On the optimal constants in the two-sided Stechkin inequalities

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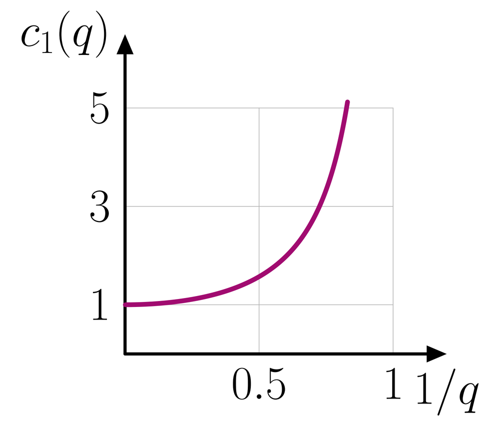
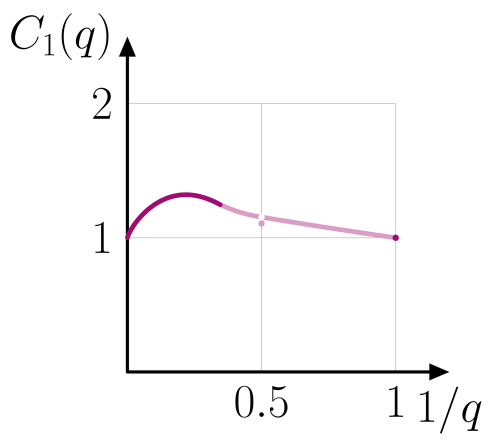
Introduction

We improve the best known constants in some equivalences of quasinorms. These inequalities serve a central task of nonlinear approximation, namely the characterization of elements of infinite-dimensional Hilbert spaces which can be approximated by n-sparse elements with a given error decay. Throughout this poster, sequences $(a_n)_{n\in\mathbb{N}}$ and functions $f:(0,\infty)\to[0,\infty)$ are assumed to be monotonically decreasing, and $1\le q\le\infty$. The constants $c_1(q),\dots,\bar{C}_{1,\infty}(q)$ are supposed to be minimal in their respective positions. *But what are these minimal values?*

Strong discrete Stechkin inequality

$$\frac{1}{c_1(q)}\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=n}^{\infty}a_k^q\right)^{\frac{1}{q}}\le\sum_{n=1}^{\infty}a_n\le C_1(q)\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=n}^{\infty}a_k^q\right)^{\frac{1}{q}},$$





Bennett 1988

$$c_1(q)=\frac{\pi}{q\sin(\frac{\pi}{q})}$$

Jahn, Ullrich 2020+ | Levin, Stechkin 1948 | Gao 2011 | De Bruijn 1958

$$C_1(q)\begin{cases}\le\left(\frac{e\ln(2)}{\sqrt{2}}\right)^{1-\frac{1}{q}}, & 1\le q\le\frac{2+\ln(2)}{2-\ln(2)},\\ \le 2\left(2^{\frac{q}{q-1}}-1\right)^{-\frac{q-1}{q}}, & \frac{2+\ln(2)}{2-\ln(2)}<q\le q_0,\\ =(q-1)^{\frac{1}{q}} & q_0<q\le\infty\end{cases}$$

where $q_0\approx 2.8855$ is a solution of the equation

$$2^{\frac{1}{q-1}}\left((q-1)^{1-\frac{1}{q}}-(q-1)\right)-\left(1+\frac{3-q}{2}\right)^{\frac{q}{q-1}}=0.$$

