Optimal divisions of a convex body

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Abstract

For a convex body C in $\mathbb{R}^d$ and a division of C into $C_1, ..., C_n$ convex subsets, we can consider $max\{F(C_1), ..., F(C_n)\}$ (respectively, $min\{F(C_1), ..., F(C_n)\}$), where F represents one of these classical geometric functionals: the diameter, the minimal width, the inradius and the circumradius.

In this work we will study the divisions of C minimizing (respectively, maximizing) the previous value. In particular, we will treat the existence, uniqueness and balancing behaviour of these *optimal divisions*, and algorithms leading to them, as well as bounds for the corresponding optimal values.

1. Motivation

- *Conway's fried potato problem* **[CFG]**: Division of a convex set C of $\mathbb{R}^d$ into n subsets by $n-1$ successive hyperplane cuts (just one subset is divided in each cut) minimizing the largest inradius of the subsets.

An optimal division is given by $n-1$ parallel (equispaced) cuts between the hyperplanes providing the minimal width of C^r (a certain rounded body of C) **[BB]**.

- *Maximum relative diameter* **[MPS]**, **[CSS]**, **[CS]**, **[CG]**: Divisions of a n-rotationally symmetric (and general) planar convex body into n subsets minimizing the largest diameter of the subsets.

⇝ These are min-Max type problems for a classical geometric functional and divisions of a convex body

2. Our problem

$C \subset \mathbb{R}^d$ convex body

Divisions of C into n subsets by means of $n-1$ successive hyperplane cuts (⇒ the subsets are convex)

Classical geometric functionals:
- Diameter
- Minimal width
- Inradius
- Circumradius

In addition to the min-Max type problems, we can also study the Max-min type problems

Our project:

- min-Max type problems: Studying the optimal divisions of C into n subsets minimizing the largest value of each of those functionals on the subsets

- Max-min type problems: Studying the optimal divisions of C into n subsets maximizing the smallest value of each of those functionals on the subsets

Questions to study
- Existence of optimal divisions
- Uniqueness of optimal divisions
- Balance behaviour of optimals division
- Algorithms leading to an optimal division
- Optimal values

3. Existence of optimal divisions

We have existence for almost all the problems

• Max-min problem for the diameter D (the optimal value is $D(C)$):

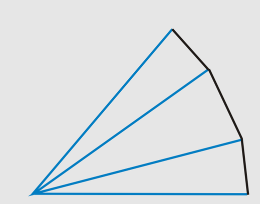
- For $d \geqslant 3$, there is always a solution:

We can consider the division given by $n-1$ hyperplanes containing a diameter segment of C (this cannot be done if $d=2$).

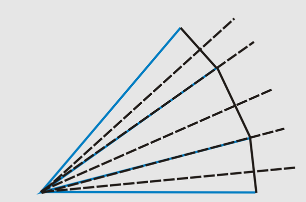
- For $d=2$, it depends on the diameter segments of C:

$$\text{Existence of a solution} \Leftrightarrow n \leq 2I + B - \delta_C,$$

where $I \equiv$ interior diameter segments, $B \equiv$ boundary diameter segments, and $\delta_C \in \{0,1\}$ correction factor (each subset must contain a diameter segment, so we need enough diameter segments)



$I=2$, $F=2$, $\delta_C=0$

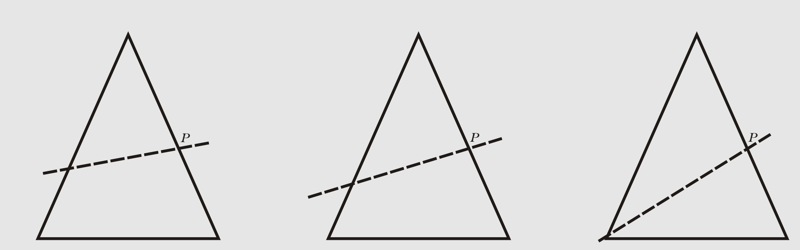


An optimal division into $n=6$ subsets

4. Uniqueness of optimal divisions

We do not have uniqueness in general

• Max-min problem for the minimal width:



For an isosceles triangle, different lines passing through P provide optimal divisions for $n=2$

• Examples also for the min-Max problem and for the Max-min problem for the diameter, for the min-Max problem for the minimal width, and for the min-Max problem for the inradius **[BB]**

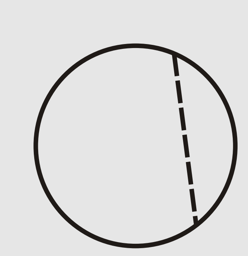
5. Balancing behaviour of optimal divisions

A division of C is balanced if all the subsets have the same value for the functional

• There are always balanced optimal divisions

• For the min-Max problem for the minimal width, all optimal divisions are balanced (Bang's Theorem)

• For the min-Max problem for the diameter, there are optimal divisions which are not balanced:



Non-balanced optimal division of the circle, for $n=2$

6. Optimal values

General fact: The corresponding optimal value is always less than or equal to $F(C)$

• Max-min problem for the diameter D:

The optimal value is $D(C)$. Not always attained ($d=2$).

If P_0 is a hyperplane containing a diameter segment, we can take a sequence of divisions by parallels hyperplanes approaching P_0

• min-Max problem for the minimal width w:

Bang's Theorem ⇒ the optimal value is $\frac{1}{n} w(C)$

• min-Max problem for the inradius **[BB]**:

The optimal value is $\rho > 0$ such that $w(C^\rho) = 2n\rho$

• min-Max problem for the diameter D:

$$\frac{1}{n} D(C) < \text{optimal value} \leq C_0,$$

where C_0 is a bound given by a particular division of C (a mesh-type division of a hypercube containing C)

7. Algorithms leading to optimal divisions

For polygons and only when the optimal value is known

• Max-min problem for the diameter: Linear algorithm (rotating calipers **[T]**)

• min-Max problem for the inradius: Quadratic algorithm (medial axis)

• min-Max problem for the minimal width: Linear algorithm

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