A discrete isoperimetric inequality

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Abstract

The isoperimetric inequality is one of the oldest and most outstanding results in mathematics, and can be summarized by saying that the Euclidean balls minimize the surface area measure $\mathrm{S}(\cdot)$ (Minkowski content) among those compact convex sets with prescribed positive volume $\mathrm{vol}(\cdot)$ (Lebesgue measure). There exist many different versions and extensions of this result, which have led to remarkable consequences in many branches of mathematics. It admits the following "*neighbourhood* form": for any compact convex set $K \subset \mathbb{R}^n$, and all $t \geq 0$,

$$\mathrm{vol}(K + tB_n) \geq \mathrm{vol}(rB_n + tB_n),$$

where $r > 0$ is such that $\mathrm{vol}(rB_n) = \mathrm{vol}(K)$ and B_n denotes the (closed) Euclidean unit ball.

In this poster we discuss and show a discrete analogue of the isoperimetric inequality in the aforementioned form for the *lattice point enumerator* $\mathrm{G}_n(K) = |K \cap \mathbb{Z}^n|$ of a bounded subset $K \subset \mathbb{R}^n$: we determine sets minimizing the functional $\mathrm{G}_n\big(K + t[-1,1]^n\big)$, for any $t \geq 0$, among those bounded sets K with given positive lattice point enumerator. We also show that this new discrete inequality implies the classical result for compact sets.

1 The classical isoperimetric inequality

The isoperimetric inequality in its classical form dates back to antiquity, and states that circles are the only closed plane curves maximizing the enclosed area for a prescribed length, or, alternatively, the only ones minimizing the length for a prescribed enclosed area. This can be succinctly expressed as

$$L^2 \geq 4\pi A, \tag{1}$$

where L is the length of the curve and A is the enclosed area.

This result was eventually generalized to arbitrary dimension in the 19th century, starting with the works of Steiner who exploited his then recently developed concept of *Steiner symmetrization* to show that if an optimal solution existed, it had to be the n-dimensional ball. The proof of existence was then completed by other authors. Its form for convex bodies in $\mathbb{R}^n$ can be stated by saying that the volume $\mathrm{vol}(\cdot)$ and surface area $\mathrm{S}(\cdot)$ (Minkowski content) of any n-dimensional convex body K satisfy

$$\left(\frac{\mathrm{S}(K)}{\mathrm{S}(B_n)}\right)^n \geq \left(\frac{\mathrm{vol}(K)}{\mathrm{vol}(B_n)}\right)^{n-1}, \tag{2}$$

where B_n denotes the Euclidean (closed) unit ball. Another equivalent formulation, analogous to (2) but in the spirit of (1), on account of the formula for the surface area of B_n, is

$$\mathrm{S}(K) \geq n\mathrm{vol}(K)^{(n-1)/n}\mathrm{vol}(B_n)^{1/n}. \tag{3}$$

During the final years of the century, works by both Hermann Brunn and Hermann Minkowski produced the currently known as Brunn-Minkowski theorem, a powerful inequality that can yield the isoperimetric inequality (2) with a fairly straightforward proof.

2 Discrete isoperimetric inequalities

In order to discretize the isoperimetric inequality we may consider the following "*neighbourhood* form" (see e.g. [5, Proposition 14.2.1]): for any n-dimensional convex body $K \subset \mathbb{R}^n$, and all $t \geq 0$, we have

$$\mathrm{vol}(K + tB_n) \geq \mathrm{vol}(rB_n + tB_n), \tag{4}$$

where rB_n, $r > 0$, is a ball of the same volume as K. In fact, by subtracting $\mathrm{vol}(K) = \mathrm{vol}(rB_n)$ and dividing by t in both sides of (4), and taking limits as $t \to 0^+$, one immediately gets (2) from (4).

The neighbourhood $K + tB_n$, $t \geq 0$, of the n-dimensional convex body K coincides with the set of all points of $\mathbb{R}^n$ having (Euclidean) distance from K at most t. Exchanging the role of the unit ball B_n in (4) by another (n-dimensional) convex body $E \subset \mathbb{R}^n$, i.e., changing the involved "distance", one is naturally led to the fact

$$\mathrm{vol}(K + tE) \geq \mathrm{vol}(rE + tE) \tag{5}$$

for all $t \geq 0$, where again $r > 0$ is such that rE has the same volume as K. Thus, the advantage of using the volume of a neighbourhood of K, instead of its surface area, is that it can be extended to other spaces in which the latter notion makes no sense; it just suffices to consider a metric and a measure on the given space. Relevant examples of spaces in which isoperimetric inequalities in this form hold are the unit sphere, the *Gauss space* or the n-dimensional discrete cube $\{0,1\}^n$ (see e.g. [5, Section 14.2]). Similar inequalities also hold in discrete metric spaces, in the settings of combinatorics and graph theory (for which we refer the reader to [2]).

