On Grünbaum type inequalities

Francisco Marín Sola (francisco.marin7@um.es) (joint work with Jesús Yepes Nicolás (jesus.yepes@um.es))

Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100 Murcia, Spain

Abstract

Let $K\subset\mathbb{R}^n$ be a compact set with positive volume $\mathrm{vol}(K)$ (i.e, with positive n-dimensional Lebesgue measure). According to a classical result by Grünbaum [3], if K is convex with centroid at the origin, then

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)}\geq\left(\frac{n}{n+1}\right)^n,$$

where K^- denotes the intersection of K with a halfspace bounded by the hyperplane $H=\{x\in\mathbb{R}^n:\langle x,u\rangle=0\}$, for any given $u\in\mathbb{S}^{n-1}$. Moreover, equality holds, for a fixed $u\in\mathbb{S}^{n-1}$, if and only if K is a cone in the direction u.

Here we show that fixing the hyperplane H, one can find a sharp lower bound for the ratio $\mathrm{vol}(K^-)/\mathrm{vol}(K)$ depending on the concavity nature of the function that gives the volumes of cross-sections (parallel to H) of K. When K is convex, this inequality recovers the previous result by Grünbaum. To this respect, we also show that the log-concave case is the limit concavity assumption for such a generalization of Grünbaum's inequality.

1. Introduction

Let $K\subset\mathbb{R}^n$ be a compact set with positive volume $\mathrm{vol}(K)$ (i.e., with positive n-dimensional Lebesgue measure). The centroid of K is the affine-covariant point

$$\mathrm{g}(K):=\frac{1}{\mathrm{vol}(K)}\int_K x\,\mathrm{d}x.$$

Furthermore, if we write $[\cdot]_1$ for the first coordinate of a vector with respect to the basis, by Fubini's theorem, we get

$$[\mathrm{g}(K)]_1=\frac{1}{\mathrm{vol}(K)}\int_a^b tf(t)\,\mathrm{d}t. \tag{1}$$

Given $u\in\mathbb{S}^{n-1}$ and a hyperplane $H=\{x\in\mathbb{R}^n:\langle x,u\rangle=0\}$, H^- and H^+ will represent the corresponding halfspaces $\big(\{x\in\mathbb{R}^n:\langle x,u\rangle\leq0\}$ and $\{x\in\mathbb{R}^n:\langle x,u\rangle\geq0\}\big)$ bounded by H, whereas K^- and K^+ will denote the intersection of K with H^- and H^+, respectively. The classical result we will focus on, originally proven in [3] and known in the literature as Grünbaum's inequality, reads as follows:

Theorem A (Grünbaum's inequality). *If $K\subset\mathbb{R}^n$ is an n-dimensional compact and convex set with centroid at the origin, then*

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)}\geq\left(\frac{n}{n+1}\right)^n. \tag{2}$$

Equality holds, for a fixed $u\in\mathbb{S}^{n-1}$, if and only if K is a cone in the direction u, i.e., the convex hull of $\{x\}\cup(K\cap(y+H))$, for some $x,y\in\mathbb{R}^n$.

The underlying key fact in the original proof of (2) (see [3]) is the following classical result (see e.g. [1, Section 1.2.1]):

Theorem B (Brunn's concavity principle). *Let $K\subset\mathbb{R}^n$ be a non-empty compact and convex set and let H be a hyperplane. Then, the function $f:H^\perp\longrightarrow\mathbb{R}_{\geq0}$ given by $f(x)=\mathrm{vol}_{n-1}\big(K\cap(x+H)\big)$ is $(1/(n-1))$-concave.*

In other words, for any given hyperplane H, the cross-sections volume function f to the power $1/(n-1)$ is concave on its support, which is equivalent (due to the convexity of K) to the well-known *Brunn-Minkowski inequality*.

Although this property cannot be in general enhanced, one can easily find compact convex sets for which f satisfies a stronger concavity, for a suitable hyperplane H. Thus, on the one hand, it is natural to wonder about a possible enhanced version of Grünbaum's inequality (2) for the family of those compact convex sets K such that (there exists a hyperplane H for which) f is p-concave, with $1/(n-1)<p$. On the other hand, one could expect to extend this inequality to compact sets K, not necessarily convex, for which f is p-concave (for some hyperplane H), with $p<1/(n-1)$.

