Metrics and Isometries for Convex Functions

Ben Li and Fabian Mussnig

Ningbo University and University of Florence

Summary

- Metrics for common spaces of convex functions are introduced.
- It is shown that convergence with respect to these metrics is equivalent to epi-convergence.
- All isometries with respect to a large class of functional analogs of the symmetric difference metric are classified.

Convex Bodies

Let $\mathcal{K}^n$ denote the set of *convex bodies*, i.e., non-empty, compact, convex subsets of $\mathbb{R}^n$. Usually this space is equipped with the *Hausdorff metric*

$$d_H(K,L) = \inf\{\varepsilon \geq 0 \colon K \subseteq L + \varepsilon B^n \text{ and } L \subseteq K + \varepsilon B^n\}$$

for every $K, L \in \mathcal{K}^n$, where $\varepsilon B^n = \{x \in \mathbb{R}^n \colon |x| \leq \varepsilon\}$ denotes the Euclidean ball of radius $\varepsilon \geq 0$ in $\mathbb{R}^n$ and where $C + D = \{x + y \colon x \in C, y \in D\}$ denotes the Minkowski sum of the convex bodies $C, D \in \mathcal{K}^n$.

For many applications also other metrics for (subsets of) convex bodies are used. Most importantly, the set of convex bodies with non-empty interiors, $\mathcal{K}^n_n$, is frequently considered together with the *symmetric difference metric*

$$d_S(K,L) = V_n(K \Delta L)$$

for $K, L \in \mathcal{K}^n_n$, where $K\Delta L = K\backslash L \cup L \backslash K$ is the symmetric difference of K and L. Note that a convex body $K \in \mathcal{K}^n$ has non-empty interior and is therefore an element of $\mathcal{K}^n_n$ if and only if $V_n(K) > 0$. It was shown by Shephard and Webster [4] that the Hausdorff metric and the symmetric difference metric are equivalent on $\mathcal{K}^n_n$, that is

$$d_H(K_i, K) \to 0 \quad \iff \quad d_S(K_i, K) \to 0$$

as $i \to \infty$ for every sequence $K_i \in \mathcal{K}^n_n$ and $K \in \mathcal{K}^n_n$.

In [2], Gruber classified all *isometries* on the metric space $(\mathcal{K}^n_n, d_S)$ and thereby characterized measure-preserving affinities. That is, a map $I : \mathcal{K}^n_n \to \mathcal{K}^n_n$ satisfies

$$d_S(I(K), I(L)) = d_S(K, L)$$

for every $K, L \in \mathcal{K}^n_n$ if and only if there exist $\phi \in \mathrm{GL}(n)$ with $|\det \phi| = 1$ and $x_0 \in \mathbb{R}^n$ such that

$$I(K) = \phi K + x_0$$

for every $K \in \mathcal{K}^n_n$.

Convex Functions

The standard space of convex functions is

$$\mathrm{Conv}(\mathbb{R}^n) = \{u : \mathbb{R}^n \to \mathbb{R} \colon u \not\equiv +\infty,\, u \text{ is lower semicontinuous}\}$$

and it is usually equipped with the topology induced by epi-convergence. Here, we say that a sequence of convex functions $u_k \in \mathrm{Conv}(\mathbb{R}^n)$ is *epi-convergent* to $u \in \mathrm{Conv}(\mathbb{R}^n)$ if for every $x \in \mathbb{R}^n$

- $\liminf_{k\to\infty} u_k(x_k) \geq u(x)$ for every sequence $x_k \in \mathbb{R}^n$ such that $x_k \to x$,
- $\limsup_{k\to\infty} u_k(x_k) \leq u(x)$ for some sequence $x_k \in \mathbb{R}^n$ such that $x_k \to x$.

In this case we will also write $u_k \xrightarrow{epi} u$. This topology is in fact metrizable but precise descriptions of such a metric seem to be cumbersome for many practical purposes.