2.1 An inequality by Radcliffe and Veomett

Recently, in [6], a discrete isoperimetric inequality has been derived for the integer lattice $\mathbb{Z}^n$ endowed with the L_∞ norm and the cardinality measure $|\cdot|$ (see also [1] for a related result in the case of the L_1 norm, where the author uses a method based on the solvability of a certain finite difference equations problem). To this aim, a suitable extension of *lattice cubes* (i.e., the intersection of cubes $[a,b]^n$ with $\mathbb{Z}^n$) is considered: we will call these sets *extended lattice cubes*, which will be denoted by $\mathcal{I}_r$ (see Definition 2), for any $r \in \mathbb{N}$. In fact, when $r = m^n$ for some $m \in \mathbb{N}$, $\mathcal{I}_r$ is indeed a *lattice cube*. Thus, the authors show that sets $\mathcal{I}_r$ minimize the cardinality of the suitable neighbourhood among all non-empty sets of fixed cardinality r. More precisely, [6, Theorem 1], combined with [6, Lemma 1], leads to the following discrete analogue of (5):

Theorem 1 (Radcliffe and Veomett, [6]) *Let $A \subset \mathbb{Z}^n$ be a non-empty finite set and let $r \in \mathbb{N}$ be such that $|\mathcal{I}_r| = |A|$. Then*

$$\Big|A + \big((m[-1,1]^n) \cap \mathbb{Z}^n\big)\Big| \geq \Big|\mathcal{I}_r + \big((m[-1,1]^n) \cap \mathbb{Z}^n\big)\Big| \tag{6}$$

for all $m \in \mathbb{N}$.

Indeed, the authors prove the above theorem for $m = 1$ in [6, Theorem 1], and a simple inductive argument using [6, Lemma 1] allows one to obtain the result for arbitrary $m \in \mathbb{N}$.

3 Defining optimal sets

Given a vector $u = (u_1 \ldots, u_n) \in \mathbb{Z}^n$ and fixing $i_u \in \{1, \ldots, n\}$, we will write

$$u' = (u_1 \ldots, u_{i_u - 1}, u_{i_u + 1}, \ldots, u_n) \in \mathbb{Z}^{n-1}.$$

With this notation, in [6] the following well-order $\prec$ in $\mathbb{Z}^n$ is defined:

Definition 1 *If $n = 1$ we define the order $\prec$ given by*

$$0 \prec 1 \prec -1 \prec 2 \prec -2 \prec \cdots \prec m \prec -m \prec \ldots$$

For $n \geq 2$ we set, for $w = (w_1, \ldots, w_n) \in \mathbb{Z}^n$,

$$M(w) = \max_{\prec}\{w_i : i = 1, \ldots, n\} \quad \text{and} \quad i_w = \min\big\{i : w_i = M(w)\big\},$$

and we define $\prec$ recursively as follows: for any $u, v \in \mathbb{Z}^n$ with $u \neq v$,

1. if $M(u) \prec M(v)$ then $u \prec v$;

2. if $M(u) = M(v)$ then $u \prec v$ if either $i_v < i_u$ or ($i_v = i_u$ and) $u' \prec v'$.

Moreover, we write $u \preceq v$ if either $u \prec v$ or $u = v$.

This order will allow us to define the extended lattice cube $\mathcal{I}_r$ of r points as the initial segment in $\mathbb{Z}^n$ with respect to $\prec$. To define the sets $\mathcal{C}_r$, which will be referred to as *extended cubes*, first we need the following definition, which can be seen as a particular case of the family of *weakly unconditional sets*, first introduced in [7] (we refer the reader to this work for further properties and relations of them with certain Brunn-Minkowski type inequalities): for any non-empty finite set $A \subset \mathbb{R}^n$, we write

$$\mathcal{C}_A = \big\{(\lambda_1 x_1, \ldots, \lambda_n x_n) \in \mathbb{R}^n : (x_1, \ldots, x_n) \in A, \lambda_i \in [0,1] \text{ for } i = 1, \ldots, n\big\}$$

(see Figure 1).