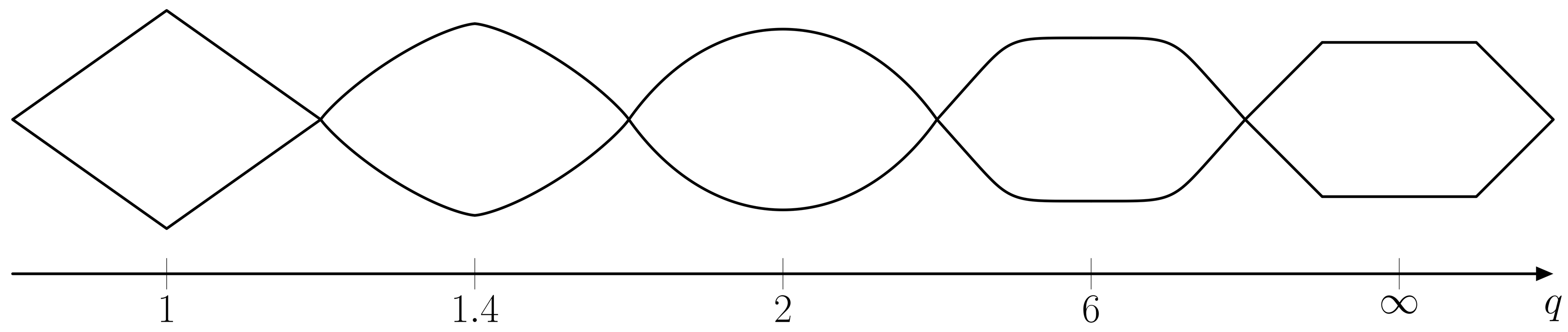
Also,

$$C_1(2)\approx 1.1064957714.$$

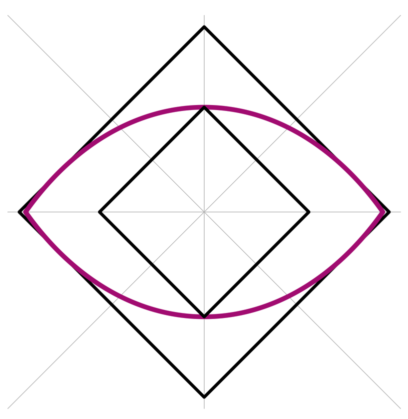
The main ingredient of our proof an adaption of Stechkin's 1951 proof for $C_1(2)\le\frac{2}{\sqrt{3}}$ (now using Hölder's inequality) to obtain upper bounds on $C_1(q)$ in terms of an auxiliary sequence, which we have to choose wisely.

Geometric explanation

If we truncate the $\sum^{\infty}$ to $\sum^{N}$, the strong discrete Stechkin inequality turns into finding the optimal scaling factors for optimal containment of the ℓ_1^N unit ball and another convex body in $\mathbb{R}^N$, whose shape varies with q.



Here is the optimal containment situation for $N=q=2$. The size of the smaller copy of the ℓ_1^N unit ball can be increased if we pay attention to the monotonicity. However, the size of the larger copy cannot be decreased. This means that monotonicity is important for $c_1(q)$ but not for $C_1(q)$.



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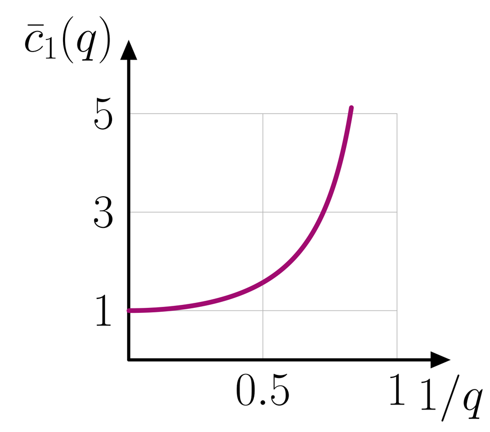
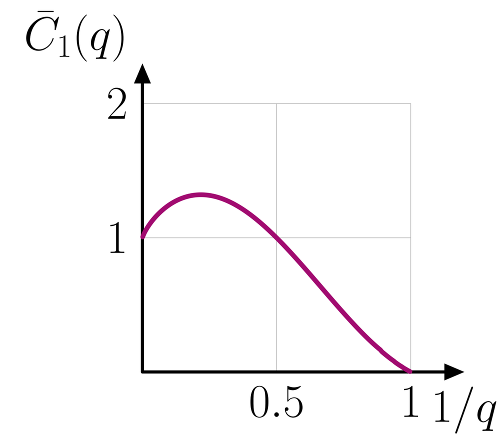
Strong continuous Stechkin inequality

$$\frac{1}{\bar{c}_1(q)}\int_0^{\infty}\left(\frac{1}{t}\int_t^{\infty}f(s)^q\mathrm{d}s\right)^{\frac{1}{q}}\mathrm{d}t\le\int_0^{\infty}f(t)\mathrm{d}t\le\bar{C}_1(q)\int_0^{\infty}\left(\frac{1}{t}\int_t^{\infty}f(s)^q\mathrm{d}s\right)^{\frac{1}{q}}\mathrm{d}t$$





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$$\bar{c}_1(q)=\frac{\pi}{q\sin(\frac{\pi}{q})}$$

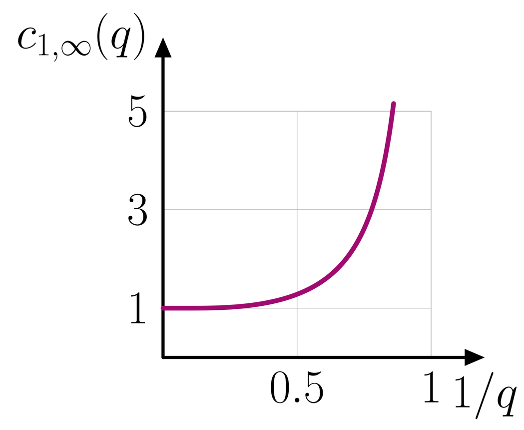
equality case: $f(t)=\frac{1}{T}\chi_{(0,T)}(t)$

