\lambda, y)\| / \lambda$. Besides, we can deduce that for $y \in \text{span}\{x\}$ and for any $\lambda > 0$, $\omega_x(\lambda, y) = 1$, since $z_x(\lambda, y)$ would be x.

Definition 1.2 (Pointwise modulus of squareness)**.** For any pair of norm one vectors x, y the *pointwise modulus of squareness at x in the direction y* is the function $\xi_{X,x,y} = \xi_{x,y} : [0, 1) \to [1, \infty)$ defined by

$$\xi_{x,y}(\beta) = \sup\{\omega_x(\lambda, \gamma y) : |\gamma| \le \beta, \lambda > 0\}.$$

Definition 1.3 (Local modulus of squareness)**.** For any norm one vector x the *local modulus of squareness at x* is the function $\xi_{X,x} = \xi_x : [0, 1) \to [1, \infty)$ defined by

$$\xi_x(\beta) = \sup\{\omega_x(\lambda, y) : \|y\| \le \beta, \lambda > 0\} = \sup_{\|y\|=1}\{\xi_{x,y}(\beta)\}.$$

Observe that for any subspace $M \subset X$ of dimension 2 containing the pair of norm one vectors x, y we have that $\xi_{x,y} = \xi_{M,x,y}$. For ξ_x we establish an analogue to (a) of theorem 1.1. Indeed,

$$\xi_x(\beta) = \sup\{\xi_{M,x}(\beta) : x \in M \subset X, \ \dim M = 2\}.$$

One can realize that for any $\beta \in [0, 1)$,

$$\xi(\beta) = \sup\{\xi_x(\beta) : x \in S_X\} = \sup\{\xi_{x,y}(\beta) : x, y \in S_X\}.$$

We shall show how these moduli are related to various geometrical properties of the norm of X. In particular, in section 3 we recall the notions of Gâteaux smoothness and Fréchet smoothness and show whether or not a normed space X is Fréchet (resp. Gâteaux) smooth depending on the behaviour of the local (resp. pointwise) modulus of squareness near zero. In section 4 we recall the notions of local uniform convexity and strict convexity and show whether or not X is locally uniformly (resp. strictly) convex depending on the behaviour of the local (resp. pointwise) modulus of squareness near one. More precisely we shall establish :

Theorem 1.4. *Let X be a normed space and x a norm one vector. Then*

(a) *X is Gâteaux smooth at x iff $\xi'_{x,y}(0) = 0$ for all y with $\|y\| = 1$.*
(b) *X is Fréchet smooth at x iff $\xi'_x(0) = 0$.*
(c) *X is strictly convex at x iff $\lim_{\beta\to 1}(1-\beta)\xi_{x,y}(\beta) = 0$ for all y with $\|y\| = 1$.*
(d) *X is locally uniformly convex at x iff $\lim_{\beta\to 1}(1-\beta)\xi_x(\beta) = 0$.*

In the following section we focus on the properties of the ratio $\omega_x(\cdot, \cdot)$.

2. Properties of $\omega_x(\lambda, y)$

For a normed space we mean the pair $(X, \|\cdot\|)$, where X is a linear space and $\|\cdot\|$ is a norm, although sometimes we will say X instead of $(X, \|\cdot\|)$. From now on we will denote by B_X and S_X the sets $B_X = \{x \in X : \|x\| \le 1\}$ and $S_X = \{x \in$

$X : \|x\| = 1\}$. We will speak indifferently of norm one vectors or of vectors lying in S_X.

The following lemma can be found in [1] as a part of the proof that ξ is locally Lipschitz continuous.

Lemma 2.1. *Let X be a normed space and x, $y \in S_X$. Then, for any $\lambda > 0$ and $0 \le \beta < \gamma < 1$,*

$$\omega_x(\lambda, \gamma y) - \omega_x(\lambda, \beta y) \le \xi_1(\gamma) - \xi_1(\beta).$$

If we fix two norm one vectors x, y, one can realize that the modulus $\xi_{x,y}$ can be expressed in a more simpler way. Indeed, we have the following result.

Proposition 2.2. *Let X be a normed space and x, y two norm one vectors. Then, for all $\beta \in [0, 1)$,*

$$\xi_{x,y}(\beta) = \sup\{\omega_x(\lambda, \pm\beta y) : \lambda > 0\}.$$

Proof. It is enough to show that for any fixed $\lambda > 0$ and any $\gamma \le \beta$ we have that $\omega_x(\lambda, \beta y) \ge \omega_x(\lambda, \gamma y)$. We use the following result which can be found in [3, 4, 8].

Lemma 2.3. *Let X be a two-dimensional normed space and let K_1, K_2 be closed convex subsets of X with nonempty interior. If $K_1 \subset K_2$ then $r(K_1) \le r(K_2)$, where $r(K_i)$ denotes the length of the circumference of K_i, $i = 1, 2$.*

This lemma can be applied to the triangles : K_1 with vertexes the origin, $z_x(\lambda, \gamma y)$ and $(1+\lambda)x$; K_2 with vertexes the origin, $z_x(\lambda, \beta y)$ and $(1+\lambda)x$. Therefore

$$\begin{aligned} r(K_1) &= \|(1+\lambda)x\| + \|z_x(\lambda, \gamma y)\| + \|(1+\lambda)x - z_x(\lambda, \gamma y)\| \\ &\le \|(1+\lambda)x\| + \|z_x(\lambda, \beta y)\| + \|(1+\lambda)x - z_x(\lambda, \beta y)\| = r(K_2) \end{aligned}$$

Simplifying and dividing by λ, we have the desired inequality. □

Proposition 2.4. *Let X be a normed space. If x, y is a pair of norm one vectors and $0 \le \beta < \gamma < 1$, then*

$$\xi_{x,y}(\gamma) - \xi_{x,y}(\beta) \le \xi_1(\gamma) - \xi_1(\beta) \tag{2.1}$$

$$\xi_x(\gamma) - \xi_x(\beta) \le \xi_1(\gamma) - \xi_1(\beta),. \tag{2.2}$$

In particular, $\xi_{x,y}$ and ξ_x are locally Lipschitz continuous functions.