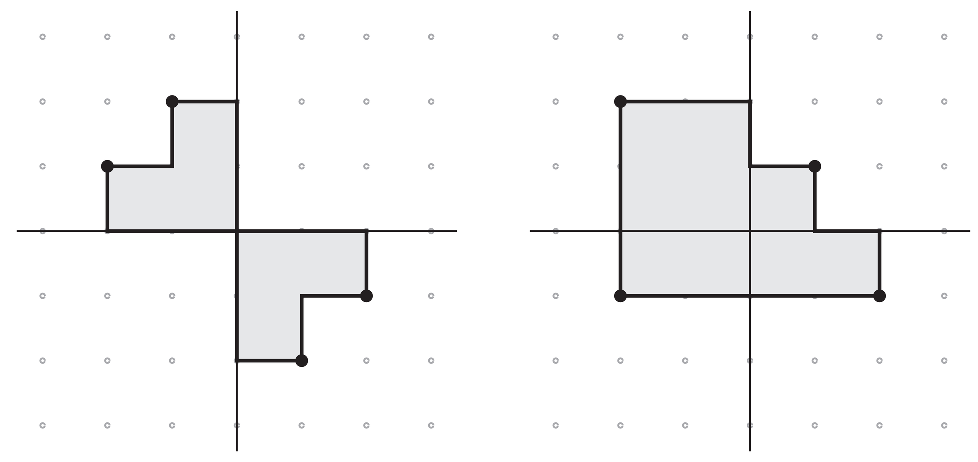

Figure 1: Sets $\mathcal{C}_A \subset \mathbb{R}^2$ for different finite sets $A \subset \mathbb{Z}^2$.

Definition 2 *Let $r \in \mathbb{N}$. By $\mathcal{I}_r$ we denote the initial segment in $(\mathbb{Z}^n, \prec)$ of length r, i.e., the set of the first r points with respect to the order $\prec$ in $\mathbb{Z}^n$ (see Figure 2, left). Moreover, by $\mathcal{C}_r$ we denote the set given by $\mathcal{C}_r := \mathcal{C}_{\mathcal{I}_r}$ (see Figure 2, right).*

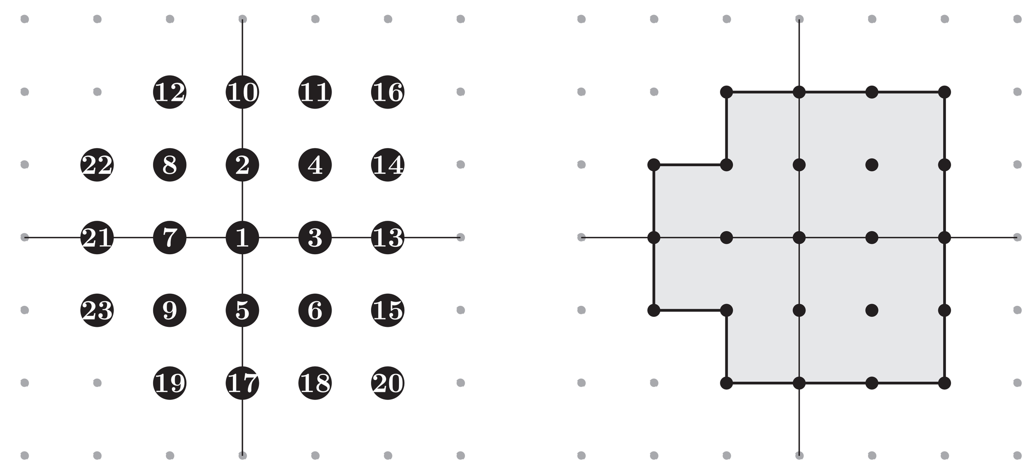

Figure 2: The extended lattice cube $\mathcal{I}_{23}$ in $\mathbb{Z}^2$ (left) and the corresponding extended cube $\mathcal{C}_{23}$ in $\mathbb{R}^2$ (right).

We note that if $r = m^n$ for some $m \in \mathbb{N}$ then $\mathcal{I}_r$ is indeed a lattice cube. More precisely, $\mathcal{I}_r = \{-m/2+1, -m/2+2, \ldots, m/2-1, m/2\}^n$ if m is even and $\mathcal{I}_r = \big\{-(m-1)/2, -(m-1)/2+1, \ldots, (m-1)/2, (m-1)/2\big\}^n$ if m is odd (cf. Figure 2, left). This further implies that $\mathcal{C}_r$ is a cube whenever $r = m^n$ for some $m \in \mathbb{N}$.