2. Auxiliary results

Observing that the equality case in Grünbaum's inequality (2) is characterized by cones, that is, those sets for which f is $(1/(n-1))$-affine (i.e., such that $f^{1/(n-1)}$ is an affine function), the following sets of revolution, associated to p-affine functions, arise as natural candidates to be the extremal sets, in some sense, of these inequalities.

Definition 1. *Let $p\in\mathbb{R}$ and let $c,\gamma,\delta>0$ be fixed. Then*

1. *if $p\neq0$, let $g_p:I\longrightarrow\mathbb{R}_{\geq0}$ be the non-negative function given by $g_p(t)=c(t+\gamma)^{1/p}$, where $I=[-\gamma,\delta]$ if $p>0$ and $I=(-\gamma,\delta]$ if $p<0$;*
2. *if $p=0$, let $g_0:(-\infty,\delta]\longrightarrow\mathbb{R}_{\geq0}$ be the non-negative function defined by $g_0(t)=ce^{\gamma t}$.*

Let $u\in\mathbb{S}^{n-1}$ be fixed. By C_p we denote the set of revolution of radius $(g_p(t)/\kappa_{n-1})^{1/(n-1)}$ with axis parallel to u.

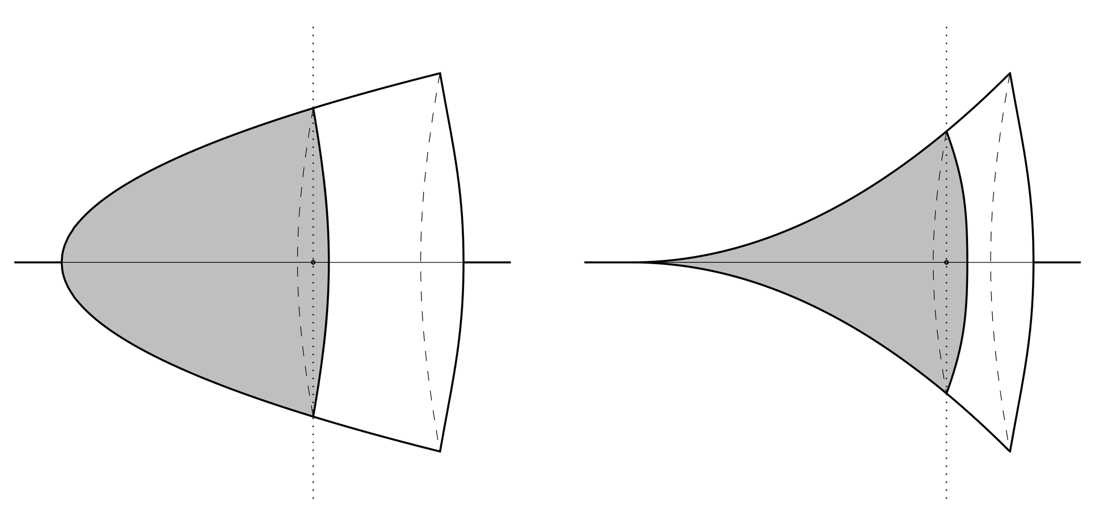

Figure 1: Sets C_p in $\mathbb{R}^3$, with centroid at the origin, and C_p^- (coloured), for $p=1$ (left) and $p=1/4$ (right).

The sets C_p associated to (cross-sections volume) functions that are p-affine seem to be possible extremal sets of such expected inequalities. So, we start by showing the precise value of the ratio $\mathrm{vol}(\cdot^-)/\mathrm{vol}(\cdot)$ for these sets.