Proof. From lemma 2.1 we deduce that $\omega_x(\lambda, \gamma y) - \xi_{x,y}(\beta) \le \xi_1(\gamma) - \xi_1(\beta)$ and, by proposition 2.2, we obtain inequality (2.1) taking suprema over $\lambda > 0$. Inequality (2.2) follows similarly from inequality (2.1), taking suprema over $y \in S_X$. □

Trying to simplify the expression for $\xi_{x,y}$ obtained in proposition 2.2, one can study the behaviour of the function $\omega_x(\cdot, y)$ for fixed $x \in S_X$ and $y \in \overset{\circ}{B}_X$. At first sight one can observe the next useful result.

Proposition 2.5. *Let X be a normed space and $x \in S_X$. Then,*

$$1 \le \omega_x(\lambda) := \sup\{\omega_x(\lambda, y) : y \in \overset{\circ}{B}_X\} \le 1 + \frac{2}{\lambda}.$$

We now prove that the limit of the function $\omega_x(\lambda, y)$ when λ goes to zero always exists and we compute it.

Recall that in a normed space X and for any pair x, $y \in X \setminus \{0\}$, one can define the right derivative of the norm at x in the direction y as the limit

$$N_+(x, y) = \lim_{\lambda \searrow 0} \frac{\|x + \lambda y\| - \|x\|}{\lambda}.$$

Proposition 2.6. *Let X be any normed space $x \in S_X$, and y with $\|y\| < 1$. Then*

$$\lim_{\lambda \searrow 0} \omega_x(\lambda, y) = \frac{\|x - y\|}{1 - N_+(x, y)}.$$

In order to prove this result we need to introduce some preliminary notation.

Let us fix a normed space X, $x \in S_X$ and $y \in \overset{\circ}{B}_X$ with $y \notin \operatorname{span}\{x\}$. We denote by $z'(\lambda)$ the unique vector which lies in $\operatorname{span}\{z_x(\lambda, y)\}$ and in the ray which starts at x and has direction y, this is

$$z'(\lambda) = \{x + \mu y : \mu \geq 0\} \cap \operatorname{span}\{z_x(\lambda, y)\}.$$

We can write $z'(\lambda) = x + \mu(\lambda) y$, for some $\mu(\lambda) \geq 0$. Denote by f_λ a continuous functional on X satisfying $f_\lambda(x) = f_\lambda(z_x(\lambda, y)) = 1$. We can also write $z_x(\lambda, y) = (1 + \lambda)x + \nu(\lambda)(y - (1 + \lambda)x)$, for some $\nu(\lambda) \in [0, 1]$.

Lemma 2.7. *Let X be a normed space, $x \in S_X$ and $y \in \overset{\circ}{B}_X$ such that $y \notin \operatorname{span}\{x\}$. Then*

(a) $\lim_{\lambda \searrow 0} z_x(\lambda, y) = x$.
(b) $\lim_{\lambda \searrow 0} \mu(\lambda) = 0$.
(c) $\lim_{\lambda \searrow 0} f_\lambda(y) = N_+(x, y)$.

Proof of lemma 2.7. For proving (a) it is enough to show that $\nu(\lambda)$ tends to 0 as $\lambda \to 0$. Firstly, let us observe that the function $\varphi(t) = \|(1 + \lambda)x + t(y - (1 + \lambda)x)\|$ is a convex function satisfying $\varphi(1) = \|y\|$ and $\varphi(0) = 1 + \lambda$. Therefore $\varphi(t) \leq (1 + \lambda) + t(\|y\| - (1 + \lambda))$ for $t \in [0, 1]$. Secondly, since $z_x(\lambda, y) \in S_X$, then $\varphi(\nu(\lambda)) = 1$, this is, $1 \leq (1 + \lambda) + \nu(\lambda)(\|y\| - (1 + \lambda))$. Finally, since $\nu(\lambda) \in [0, 1]$, we obtain $\lim_{\lambda \searrow 0} \nu(\lambda) = 0$ and (a) is proved.

For proving (b), observe that $z_x(\lambda, y) = (1 + \lambda)(1 - \nu(\lambda))x + \nu(\lambda)y$. Since $z'(\lambda)$ lies in $\operatorname{span}\{z_x(\lambda, y)\}$, there exists $\alpha(\lambda) \in \mathbb{R}$ such that

$$x + \mu(\lambda)y = z'(\lambda) = \alpha(\lambda) z_x(\lambda, y),$$

from which $\alpha(\lambda) = (1 + \lambda)^{-1}(1 - \nu_x(\lambda))^{-1}$ and then

$$\mu(\lambda) = \nu(\lambda)/[(1 + \lambda)(1 - \nu(\lambda))].$$

Since $\nu(\lambda)$ converges to 0 as $\lambda \to 0$, then (b) is proved.