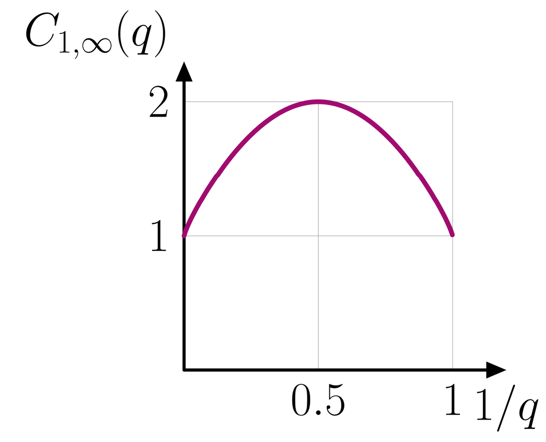
Hardy, Littlewood, Pólya 1934

$$\bar{C}_1(q)=(q-1)^{\frac{1}{q}}$$

Weak discrete Stechkin inequality

$$\frac{1}{c_{1,\infty}(q)}\sup_{n\in\mathbb{N}}n\left(\frac{1}{n}\sum_{k=n}^{\infty}a_k^q\right)^{\frac{1}{q}}\le\sup_{n\in\mathbb{N}}na_n\le C_{1,\infty}(q)\sup_{n\in\mathbb{N}}n\left(\frac{1}{n}\sum_{k=n}^{\infty}a_k^q\right)^{\frac{1}{q}}$$





Jahn, Ullrich 2020+

$$c_{1,\infty}=\zeta(q)^{\frac{1}{q}}$$

equality case: $a_n=\frac{1}{n}$

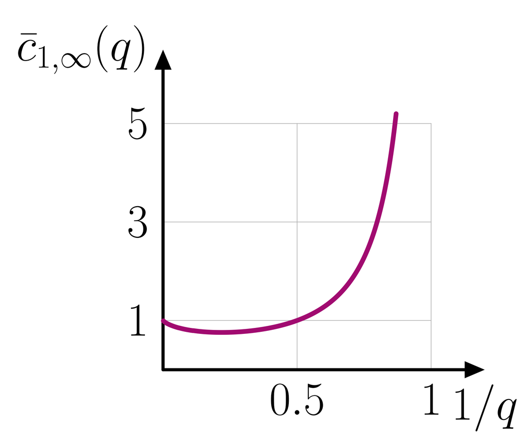
Jahn, Ullrich 2020+

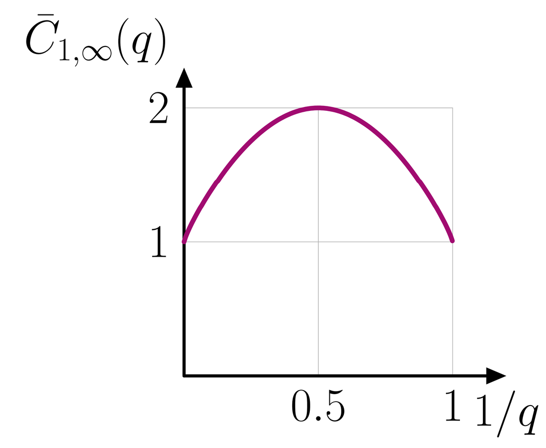
$$C_{1,\infty}=\left(\frac{1}{q}\right)^{-\frac{1}{q}}\left(1-\frac{1}{q}\right)^{-(1-\frac{1}{q})}$$

asymptotic equality case: $a_n=\frac{1}{N}\chi_{\{1,\dots,N\}}(n)$

Weak continuous Stechkin inequality

$$\frac{1}{\bar{c}_{1,\infty}(q)}\sup_{t>0}t\left(\frac{1}{t}\int_t^{\infty}f(s)^q\mathrm{d}s\right)^{\frac{1}{q}}\le\sup_{t>0}tf(t)\le\bar{C}_{1,\infty}(q)\sup_{t>0}t\left(\frac{1}{t}\int_t^{\infty}f(s)^q\mathrm{d}s\right)^{\frac{1}{q}}$$





Jahn, Ullrich 2020+

$$\bar{c}_{1,\infty}=(q-1)^{-\frac{1}{q}}$$

equality case: $f(t)=\frac{1}{t}$

Jahn, Ullrich 2020+

$$\bar{C}_{1,\infty}=\left(\frac{1}{q}\right)^{-\frac{1}{q}}\left(1-\frac{1}{q}\right)^{-(1-\frac{1}{q})}$$

equality case: $f(t)=\frac{1}{T}\chi_{(0,T)}(t)$

Acknowledgments

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