4 Main results

We are interested in studying an analogue of the discrete isoperimetric inequality obtained in Theorem 1 in the setting of arbitrary non-empty bounded sets in $\mathbb{R}^n$ endowed with (the L_∞ norm and) the lattice point enumerator $\mathrm{G}_n(\cdot)$. In this way, one may consider neighbourhoods of a given set at any distance $t \geq 0$, not necessarily integer (cf. (6)). We show that the extremal sets will be the *extended cubes* $\mathcal{C}_r$ (cf. Definition 2), which satisfy $\mathcal{C}_r \cap \mathbb{Z}^n = \mathcal{I}_r$ (and thus $\mathrm{G}_n(\mathcal{C}_r) = |\mathcal{I}_r| = r$) and are furthermore characterized as the largest sets for which $\mathcal{C}_r + (-1,1)^n = \mathcal{I}_r + (-1,1)^n$, for any $r \in \mathbb{N}$:

Theorem 2 ([3, Theorem 1.2]) *Let $K \subset \mathbb{R}^n$ be a bounded set with $\mathrm{G}_n(K) > 0$ and let $r \in \mathbb{N}$ be such that $\mathrm{G}_n(\mathcal{C}_r) = \mathrm{G}_n(K)$. Then*

$$\mathrm{G}_n\big(K + t[-1,1]^n\big) \geq \mathrm{G}_n\big(\mathcal{C}_r + t[-1,1]^n\big) \tag{7}$$

for all $t \geq 0$.

Remark 1 *From the proof of the previous result we note that the role of the extended cubes $\mathcal{C}_r$ could be played by other sets L_r, with $\mathrm{G}_n(L_r) = r$, such that $L_r + (-1,1)^n \subset \mathcal{I}_r + (-1,1)^n$. However, $\mathcal{C}_r$ are the largest sets (with respect to set inclusion) satisfying this property (cf. [3, (2.8)]). Indeed, for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ such that $x + (-1,1)^n \subset \mathcal{I}_r + (-1,1)^n$, it is enough to consider the point $y = (y_1, \ldots, y_n) \in \big(x + (-1,1)^n\big) \cap \mathbb{Z}^n \subset \big(\mathcal{I}_r + (-1,1)^n\big) \cap \mathbb{Z}^n = \mathcal{I}_r$ given by*

$$y_i = \begin{cases} \lceil x_i \rceil & \text{if } x_i > 0, \\ 0 & \text{if } x_i = 0, \\ \lfloor x_i \rfloor & \text{otherwise,} \end{cases}$$

which yields $x \in \mathcal{C}_{\{y\}} \subset \mathcal{C}_r$.

Remark 2 *Theorem 2 can be extended to the setting of an arbitrary n-dimensional lattice $\Lambda \subset \mathbb{R}^n$. Indeed, if $\mathcal{B} = \{v_1 \ldots, v_n\}$ is a basis of Λ, we may consider $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ the linear (bijective) map given by $\varphi(x) = \sum_{i=1}^n x_i v_i$ for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Then, denoting by $\mathrm{G}_\Lambda(M) = |M \cap \Lambda|$, Theorem 2 yields*

$$\mathrm{G}_\Lambda\Big(K + t\varphi\big([-1,1]^n\big)\Big) \geq \mathrm{G}_\Lambda\Big(\varphi(\mathcal{C}_r) + t\varphi\big([-1,1]^n\big)\Big)$$

for any bounded set $K \subset \mathbb{R}^n$ with $\mathrm{G}_\Lambda(K) > 0$ and all $t \geq 0$, where $r \in \mathbb{N}$ is such that $\mathrm{G}_\Lambda\big(\varphi(\mathcal{C}_r)\big) = \mathrm{G}_\Lambda(K)$.

Finally, we also show that the classical isoperimetric inequality (5), in the setting of non-empty compact sets, can be derived as a consequence of this new discrete inequality for the lattice point enumerator $\mathrm{G}_n(\cdot)$:

Theorem 3 ([3, Theorem 1.4]) *The discrete isoperimetric inequality (7) implies the classical isoperimetric inequality (5), with $E = [-1,1]^n$, for non-empty compact sets.*

References

[1] N. Hamamuki, A discrete isoperimetric inequality on lattices, *Discrete Comput. Geom.* **52** (2) (2014), 221–239.

[2] L. H. Harper, *Global methods for combinatorial isoperimetric problems.* Cambridge Studies in Advanced Mathematics, 90. Cambridge: Cambridge University Press, 2004.

[3] D. Iglesias, E. Lucas and J. Yepes Nicolás, On discrete Brunn-Minkowski and isoperimetric type inequalities. Submitted.

[4] D. Iglesias and J. Yepes Nicolás, On discrete Borell-Brascamp-Lieb inequalities, *Rev. Mat. Iberoam.* **36** (3) (2020), 711–722.

[5] J. Matoušek, *Lectures on discrete geometry.* Graduate Texts in Mathematics, 212. New York: Springer-Verlag, 2002.

[6] A. J. Radcliffe and E. Veomett, Vertex Isoperimetric Inequalities for a Family of Graphs on $\mathbb{Z}^k$, *Electr. J. Comb.* 19 (2) (2012), P45.

[7] M. Ritoré and J. Yepes Nicolás, Brunn-Minkowski inequalities in product metric measure spaces, *Adv. Math.* **325** (2018), 824–863.

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