In order to show (c), observe that, since (b), we have

$$N_+(x, y) = \lim_{\lambda \searrow 0} \frac{\|x + \mu(\lambda)y\| - \|x\|}{\mu(\lambda)} = \lim_{\lambda \searrow 0} \frac{\|z'(\lambda)\| - \|x\|}{\mu(\lambda)}.$$

Since $z'(\lambda) \in \operatorname{span}\{z\}$, $\|z'(\lambda)\| = f_\lambda(z'(\lambda))$. Hence, since $f_\lambda(x) = \|x\|$,

$$N_+(x, y) = \lim_{\lambda \searrow 0} \frac{f_\lambda(z'(\lambda)) - f_\lambda(x)}{\mu(\lambda)} = \lim_{\lambda \searrow 0} \frac{\mu(\lambda) f_\lambda(y)}{\mu(\lambda)} = \lim_{\lambda \searrow 0} f_\lambda(y). \qquad \square$$

Proof of Proposition 2.6. First of all, if $y \in \operatorname{span}\{x\}$ then $1 - N_+(x, y) = \|x - y\|$, and since $\omega_x(\lambda, y) = 1$, this case is clear. So, let us assume that $y \notin \operatorname{span}\{x\}$ and consider $w(\lambda)$ the unique vector satisfying the conditions $f_\lambda(w(\lambda)) = 1$ and $w(\lambda) \in \{\mu((1+\lambda)x - y) : \mu \geq 0\}$. One can easily see, by comparing similar triangles, that $\omega_x(\lambda, y) = \|w(\lambda)\|$. Since $f_\lambda(w(\lambda)) = 1$, it is clear that

$$w(\lambda) = (1 + \lambda - f_\lambda(y))^{-1}[(1 + \lambda)x - y],$$

this is, $\omega_x(\lambda, y) = \frac{\|(1+\lambda)x - y\|}{1 + \lambda - f_\lambda(y)}$.

Using the continuity of the norm and the item (c) of the previous lemma we obtain the desired equality. □

Remark 2.8. However this last fact does not help to compute $\xi_{x,y}(\beta)$, since the function $\omega_x(\cdot, y)$ is neither convex nor monotonic as the following example shows.

Example 2.9. For any $0 < \varepsilon < 1/2$, let us consider in $\mathbb{R}^2$ the norm defined by $\|x\| = \max\{(1-\varepsilon)^{-1}\|x\|_\infty, \|x\|_1\}$, and the vectors $x = (1-\varepsilon, 0)$ and $y = (\varepsilon, 1-\varepsilon)$. Let us also fix $\beta \geq 1-\varepsilon$. Here is the graph of the function $\omega_x(\cdot, \beta y)$ for $\varepsilon = 0.2$ and $\beta = 0.88$.

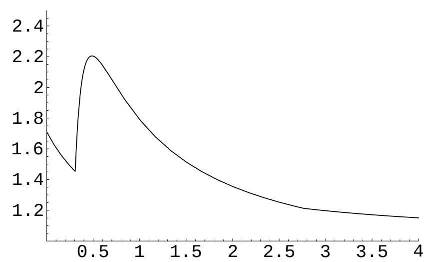

FIGURE 1

3. On Differentiability and localized squareness moduli

Throughout this section X will be a normed space endowed with the norm $\|\cdot\|$. The collection of support functionals for a norm one vector x is defined as

$$\mathcal{D}(x) = \{f \in X^* : \|f\| = 1, f(x) = \|x\| = 1\}.$$

We recall that the *modulus of smoothness* of a normed space is the function $\varrho : [0,\infty) \to \mathbb{R}^+$ defined by

$$\varrho(\beta) = \sup\Big\{\big(\,\|x+\beta y\| + \|x-\beta y\|\,\big)/2 - 1 : \|x\| = \|y\| = 1\Big\}.$$

The different localizations of this modulus are the *local modulus of smoothness*, which is defined for any $x \in S_X$ and for all $\beta \in [0,\infty)$ by

$$\varrho_x(\beta) = \sup\Big\{\big(\,\|x+\beta y\| + \|x-\beta y\|\,\big)/2 - 1 : \|y\| = 1\Big\},$$

and the *pointwise modulus of smoothness* , which is defined for any pair of norm one vectors x, y and for all $\beta \in [0,\infty)$ by

$$\varrho_{x,y}(\beta) = \big(\,\|x+\beta y\| + \|x-\beta y\|\,\big)/2 - 1.$$

Let us recall that a normed space is: *Gâteaux smooth at $x \in S_X$ in the direction $y \in S_X$* iff $\varrho_{x,y}(\beta)/\beta \to 0$ as $\beta \to 0$; *Gâteaux smooth at $x \in S_X$* iff it is Gâteaux smooth at x in every direction $y \in S_X$; *Gâteaux smooth* iff it is Gâteaux smooth at any $x \in S_X$; *Fréchet smooth at $x \in S_X$* iff $\varrho_x(\beta)/\beta \to 0$ as $\beta \to 0$; and *Fréchet smooth* iff it is Fréchet smooth at any $x \in S_X$.

For any pair of norm one vectors x, y, we define the function $\varepsilon_{x,y} : [0,\infty) \to [0,\infty)$ by the formula

$$\varepsilon_{x,y}(\beta) = \sup\left\{\frac{\|x+\beta w\| - \|x\|}{\beta} - f(w) : w \in Y, f \in \mathcal{D}_Y(x)\right\},$$

where $Y = \operatorname{span}\{x, y\}$ and $\mathcal{D}_Y(x)$ denotes the set $\{f|_Y : f \in \mathcal{D}(x)\}$. One can observe that this function is increasing and that the space is Gâteaux smooth at x in the direction y if and only if $\varepsilon_{x,y}(\beta) \to 0$ as $\beta \to 0$. Let us show the relation between $\varepsilon_{x,y}$ and the pointwise modulus of squareness $\xi_{x,y}$.

