A Characterization of centrally symmetric convex body in terms of visual cones

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In this work we prove the following result: Let K be a strictly convex body in the Euclidean space $\mathbb{E}^n, n \geq 3$, and let L be a hypersurface, which is the image of an embedding of the sphere $\mathbb{S}^{n-1}$, such that K is contained in the interior of L. Suppose that, for every $x \in L$, there exists $y \in L$ such that the support double-cones with apexes at x and y of K differ by a translation. Then K and L are centrally symmetric and concentric.

Introduction

A classical problem in convexity is to determine properties of a convex body $K \subset \mathbb{R}^n$ from the information of its orthogonal projections, for instance, in dimension 3 one can prove that if all the orthogonal projections of a body K in $\mathbb{R}^3$ are circles, then K is a sphere. One can see this problem from the following perspective: to consider the family of the cylinders where K is inscribed and to impose a condition in a particular section of each of them, which is obtained with a hyperplane perpendicular to the lines which generates the cylinder. In our example, this means that we have a convex body $K \subset \mathbb{R}^3$ such that for every cylinder Ω, where K is inscribed, the section $H \cap \Omega$ is a circle, where H is a plane orthogonal to the lines which determines Ω.

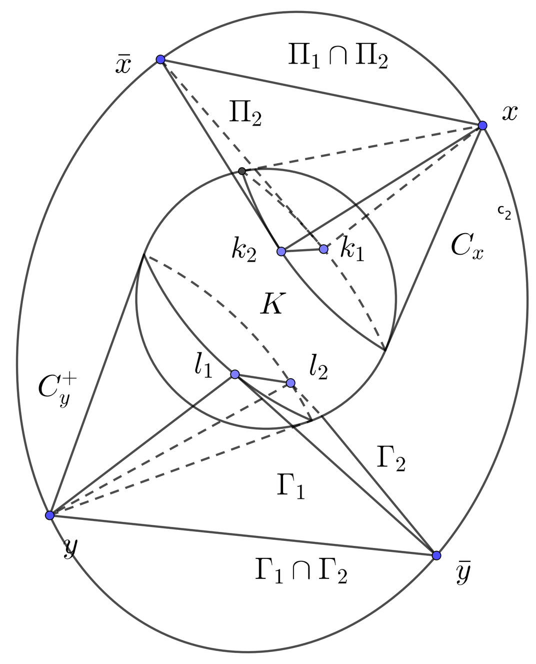
The general problem for cylinders

We can formulate the following general problem:

(I) *Given a subgroup G of the general lineal group* $\mathrm{GL}(\mathbb{R}, n)$ *to determine a convex body $K \subset \mathbb{R}, n \geq 3$ such that, for every couple of different cylinders Λ, Γ, which circumscribes K, there exists an element $\Phi \in G$ such that $\Phi(\Lambda) = \Gamma$.*

Kuzminyh [1] proved, for $n = 3$, that the assumption $G = O(\mathbb{R}, 3)$ implies that K is a sphere, where $O(\mathbb{R}, 3)$ is the real orthogonal group. On the other hand, if $K \subset \mathbb{R}^n, n \geq 3$, is centrally symmetric, in virtue of the Aleksandrov Uniqueness Theorem, it follows that K is a sphere since all the projections have the same volume. Recently L. Montejano [2] has considered the case where G is the affine subgroup $A(\mathbb{R}, n)$ and he has obtained that K is an ellipsoid.

In virtue that the cylinders are cones with apexes at the infinity, the original problem can be generalized in the following manner: *To determine properties of convex bodies imposing conditions on the sections of cones where K is inscribed and whose apexes are contained in a hyperplane.*



Naturally, we can replace in the aforesaid problem the condition that that set of apexes is situated in a hyperplane, instead we can suppose that it is contained in a hypersurface S, in particular, we can assume that S is the boundary of a convex body $M \subset \mathbb{R}^n$ such that $K \subset \operatorname{int} M$. An interesting example of this type is the well known Matsuura's Theorem [3] where S is a sphere.

The general problem for cones

Finally we present our version of the Problem I for cones.

(II) *Given a subgroup G of the general lineal group* $\mathrm{GL}(\mathbb{R}, n)$ *to determine a convex body $K \subset \mathbb{R}^n, n \geq 3$, and an hypersurface S, which is the image of an embedding of $\mathbb{S}^{n-1}$, such that, for every couple of different cones Λ, Γ, which circumscribes K and with apexes in S, there exists an element $\Phi \in G$ such that $\Phi(\Lambda) = \Gamma$.*

A particularly interesting case of problem II is when G is equal to $O(\mathbb{R}, n)$, i.e., *we know that all the cones which circumscribes K and with apexes in S are congruentes.*

We denote by $T(\mathbb{R}, n)$ the family of the translations of $\mathbb{R}^n$. The main result of this work was inspired by the Problem II, however, we involve $T(\mathbb{R}, n)$ which is not a subgroup of $GL(\mathbb{R}, n)$ nevertheless it is an isometry of $\mathbb{R}^n$. Our main theorem claim that if $K \subset \mathbb{E}^n, n \geq 3$, is a strictly convex body and L is a hypersurface, which is the image of an embedding of the sphere $\mathbb{S}^{n-1}$, $K \subset \operatorname{int} L$, and for every $x \in L$, there exists $y \in L$ and $\Phi \in T(\mathbb{R}, n)$ such that $C_y = \Phi(C_y)$, then K and L are centrally symmetric and concentric.

The main result

Let $K \subset \mathbb{R}^n$ be a convex body, $n \geq 3$, and let $x \in \mathbb{R}^n \backslash K$. We call the set

$$\bigcup_{y \in K} \operatorname{aff}\{x, y\}$$

the *solid cone* generated by K and x, where $\operatorname{aff}\{x, y\}$ denotes the affine hull of x and y. The boundary of the the solid cone generated by K and x will be called the cone that *circumscribes K with vertex at x* and it will be denoted by C_x.

Our main result in this work is the following theorem.

Theorem. *Let $K \subset \mathbb{R}^n, n \geq 3$, be a strictly convex body and let L be hypersurface which is an embedding of $\mathbb{S}^{n-1}$ such that $K \subset \operatorname{int} L$. Suppose that for every $x \in L$ there exists $y \in L$ and $p \in \mathbb{R}^n$ such that $C_y = p + C_x$. Then K and L are centrally symmetric and concentric.*

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