Lemma 1 ([4]). *Let $p\in(-\infty,-1)\cup[0,\infty)$ and let H be a hyperplane with unit normal vector $u\in\mathbb{S}^{n-1}$. Let g_p and C_p, with axis parallel to u, be as in Definition 1, for any fixed $c,\gamma,\delta>0$. If C_p has centroid at the origin then*

$$\frac{\mathrm{vol}(C_p^-)}{\mathrm{vol}(C_p)}=\left(\frac{p+1}{2p+1}\right)^{(p+1)/p}$$

where, if $p=0$, the above identity must be understood as

$$\frac{\mathrm{vol}(C_0^-)}{\mathrm{vol}(C_0)}=\lim_{p\to0^+}\left(\frac{p+1}{2p+1}\right)^{(p+1)/p}=e^{-1}.$$

Before showing the general case, we have that if the cross-sections volume function f associated to a compact set K is increasing in the direction of the normal vector of H, then the minimum of the ratios $\mathrm{vol}(K^-)/\mathrm{vol}(K)$ and $\mathrm{vol}(K^+)/\mathrm{vol}(K)$ is attained at $\mathrm{vol}(K^-)/\mathrm{vol}(K)$, independently of the concavity nature of f.

Proposition 2 ([4]). *Let $K\subset\mathbb{R}^n$ be a compact set with non-empty interior and with centroid at the origin. Let H be a hyperplane, with unit normal vector $u\in\mathbb{S}^{n-1}$, such that the function $f:H^\perp\longrightarrow\mathbb{R}_{\geq0}$ given by $f(x)=\mathrm{vol}_{n-1}\big(K\cap(x+H)\big)$ is quasi-concave with $f(bu)=\max_{x\in H^\perp}f(x)$, where $[au,bu]=K|H^\perp$. Then*

$$\frac{\mathrm{vol}(K^+)}{\mathrm{vol}(K)}\geq\frac{1}{2}.$$

3. Main results

In this poster we show that the above-mentioned problem has a positive answer in the full range of $p\in[0,\infty]$. Moreover, we also prove that the log-concave case is the limit concavity assumption for this kind of generalization of Grünbaum's inequality.

3.1. Grünbaum's inequality for sets with a p-concave cross-sections function

Denoting by $\sigma_{H^\perp}$ the *Schwarz symmetrization* with respect to $H^\perp$, our main result reads as follows:

Theorem 3 ([4]). *Let $K\subset\mathbb{R}^n$ be a compact set with non-empty interior and with centroid at the origin. Let H be a hyperplane such that the function $f:H^\perp\longrightarrow\mathbb{R}_{\geq0}$ given by $f(x)=\mathrm{vol}_{n-1}\big(K\cap(x+H)\big)$ is p-concave, for some $p\in[0,\infty)$. If $p>0$ then*

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)}\geq\left(\frac{p+1}{2p+1}\right)^{(p+1)/p}$$

with equality if and only if $\sigma_{H^\perp}(K)=C_p$. If $p=0$ then

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)}\geq e^{-1}.$$

The inequality is sharp; that is, the quotient $\mathrm{vol}(K^-)/\mathrm{vol}(K)$ comes arbitrarily close to e^{-1}.

Assuming that the support of the cross-sections volume function is symmetric with respect to the origin, instead of dealing with sets with centroid at the origin, we get the following:

Corollary 4. *Let $K\subset\mathbb{R}^n$ be a compact set with non-empty interior. Let H be a hyperplane such that the function $f:H^\perp\longrightarrow\mathbb{R}_{\geq0}$ given by $f(x)=\mathrm{vol}_{n-1}\big(K\cap(x+H)\big)$ is p-concave, for some $p\in(0,\infty)$. If the support of f is symmetric with respect to the origin, then*

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)}\geq\left(\frac{1}{2}\right)^{(p+1)/p}$$

with equality if and only if $\sigma_{H^\perp}(K)=C_p$.