Proposition 3.1. *For every pair of norm one vectors x, y and for all $\beta \in [0,1)$*

$$\xi_{x,y}(\beta) \leq 1 + \frac{2\beta}{(1-\beta)^2}\varepsilon_{x,y}\left(\frac{2\beta}{1-\beta}\right).$$

Proof. Let us fix vectors x, $y \in S_X$, values $\lambda > 0$, $\beta \in [0,1)$ and a linear functional $f \in \mathcal{D}_Y(x)$. Then, there exists $z_0 \in [\beta y, (1+\lambda)x]$ such that $f(z_0) = 1$. One can take a vector u such that $f(u) = 0$ and $z_0 \in [u, (1+\lambda)x]$. It follows that there exists $\mu \geq 0$ such that $u = (1-\mu)(1+\lambda)x + \mu\beta y$ and, since $f(u) = 0$, that $\mu = (1+\lambda)/(1+\lambda-\beta f(y))$. Thus, one can estimate

$$\|u\| \leq \frac{(1+\lambda)\beta}{1+\lambda-\beta f(y)}(|f(y)|+1) \leq \frac{2\beta}{1-\beta}.$$

Since $z_0 \in [u, (1+\lambda)x]$, there exists $\alpha \in (0,1)$ such that $z_0 = (1-\alpha)(1+\lambda)x + \alpha u$. Using that $f(z_0) = 1$, it is easily seen that $\alpha = \lambda/(1+\lambda)$. Therefore

$$\frac{\|z_0 - x\|}{\lambda} = \frac{\|u\|}{1+\lambda} \leq \|u\| \leq \frac{2\beta}{1-\beta}, \tag{3.1}$$

$$\|z_0 - x\| = \frac{\lambda}{1+\lambda}\|u\| \leq \|u\| \leq \frac{2\beta}{1-\beta}. \tag{3.2}$$

Let us observe now that, from the definition of $\varepsilon_{x,y}$, it follows

$$\|(1+\lambda)x - z_0\| - \|\lambda x\| \leq \|x - z_0\|\,\varepsilon_{x,y}\left(\frac{\|x - z_0\|}{\lambda}\right).$$

Dividing by λ, and using (3.1) one obtains the inequality

$$\frac{\|(1+\lambda)x - z_0\|}{\lambda} \leq 1 + \frac{2\beta}{1-\beta}\varepsilon_{x,y}\left(\frac{2\beta}{1-\beta}\right). \tag{3.3}$$

Now, let us put $z = z_x(\lambda, \beta y)$ and denote by ξ_X the modulus of squareness of X. One can realize easily that $\|z - z_0\| \leq (\|z_0\| - 1)\xi_X(\beta)$, and $(\|z_0\| - 1) \leq \|x - z_0\|\,\varepsilon_{x,y}(\|x - z_0\|)$. Putting both together, and using (3.1), (3.2) and $\xi_X \leq \xi_1$, one has

$$\frac{\|z - z_0\|}{\lambda} \leq \xi_1(\beta)\left(\frac{2\beta}{1-\beta}\right)\varepsilon_{x,y}\left(\frac{2\beta}{1-\beta}\right). \tag{3.4}$$

Finally, since

$$\omega_x(\lambda, \beta y) \leq \frac{\|(1+\lambda)x - z_0\|}{\lambda} + \frac{\|z - z_0\|}{\lambda},$$

using (3.3) and (3.4) one obtains

$$\omega_x(\lambda, \beta y) \leq 1 + \frac{2\beta}{1-\beta}\varepsilon_{x,y}\left(\frac{2\beta}{1-\beta}\right)(1 + \xi_1(\beta)),$$

which, taking suprema over $\lambda > 0$, finishes the proof. ∎

Now we establish the relation between the pointwise modulus of squareness $\xi_{x,y}$, and the pointwise modulus of smoothness $\rho_{x,y}$.

Proposition 3.2. *For any pair of norm one vectors x, y and for every $\beta \in [0,1)$:*

$$\varrho_{x,y}(\beta) \leq \xi_{x,y}(\beta) - 1, \tag{3.5}$$

$$\varrho_{x}(\beta) \leq \xi_{x}(\beta) - 1, \tag{3.6}$$

Proof. Observe that the second inequality follows from the first one taking suprema over $y \in S_X$. Therefore we just have to show inequality (3.5). In order to do so, let us fix two norm one vectors x, y. For a fixed $\beta \in [0,1)$ and for $\lambda > 0$, we denote by $y_1 = y_1(\lambda, \beta y) = -(1+\lambda)\beta y$, $y_2 = y_2(\lambda, \beta y) = (1+\lambda)\beta y$, $x' = (1+\lambda)x$ and $z_i = (1-\alpha_i)x' + \alpha_i y_i$, where $\alpha_i \in [0,1]$ for $i = 1, 2$.

On one hand, let us observe that $1 = \|z_i\| \geq f(z_i)$ for any $f \in \mathcal{D}(x)$. Therefore $\alpha_i \geq \lambda/(1+\lambda - f(y_i))$. On the other hand, $\|x' - y_i\| = (1+\lambda)\|x \pm \beta y\|$. Since, for $\lambda < (1-\beta)/\beta$,

$$\frac{\alpha_i(\lambda)\|x' - y_i\|}{\lambda} = \omega_x(\lambda, \pm(1+\lambda)\beta y) \leq \xi_{x,y}((1+\lambda)\beta),$$

we have that

$$\|x' - y_1\| + \|x' - y_2\| \leq \xi_{x,y}((1+\lambda)\beta)\left(\frac{\lambda}{\alpha_1} + \frac{\lambda}{\alpha_2}\right).$$

Since $\alpha_i \geq \lambda/(1+\lambda - f(y_i))$ we deduce that

$$\begin{aligned}
\|x' - y_1\| + \|x' - y_2\| &\leq \xi_{x,y}((1+\lambda)\beta)(2 + 2\lambda - (f(y_1) + f(y_2))) \\
&= \xi_{x,y}((1+\lambda)\beta)(2 + 2\lambda) \\
&= 2\xi_{x,y}((1+\lambda)\beta)(1+\lambda),
\end{aligned}$$

and therefore

$$\begin{aligned}
\|x + \beta y\| + \|x - \beta y\| &\leq \frac{\|x' - y_1\| + \|x' - y_2\|}{(1+\lambda)} \\
&\leq 2\xi_{x,y}((1+\lambda)\beta),
\end{aligned}$$

which means that

$$\varrho_{x,y}(\beta) \leq \xi_{x,y}((1+\lambda)\beta) - 1.$$

Since it is true for $\lambda < (1-\beta)/\beta$, we can take the limit as λ tends to 0 and, by the continuity of $\xi_{x,y}$, we obtain the desired inequality. ☐