A functional analog of the volume is given by the map

$$u \mapsto \int_{\mathbb{R}^n} e^{-u(x)} \, \mathrm{d}x$$

for $u \in \mathrm{Conv}(\mathbb{R}^n)$. Similar to the space $\mathcal{K}^n_n$, one is often interested in the space

$$\begin{aligned}\mathrm{Conv}^n_{\mathrm{c}}(\mathbb{R}^n) &= \{u \in \mathrm{Conv}(\mathbb{R}^n) \colon 0 < \textstyle\int_{\mathbb{R}^n} e^{-u(x)} \, \mathrm{d}x < +\infty\}\\ &= \{u \in \mathrm{Conv}(\mathbb{R}^n) \colon \lim_{|x|\to\infty} u(x) = +\infty,\ \dim \mathrm{dom}\, u = n\},\end{aligned}$$

the space of proper, l.s.c., convex, coercive functions with full-dimensional *domain*, $\mathrm{dom}\, u = \{x \in \mathbb{R}^n \colon u(x) < +\infty\}$.

Main Results

Let

$$M^{n-1} := \{\zeta : \mathbb{R} \to (0,\infty) : \zeta \text{ is continuous, strictly decreasing, } \textstyle\int_0^\infty \zeta(t) t^{n-1} \, \mathrm{d}t < +\infty\}.$$

A functional analog of the symmetric difference metric on $\mathrm{Conv}^n_{\mathrm{c}}(\mathbb{R}^n)$ is given by

$$\delta_\zeta(u,v) := \|\zeta(u) - \zeta(v)\|_1 = \int_{\mathbb{R}^n} |\zeta(u(x)) - \zeta(v(x))| \, \mathrm{d}x$$

for $u, v \in \mathrm{Conv}^n_{\mathrm{c}}(\mathbb{R}^n)$. We have the following result.

Theorem

For $\zeta \in M^{n-1}$ the functional δ_ζ defines a metric on $\mathrm{Conv}^n_{\mathrm{c}}(\mathbb{R}^n)$. Furthermore, convergence with respect to this metric is equivalent to epi-convergence, that is, for every $u_k, u \in \mathrm{Conv}^n_{\mathrm{c}}(\mathbb{R}^n)$ we have

$$\delta_\zeta(u_k, u) \to 0 \quad \iff \quad u_k \xrightarrow{epi} u$$

as $k \to \infty$.

The metric δ_ζ corresponds to a measure of the symmetric difference of the epigraphs of the functions. Following ideas of Gruber [2] and also Cavallina and Colesanti [1] we give a full classification of all isometries on $(\mathrm{Conv}^n_{\mathrm{c}}(\mathbb{R}^n), \delta_\zeta)$. For $\zeta \in M^{n-1}$ let

$$\Phi(\zeta) := \{\phi \in \mathrm{GL}(n) \colon t \mapsto \zeta^{-1}(\zeta(t)/|\det \phi|) \text{ is well-defined and convex on } \mathbb{R}\}.$$

Observe that $\Phi(\zeta)$ is not empty since every $\phi \in \mathrm{GL}(n)$ with $|\det \phi| = 1$ satisfies the necessary conditions.

Theorem

Let $\zeta \in M^{n-1}$. A map $I : (\mathrm{Conv}^n_{\mathrm{c}}(\mathbb{R}^n), \delta_\zeta) \to (\mathrm{Conv}^n_{\mathrm{c}}(\mathbb{R}^n), \delta_\zeta)$ is an isometry if and only if there exist $\phi \in \Phi(\zeta)$ and $x_0 \in \mathbb{R}^n$ such that

$$I(u) = f(u \circ \alpha^{-1})$$

for every $u \in \mathrm{Conv}^n_{\mathrm{c}}(\mathbb{R}^n)$, where $\alpha(x) = \phi(x) + x_0$ for $x \in \mathbb{R}^n$ and $f(t) = \zeta^{-1}(\zeta(t)/|\det \phi|)$ for $t \in \mathbb{R}$.

Further Results

We also consider a generalization of δ_ζ where for $p \in [1,\infty)$ the L_p distance of the associated functions $\zeta(u)$ and $\zeta(v)$, $u, v \in \mathrm{Conv}^n_{\mathrm{c}}(\mathbb{R}^n)$, is considered. Moreover, we introduce further metrics on related function spaces which can be interpreted as functional analogs of the Hausdorff metric. For all these metrics we show that convergence is equivalent to epi-convergence. In addition, we establish a functional analog of the Blaschke selection theorem. For details we refer to [3].

References

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Ben Li: liben@nbu.edu.cn Fabian Mussnig: fabian.mussnig@alumni.tuwien.ac.at