Note that the "limit case" $p=\infty$ in Theorem 3 is also trivially fulfilled. Indeed, if f is ∞-concave then f is constant on $[a,b]$ and thus $0=[\mathrm{g}(K)]_1=b+a$ (see (1)), which yields that $a=-b$ and hence

$$\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)}=\frac{1}{2}=\lim_{p\to\infty}\left(\frac{p+1}{2p+1}\right)^{(p+1)/p}.$$

We point out that Theorem 3 can be obtained from recent engaging results in the functional setting (more precisely, the case $p>0$ is derived from [6, Theorem 1] whereas the case $p=0$ follows from [5, Theorem in p. 746] -see also [1, Lemma 2.2.6]). Our goal here is to provide with a simpler geometric proof, inspired by the role of Brunn's concavity principle and comparing with the sets C_p, in the spirit of Grünbaum's approach in [3]. Here we also consider the range of $p\in[-\infty,0)$ and we prove that $[0,\infty]$ is the largest set (where the parameter p lies) in which C_p provides us with the infimum value for such a Grünbaum type inequality.

3.2. The case of $p\in[-\infty,0)$

Let $K\subset\mathbb{R}^n$ be a compact set with non-empty interior and with centroid at the origin, such that its cross-sections volume function f is p-concave, with respect to a given hyperplane H. Moreover, if $p\in(-\infty,-1)\cup(-1/2,\infty)$, we write for short

$$\alpha_p:=\left(\frac{p+1}{2p+1}\right)^{(p+1)/p},$$

where, if $p=0$, α_0 is the value that is obtained "by continuity", that is,

$$\alpha_0=\lim_{p\to0}\left(\frac{p+1}{2p+1}\right)^{(p+1)/p}=e^{-1}.$$

We show that Theorem 3 cannot be extended to the range of $p\in[-\infty,-1)$. In fact, we have a more general result:

Proposition 5 ([4]). *Let $p\in[-\infty,-1)$. There exists no positive constant β_p such that*

$$\min\left\{\frac{\mathrm{vol}(K^-)}{\mathrm{vol}(K)},\frac{\mathrm{vol}(K^+)}{\mathrm{vol}(K)}\right\}\geq\beta_p$$

for all compact sets $K\subset\mathbb{R}^n$ with non-empty interior and with centroid at the origin, for which there exists H such that $f(x)=\mathrm{vol}_{n-1}\big(K\cap(x+H)\big)$, $x\in H^\perp$, is p-concave.

We conclude the poster by showing that the statement of Theorem 3 cannot be extended to the range of $p\in(-1/2,0)$ either. To this aim, note that if $p<q$ are parameters for which β_p and β_q are such sharp lower bounds for the ratio $\mathrm{vol}(K^-)/\mathrm{vol}(K)$ then $\beta_p\leq\beta_q$, because every q-concave function is also p-concave. We notice however that, if $p\in(-1/2,0)$, the value obtained by C_p is not α_p but $1-\alpha_p$ and then $1-\alpha_p\geq1-\alpha_0>1/2$ for any $p\in(-1/2,0)$ whereas $\alpha_p\leq1/2$ for all $p\geq0$.

Therefore, this fact (jointly with the case in which $p\in(-\infty,-1)$, collected in Proposition 5) gives that $[0,\infty]$ is the largest subset of the real line (with respect to set inclusion) for which C_p provides us with the infimum value for the ratio $\mathrm{vol}(\cdot^-)/\mathrm{vol}(\cdot)$, among all compact sets with (centroid at the origin and) p-concave cross-sections volume function.

References

[1] S. Brazitikos, A. Giannopoulos, P. Valettas and B. H. Vritsiou, *Geometry of isotropic convex bodies*, Mathematical Surveys and Monographs, 196. American Mathematical Society, Providence, RI, 2014.

[2] R. J. Gardner, The Brunn-Minkowski inequality, *Bull. Amer. Math. Soc.* **39** (3) (2002): 355–405.

[3] B. Grünbaum, Partitions of mass-distributions and of convex bodies by hyperplanes, *Pacific J. Math.* **10** (1960): 1257–1261.

[4] F. Marín Sola and J. Yepes Nicolás, On Grünbaum type inequalities, *To appear in J. Geom. Anal.*

[5] M. Meyer, F. Nazarov, D. Ryabogin and V. Yaskin, Grünbaum-type inequality for log-concave functions, *Bull. Lond. Math. Soc.* **50** (4) (2018): 745–752.

[6] S. Myroshnychenko, M. Stephen and N. Zhang, Grünbaum's inequality for sections, *J. Funct. Anal.* **275** (9) (2018): 2516–2537.