Theorem 3.3. *Let ξ_x and $\xi_{x,y}$ be the localized squareness moduli of X. Then*

(a) *X is Gâteaux smooth at $x \in S_X$ in the direction $y \in S_X$ if and only if ${\xi_{x,y}}'(0) = 0$.*
(b) *X is Gâteaux smooth at $x \in S_X$ if and only if ${\xi_{x,y}}'(0) = 0$ for all $y \in S_X$.*
(c) *X is Gâteaux smooth if and only if ${\xi_{x,y}}'(0) = 0$ for all pairs x, $y \in S_X$.*
(d) *X is Fréchet smooth at $x \in S_X$ if and only if ${\xi_x}'(0) = 0$.*
(e) *X is Fréchet smooth if and only if ${\xi_x}'(0) = 0$ for all $x \in S_X$.*

Proof. (a) Firstly, considering inequality (3.5) of proposition 3.2, it is straightforward that if $\xi'_{x,y}(0) = 0$ then $\varrho_{x,y}(\beta)/\beta$ tends to 0 when β goes to 0, i.e. the norm is differentiable at x in the direction y.

Secondly, let us assume that X is Gâteaux smooth at x in the direction y. If x and y are linearly dependent the result is trivial. Let us suppose then that x and y are linearly independent, then applying proposition 3.1 one has that

$$\frac{\xi_{x,y}(\beta) - 1}{\beta} \leq \frac{2}{(1-\beta)^2}\varepsilon_{x,y}\left(\frac{2\beta}{1-\beta}\right).$$

Since the norm of X is Gâteaux smooth at x in the direction y, we have $\varepsilon_{x,y}(t) \to 0$ as $t \to 0$. This implies that $\xi_{x,y}{}'(0) = 0$.

(b) It follows from (a) since for convex functions the existence of all directional derivatives at x implies Gâteaux smoothness at x.

(c) Evident from (b).

(d) On one hand, considering inequality (3.6), of proposition 3.2, it is clear that if $\xi_x{}'(0) = 0$ then $\varrho_x(\beta)/\beta$ tends to 0 when β goes to 0, i.e. the space is Fréchet smooth at x.

On the other hand, if we assume that X is Fréchet smooth at x, then applying proposition 3.1, for any $y \in S_X$ we have

$$\frac{\xi_{x,y}(\beta) - 1}{\beta} \leq \frac{2}{(1-\beta)^2} \varepsilon_{x,y} \left(\frac{2\beta}{1-\beta} \right).$$

Taking suprema over $y \in S_X$ we obtain

$$\frac{\xi_x(\beta) - 1}{\beta} \leq \frac{2}{(1-\beta)^2} \sup_{y \in S_X} \left\{ \varepsilon_{x,y} \left(\frac{2\beta}{1-\beta} \right) \right\}.$$

Since the space is Fréchet smooth at x, the righthand side of the inequality tends to 0 as β goes to 0. Therefore $\xi_x{}'(0) = 0$.

(e) It follows from (d). □

4. On convexity and localized squareness moduli

This section is devoted to show the relation between the behaviour of the localized moduli of squareness near one and the properties of convexity of a normed space X. In the first subsection the local modulus of squareness ξ_x is related with local uniform convexity and in the second subsection the pointwise modulus of squareness $\xi_{x,y}$ is related with strict convexity.

4.1. **Local Uniform Convexity.**

Let us fix a normed space X and $x \in S_X$. The space X is said to be *locally uniformly convex at* x if its *local modulus of convexity*

$$\delta_x(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : \|y\| = 1, \|x-y\| \geq \varepsilon \right\}$$

is strictly positive, for each $\varepsilon > 0$. The number $\varepsilon_0(x) = \sup\{\varepsilon : \delta_x(\varepsilon) = 0\}$ will be called the *characteristic of convexity at* x. Obviously, a normed space is locally uniformly convex at x if and only if $\varepsilon_0(x) = 0$.

One defines $D(x,\beta) = \mathrm{co}(\{x\} \cup \beta B_X)$ as the *drop* of βB_X with respect to the point x, and $R(x,\beta) = D(x,\beta) \setminus \beta B_X$ as the *residue*. In [1] the authors observe that X is locally uniformly convex at x iff $\operatorname{diam} R(x,\beta) \to 0$ as $\beta \to 0$.

Recall that the *radius* of a set A relative to a point x is defined by $\mathrm{rad}(x,A) = \sup_{a\in A} \|x-a\|$. It is clear that $\mathrm{diam}(A)/2 \leq \mathrm{rad}(x,A) \leq \mathrm{diam}(A)$ whenever $x \in A$. For $\|x\| = 1$ and $0 < \beta < 1$, Kadets [6] defined the set $G(x,\beta) = \{y : [y,z] \subset B_X \setminus \beta \overset{\circ}{B}_X\}$, and noted that X is locally uniformly convex at x iff $\mathrm{rad}(x, G(x,\beta)) \to 0$ as $\beta \to 1$. Moreover it is known that the function $\epsilon(x,\beta) = \mathrm{rad}(x, G(x,\beta))$ is uniformly continuous on the set $S_X \times [0,r]$ for all $r < 1$ and that ϵ is continuous at $(x,1)$ if the norm is locally uniformly convex at $x \in S_X$ (see [2, 5]).

It is also well known that the norm is locally uniformly convex at x if and only if whenever a sequence $\{x_n\}_n$ satisfies

$$\lim_{n\to\infty} 2(\|x\|^2 + \|x_n\|^2) - \|x + x_n\|^2 = 0,$$

then $lim_n \|x_n - x\| = 0$. This can be shown easily using the local modulus of convexity defined above. Finally, we say that the norm of X is locally uniformly convex if it is locally uniformly convex at all $x \in S_X$.

Lemma 4.1. *If a normed space is locally uniformly convex at $x \in S_X$, then*

$$\lim_{\lambda\to 0} \sup_{y\in \overset{\circ}{B}_X} \|x - z_x(\lambda, y)\| = 0.$$

Proof. Observe that for any $\lambda > 0$ and y with $\|y\| < 1$ all points of the segment $[(1+\lambda)x, z_x(\lambda, y)]$ different of $z_x(\lambda, y)$ are outside of the closed unit ball. Indeed, the function $f(\alpha) = \|\alpha(1+\lambda)x + (1-\alpha)z(\lambda, y)\|$ satisfies $f(0) = 1$ and there exists $\alpha_0 < 0$ such that $f(\alpha_0) = \|y\| < 1$. Since f is a convex function we obtain that $f(\alpha) > 1$ whenever $\alpha > 0$. In particular, $f(1/2) = \frac{(1+\lambda)}{2}\left\|x + \frac{z_x(\lambda,y)}{1+\lambda}\right\| > 1$.

Therefore,

$$\begin{aligned} 0 \le 2\|x\|^2 + 2\left\|\frac{z_x(\lambda, y)}{1+\lambda}\right\|^2 - \left\|x + \frac{z_x(\lambda, y)}{1+\lambda}\right\|^2 &< 2 + \frac{1}{(1+\lambda)^2} - \frac{4}{(1+\lambda)^2} \\ &= 2 - \frac{2}{(1+\lambda)^2} \end{aligned}$$

whose last term tends to 0 uniformly over all $y \in \overset{\circ}{B}_X$ and, since the space is locally uniformly convex at x, $z_x(\lambda, y)$ converges to x uniformly over $y \in \overset{\circ}{B}_X$. □

Theorem 4.2. *For any normed space X and for any $x \in S_X$, the following are equivalent :*

(a) *X is locally uniformly convex at x.*
(b) $\operatorname{diam} G(x, \beta) \to 0$ *as* $\beta \to 1$.
(c) $\operatorname{diam} R(x, \beta) \to 0$ *as* $\beta \to 1$.
(d) $\limsup_{\beta\to 1}(1-\beta)\xi_x(\beta) = 0$.
(e) $\liminf_{\beta\to 1}(1-\beta)\xi_x(\beta) = 0$.

Moreover, $\liminf_{\beta\to 1}(1-\beta)\xi_x(\beta) \ge \varepsilon_0(x)$.

Proof. The equivalence between (a), (b) and (c) are known. We claim that for all $0 \le \beta < 1$,

$$\varepsilon_0(x) - 1 + \beta \le (1-\beta)\xi_x(\beta). \tag{4.1}$$

Letting β go to 1, this inequality proves the last assertion and (e)⇒(a).

Inequality (4.1) is trivial if $\varepsilon_0(x) = 0$, so suppose that X is not locally uniformly convex at x. This means that, given any $\lambda > 0$ we can find a norm one vector y, at distance at least $\varepsilon_0(x)$ from x, and such that for all $\gamma, \mu \ge 0$

$$(1+\lambda^2)\|\gamma x + \mu y\| \ge \gamma + \mu.$$

Let us consider $x' = (1+\lambda)x$ and $y' = \beta y$, so that $\|x' - y'\| \ge \varepsilon_0(x) - \lambda - (1-\beta)$. Then $z = z_x(\lambda, y') = (1-\alpha)x' + \alpha y'$ must satisfy

$$1 = \|z\| \ge \frac{1 + \lambda - \alpha(1+\lambda-\beta)}{1+\lambda^2} \text{ and so } \alpha \ge \frac{\lambda - \lambda^2}{1+\lambda-\beta}.$$

But then
$$\frac{\|x'-z\|}{\lambda} = \frac{\alpha\|x'-y'\|}{\lambda} \geq \frac{(1-\lambda)(\varepsilon_0(x)-\lambda-(1-\beta))}{1+\lambda-\beta}.$$
Letting λ go to 0, we see that $\xi_x(\beta) \geq (\varepsilon_0(x)-1+\beta)/(1-\beta)$. And the inequality has been obtained.

It is obvious that (d) implies (e). Then, it only remains to show the implication (a)⇒(d). In order to do so, let us consider a sequence $\{\beta_n\}_n$ tending to 1, and the following sequences: $\{\delta_n\}_n$ tending to 0, $\lambda_n > 0$ and vectors $y_n \in \beta_n B_X$ in such a way
$$\xi_x(\beta_n) < \omega_x(\lambda_n, y_n) + \delta_n.$$
We have to distinguish between two cases:

(a) If $\liminf_n \lambda_n > 0$, lemma 2.5 shows that $M = \sup_n\{\omega_x(\lambda_n)\} < \infty$ and then
$$\xi_x(\beta_n) < \omega_x(\lambda_n, y_n) + \delta_n \leq \omega_x(\lambda_n) + \delta_n \leq M + \delta_n.$$
Therefore,
$$\limsup_{n\to\infty}(1-\beta_n)\xi_x(\beta_n) \leq \lim_{n\to\infty}(1-\beta_n)(M+\delta_n) = 0.$$

(b) If $\liminf_n \lambda_n = 0$, we can assume, passing to a subsequence, that $\lambda_n \to 0$. If it is necessary we can choose y'_n in such a way $\|y'_n\| = \beta_n$ and $y'_n \in [y_n, (1+\lambda_n)x] \cap G(z_x(\lambda_n, y_n), \beta_n))$. Let us write $z_n = z_x(\lambda_n, y_n) = \alpha_n(1+\lambda_n)x + (1-\alpha_n)y'_n$. Then, $1 = \|z_n\| \leq \alpha_n(1+\lambda_n) + (1-\alpha_n)\beta_n$, from which it follows that $(1-\alpha_n)(1-\beta_n) \leq \alpha_n\lambda_n$ and
$$\begin{aligned}(1-\alpha_n)(1-\beta_n)\omega_x(\lambda_n, y'_n) &\leq \alpha_n\|(1+\lambda_n)x - z_n\| = (1-\alpha_n)\|y'_n - z_n\| \\ &\leq (1-\alpha_n)rad(z_n, G(z_n,\beta)).\end{aligned}$$
This is, $(1-\beta_n)\omega_x(\lambda_n, y_n) = (1-\beta_n)\omega_x(\lambda_n, y'_n) \leq \epsilon(z_n, \beta_n)$. Lemma 4.1 tells that z_n tends to x and therefore, since $\epsilon(\cdot,\cdot)$ is continuous at $(x, 1)$ we have that
$$\begin{aligned}\limsup_{n\to\infty}(1-\beta_n)\xi_x(\beta_n) &\leq \limsup_{n\to\infty}(1-\beta_n)\omega_x(\lambda_n, y_n) \\ &\leq \lim_{n\to\infty}\epsilon(z_n, \beta_n) = \epsilon(x, 1) = 0,\end{aligned}$$
which is what we want to show. □

From this proposition arises a new characterization of what a locally uniformly convex space is.

Corollary 4.3. *For any normed space X the following are equivalent :*

(a) *X is locally uniformly convex.*
(b) $\operatorname{diam} G(x,\beta) \to 0$ *as* $\beta \to 1$ *for all* $x \in S_X$*.*
(c) $\operatorname{diam} R(x,\beta) \to 0$ *as* $\beta \to 1$ *for all* $x \in S_X$*.*
(d) $\limsup_{\beta\to 1}(1-\beta)\xi_x(\beta) = 0$ *for all* $x \in S_X$*.*
(e) $\liminf_{\beta\to 1}(1-\beta)\xi_x(\beta) = 0$ *for all* $x \in S_X$*.*

4.2. **Strict Convexity.**

Let X be a normed space and x, $w \in S_X$. The norm of X is said to be *strictly convex at x in the direction w* if there is no proper segment included on the unit sphere starting at x with direction w. Similarly, it is said to be *strictly convex at x* if there is no proper segment included in the unit sphere starting at x in any direction. X is said to be *strictly convex* if it is strictly convex at all its norm one vectors. We define the number $\varepsilon_0(x, w)$ as the supremum of all those $\varepsilon > 0$ such that the segment $[x, x+\varepsilon w]$ or $[x, x-\varepsilon w]$ is included on the unit sphere. We also define the set $C_x^w = \{y \in S_X : \exists \lambda \in \mathbb{R},\, y = x + \lambda w\}$.

Proposition 4.4. *Let X be a normed space and x, w two norm one vectors. If $\liminf_{\beta\to 1}(1-\beta)\xi_{x,y}(\beta) = 0$ for all $y \in C_x^w$, then X is strictly convex at x in the direction w. Moreover, $\sup_{y\in C_x^w} \liminf_{\beta\to 1}(1-\beta)\xi_{x,y}(\beta) \geq \varepsilon_0(x,w)$.*

Proof. Let us assume that X is not strictly convex at x in the direction w. It means that $\varepsilon_0(x,w) > 0$, and that for any $\varepsilon_0(x,w) > \delta > 0$ there exits $y \in C_x^w$ such that $\|y - x\| \geq \varepsilon_0(x,w) - \delta$. Let us denote $z = z_x(\lambda, \beta y)$. There exists $\alpha \in [0,1]$ such that $z = (1-\alpha)(1+\lambda)x + \alpha\beta y$. Let us compute α. Fix $f \in \mathcal{D}(x)$ such that $f([x,y]) = 1$. We have $1 = f(z) = (1-\alpha)(1+\lambda) + \alpha\beta$. Therefore $\alpha = \lambda/(1+\lambda-\beta)$.

On the other hand, $\|(1+\lambda)x - \beta y\| \geq \|x-y\| - \|\lambda x + (1-\beta)y\| \geq \varepsilon_0(x,w) - \delta - \lambda - (1-\beta)$. Therefore,

$$\xi_{x,y}(\beta) \geq \omega_x(\lambda, \beta y) = \alpha\frac{\|(1+\lambda)x - \beta y\|}{\lambda} \geq \frac{\varepsilon_0(x,w) - \delta - \lambda - (1-\beta)}{1+\lambda-\beta}.$$

Taking the limit as $\lambda \to 0$, we obtain $(1-\beta)\xi_{x,y}(\beta) \geq \varepsilon_0(x,w) - \delta - (1-\beta)$. Therefore

$$\liminf_{\beta\to 0}(1-\beta)\xi_{x,y}(\beta) \geq \varepsilon_0(x,w) - \delta.$$

This implies that $\liminf_{\beta\to 0}(1-\beta)\xi_{x,y}(\beta) > 0$, which shows the first and, whenever $\varepsilon_0(x,w) > 0$, second assertion of the theorem. The proof is finished, since the second assertion is clear when $\varepsilon_0(x,w) = 0$. □

Theorem 4.5. *For any normed space X and for any $x \in S_X$ the following are equivalent :*

(a) *X is strictly convex at x.*
(b) $\limsup_{\beta\to 1}(1-\beta)\xi_{x,y}(\beta) = 0$*, for all* $y \in S_X$.
(c) $\liminf_{\beta\to 1}(1-\beta)\xi_{x,y}(\beta) = 0$*, for all* $y \in S_X$.

Proof. The implication (b)⇒(c) is evident. The implication (c)⇒(a) follows from proposition 4.4. In order to see (a)⇒(b), let us fix $y \in S_X$, consider a $\{\beta_n\}_n$ tending to 1, and the following sequences: $\{\delta_n\}_n$ tending to 0, $\lambda_n > 0$ and vectors $y_n = \gamma_n y \in \beta_n B_X$ in such a way

$$\xi_{x,y}(\beta_n) < \omega_x(\lambda_n, y_n) + \delta_n.$$

We have to distinguish between two cases:

(a) If $\liminf_n \lambda_n > 0$, lemma 2.5 shows that $M = \sup_n\{\omega_x(\lambda_n)\} < \infty$ and then

$$\xi_{x,y}(\beta_n) < \omega_x(\lambda_n, y_n) + \delta_n \leq \omega_x(\lambda_n) + \delta_n \leq M + \delta_n.$$

Therefore,

$$\limsup_{n\to\infty}(1-\beta_n)\xi_{x,y}(\beta_n) \leq \lim_{n\to\infty}(1-\beta_n)(M+\delta_n) = 0.$$

(b) If $\liminf_n \lambda_n = 0$, we can assume, passing to a subsequence, that $\lambda_n \to 0$. If it is necessary we can choose y_n' in such a way $\|y_n'\| = \beta_n$ and $y_n' \in [y_n, (1+\lambda_n)x] \cap G_Y(z_x(\lambda_n, y_n), \beta_n))$, where $Y = \operatorname{span}\{x,y\}$. Let us write $z_n = z_x(\lambda_n, y_n) = \alpha_n(1+\lambda_n)x + (1-\alpha_n)y_n'$. Then, $1 = \|z_n\| \leq \alpha_n(1+\lambda_n) + (1-\alpha_n)\beta_n$, from which it follows that $(1-\alpha_n)(1-\beta_n) \leq \alpha_n\lambda_n$ and

$$\begin{aligned}(1-\alpha_n)(1-\beta_n)\omega_x(\lambda_n, y_n') &\leq \alpha_n\|(1+\lambda_n)x - z_n\| = (1-\alpha_n)\|y_n' - z_n\| \\ &\leq (1-\alpha_n)rad(z_n, G_Y(z_n, \beta)).\end{aligned}$$

This is, $(1-\beta_n)\omega_x(\lambda_n, y_n) = (1-\beta_n)\omega_x(\lambda_n, y_n') \leq \epsilon_Y(z_n, \beta_n)$. Since Y is locally uniformly convex at x, lemma 4.1 tells that z_n tends to x and therefore, since $\epsilon_Y(\cdot,\cdot)$ is continuous at $(x, 1)$, we have that

$$\begin{aligned}\limsup_{n\to\infty}(1-\beta_n)\xi_{x,y}(\beta_n) &\leq \limsup_{n\to\infty}(1-\beta_n)\omega_x(\lambda_n, y_n)\\ &\leq \lim_{n\to\infty}\epsilon_Y(z_n,\beta_n) = \epsilon_Y(x,1) = 0,\end{aligned}$$

which is what we want to show. □

From this theorem one can easily deduce the following one.

Theorem 4.6. *For any normed space X the following are equivalent :*

(a) *X is strictly convex.*
(b) $\limsup_{\beta\to 1}(1-\beta)\xi_{x,y}(\beta) = 0$*, for all $x, y \in S_X$.*
(c) $\liminf_{\beta\to 1}(1-\beta)\xi_{x,y}(\beta) = 0$*, for all $x, y \in S_X$.*

Acknowledgment. The author wishes to express his gratitude to Professor R. Deville for helpful comments and suggestions which improved the content of this paper.

References

1. C. Benítez, K. Przesławski, and D. Yost, *A universal modulus for normed spaces*, Studia Math. **127** (1998), no. 1, 21–46. MR MR1488142 (99i:46005)
2. C. Bessaga and A. Pełczyński, *Selected topics in infinite-dimensional topology*, PWN—Polish Scientific Publishers, Warsaw, 1975, Monografie Matematyczne, Tom 58. [Mathematical Monographs, Vol. 58]. MR MR0478168 (57 #17657)
3. H. Busemann, *The geometry of geodesics*, Academic Press Inc., New York, N. Y., 1955. MR MR0075623 (17,779a)
4. J. Gao and K. Lau, *On two classes of Banach spaces with uniform normal structure*, Studia Math. **99** (1991), no. 1, 41–56. MR MR1120738 (92h:46017)
5. A. J. Guirao, *Topological clasification of Banach spaces*, Published by University of Murcia.
6. M. I. Kadec, *A proof of the topological equivalence of all separable infinite-dimensional Banach spaces*, Funkcional. Anal. i Priložen. **1** (1967), 61–70. MR MR0209804 (35 #700)
7. K. Przesławski and D. Yost, *Lipschitz retracts, selectors, and extensions*, Michigan Math. J. **42** (1995), no. 3, 555–571. MR MR1357625 (97g:54034)
8. J. J. Schäffer, *Geometry of spheres in normed spaces*, Marcel Dekker Inc., New York, 1976, Lecture Notes in Pure and Applied Mathematics, No. 20. MR MR0467256 (57 #7120)
9. I. Şerb, *On the behaviour of the tangential modulus of a Banach space. I*, Rev. Anal. Numér. Théor. Approx. **24** (1995), no. 1-2, 241–248. MR MR1608428 (99d:46018)
10. ———, *On the behaviour of the tangential modulus of a Banach space. II*, Mathematica **38(61)** (1996), no. 1-2, 199–207. MR MR1606805 (99d:46019)
11. ———, *A Day-Nordlander theorem for the tangential modulus of a normed space*, J. Math. Anal. Appl. **209** (1997), no. 2, 381–391. MR MR1474615 (98i:46015)

Departamento de Matemáticas, Universidad de Murcia, 30100 Espinardo (Murcia), Spain
E-mail address: ajguirao@um.es